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# Common Fixed Point of Weakly Compatible Mappings in Quasi-Gauge Space

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**Abstract.** The aim of this paper is to prove common fixed point theorems for four mappings under the condition of weak compatible mappings in Quasi-gauge space. We point out that the continuity of any mapping for the existence of the fixed point is not required.

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## 1. Introduction

Pathak, Chang and Cho [3] proved fixed point theorems for compatible mappings of type (P). Rao and Murty [4] extended results on common fixed points of self maps by replacing the domain "complete metric space" with "Quasi-gauge space". But in both theorems continuity of any mapping was the necessary condition for the existence of the fixed point. We improve results of Rao and Murty [4] and show that the continuity of any mapping for the existence of the fixed point is not required.

**Definition 1.1.** *A Quasi-pseudo-metric on a set $X$ is a non negative real valued function $p$ on $X \times X$ such that*

(i) $p(x,x) = 0$ *for all* $x \in X$.
(ii) $p(x,z) \leq p(x,y) + p(y,z)$ *for all* $x, y, z \in X$.

**Definition 1.2.** *A Quasi-gauge structure for a topological space $(X,T)$ is a family $\mathcal{P}$ of quasi-pseudo-metrics on $X$ such that $T$ has as a subbase the family*

$$\{B(x,p,\varepsilon) : x \in X, p \in \mathcal{P}, \varepsilon > 0\}$$

*where $B(x,p,\varepsilon)$ is the set $\{y \in X : p(x,y) < \varepsilon\}$. If a topological space has a Quasi-gauge structure, it is called a Quasi-gauge space.*

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**Definition 1.3.** [5] *A sequence $\{x_n\}$ in a Quasi-gauge space $(X, \mathcal{P})$ is said to be P-Cauchy, if for each $\varepsilon > 0$ and $p \in \mathcal{P}$ there is an integer $k$ such that $p(x_m, x_n) < \varepsilon$ for all $m, n \geq k$.*

**Definition 1.4.** [5] *A Quasi-gauge space $(X, \mathcal{P})$ is sequentially complete iff every P-Cauchy sequence in $X$ is convergent in $X$.*

We now propose the following characterization. Let $(X, \mathcal{P})$ be a Quasi-gauge space. $X$ is a $T_0$ Space iff $p(x, y) = p(y, x) = 0$ for all $p$ in $\mathcal{P}$ implies $x = y$. Antony [1] introduced the concept of weak compatibility for a pair of mappings on Quasi-gauge Space.

**Definition 1.5.** [1] *Let $(X, \mathcal{P})$ be a Quasi-gauge Space. The self maps $f$ and $g$ are said to be $(f, g)$ weak compatible if $\lim gfx_n = fz$ for some $z \in X$ whenever $x_n$ is sequence in $X$ such that $\lim fx_n = \lim gx_n = z$ and $\lim fgx_n = \lim ffx_n = fz$.*

$f$ and $g$ are said to be weak compatible to each other if $(f, g)$ and $(g, f)$ are weak compatible. Compatibility implies weak compatibility but the converse is not true in view of the following.

**Example 1.1.** Let $X = \mathbb{R}$ equipped with the usual metric. Define self maps $f$ and $g$ on $X$ as

$$fx = \begin{cases} \frac{10}{32}, & \text{if} x < \frac{3}{8} \\ \frac{3}{8}, & \text{if } \frac{3}{8} \leq x < \frac{1}{2}, \\ 1, & \text{if } x \geq \frac{1}{2} \end{cases} \qquad gx = \begin{cases} \frac{11}{32}, & \text{if } x < \frac{3}{8} \\ \frac{1+x}{4}, & \text{if } \frac{3}{8} \leq x < \frac{1}{2}. \\ \frac{1+x}{2}, & \text{if } x \geq \frac{1}{2} \end{cases}$$

Consider sequences $x_n \to \frac{1}{2}^-$ with $x_n \geq \frac{3}{8}$, clearly, $\lim fx_n = \lim gx_n = \frac{3}{8}$ and also $\lim gfx_n = \lim ggx_n = \frac{11}{32}$ while $\lim fgx_n = \frac{10}{32} \neq \frac{3}{8} = \lim ffx_n$. This shows that $f$ and $g$ are $(g, f)$ weak compatible but not $(f, g)$ weak compatible and hence not compatible.

The following is useful in establishing our result.

**Lemma 1.1.** [2] *Suppose that $\psi : [0, \infty) \to [0, \infty)$ is non-decreasing and upper semi continuous from the right. If $\psi(t) < t$ for every $t > 0$, then $\lim \psi^n(t) = 0$.*

## 2. Main results

Rao and Murty [4] proved the following.

**Theorem 2.1.** *Let $A, B, S$ and $T$ be self maps on a left (right) sequentially complete Quasi-gauge $T_0$ Space $(X, \mathcal{P})$ such that*

(i) *$(A, S), (B, T)$ are weakly compatible pairs of maps with $T(X) \subseteq A(X); S(X) \subseteq B(X)$;*
(ii) *$A$ and $B$ are continuous;*
(iii)

$$\max\{p^2(Sx, Ty), p^2(Ty, Sx)\}$$
$$\leq \phi\{\, p(Ax, Sx)\, p(By, Ty), p(Ax, Ty)\, p(By, Sx),$$

$$p(Ax, Sx)\,p(Ax, Ty), p(By, Sx)\,p(By, Ty),$$
$$p(By, Sx)\,p(Ax, Sx), p(By, Ty)\,p(Ax, Ty)\ \}$$

*for all $x, y \in X$ and for all $p$ in $\mathcal{P}$, where $\phi : [0,\infty)^6 \to (0,+\infty)$ satisfies the following:*

(iv) *$\phi$ is non-decreasing and upper semi-continuous in each coordinate variable and for each*

$$t > 0, \psi(t) = \max\{\phi(t,0,2t,0,0,2t), \phi(t,0,0,2t,2t,0), \phi(0,t,0,0,0,0)\} < t;$$

*then $A, B, S$ and $T$ have a unique common fixed point.*

**Theorem 2.2.** *Let $A, B, S$ and $T$ be self maps on a left (right) sequentially complete Quasi-gauge $T_0$ space $(X, \mathcal{P})$ with condition* (iii) *and* (iv) *of Theorem 2.1, such that*

(i) *$(S, A), (A, S), (B, T)$ and $(T, B)$ are weakly compatible pairs of maps with $T(X) \subseteq A(X); S(X) \subseteq B(X)$;*

(ii) *One of $A, B, S$ and $T$ is continuous;*

*then the same conclusions of Theorem 2.1 holds.*

We prove Theorem 2.1 and Theorem 2.2 without assuming that any function is continuous.

We prove the following:

**Theorem 2.3.** *Let $A, B, S$ and $T$ be self maps on a left (right) sequentially complete Quasi-gauge $T_0$ Space $(X, \mathcal{P})$ such that*

(2.1) *$(A, S)$ and $(B, T)$ are weakly compatible pairs of mappings with $T(X) \subseteq A(X); S(X) \subseteq B(X)$;*

$$\max\{p^2(Sx,Ty), p^2(Ty,Sx)\} \le \phi\,\{p(Ax,Sx)\,p(By,Ty), p(Ax,Ty)\,p(By,Sx),$$
$$p(Ax,Sx)\,p(Ax,Ty), p(By,Sx)\,p(By,Ty),$$
$$p(By,Sx)\,p(Ax,Sx), p(By,Ty)\,p(Ax,Ty)\}\,; \tag{2.2}$$

*for all $x, y \in X$ and for all $p$ in $\mathcal{P}$, where $\phi : [0,\infty)^6 \to (0,+\infty)$ satisfies the following :*

(2.3) *$\phi$ is non-decreasing and upper semi-continuous in each coordinate variable and for each $t > 0$ :*

$$\psi(t) = \max\{\phi(t,0,2t,0,0,2t), \phi(t,0,0,2t,2t,0), \phi(0,t,0,0,0,0),$$
$$\phi(0,0,0,0,0,t), \phi(0,0,0,0,t,0)\} < t.$$

*Then $A, B, S$ and $T$ have a unique common fixed point.*

*Proof.* Let $x_0$ be an arbitrary point in $X$ . Since (2.1) holds we can choose $x_1, x_2$ in $X$ such that $Bx_1 = Sx_0$ and $Ax_2 = Tx_1$; in general we can choose $x_{2n+1}$ and $x_{2n+2}$ in $X$ such that

$$y_{2n} = Bx_{2n+1} = Sx_{2n} \text{ and } y_{2n+1} = Ax_{2n+2} = Tx_{2n+1}; n = 0, 1, 2 \ldots\ldots\ldots \tag{2.4}$$

We denote $d_n = p(y_n, y_{n+1})$ and $e_n = p(y_{n+1}, y_n)$; now applying (2.2) we get

$$\begin{aligned}
&\max\{d_{2n+2}^2, e_{2n+2}^2\}\\
&= \max\{p^2(Sx_{2n+2}, Tx_{2n+3}), p^2(Tx_{2n+3}, Sx_{2n+2})\}\\
&\leq \Phi\{\, p(Ax_{2n+2}, Sx_{2n+2})\, p(Bx_{2n+3}, Tx_{2n+3}),\\
&\quad p(Ax_{2n+2}, Tx_{2n+3})\, p(Bx_{2n+3}, Sx_{2n+2}),\\
&\quad p(Ax_{2n+1}, Sx_{2n+2})\, p(Ax_{2n+2}, Tx_{2n+1}),\\
&\quad p(Bx_{2n+1}, Sx_{2n+2})\, p(Bx_{2n+2}, Tx_{2n+3}),\\
&\quad p(Bx_{2n+3}, Sx_{2n+2})\, p(Ax_{2n+2}, Sx_{2n+1}),\\
&\quad p(Bx_{2n+3}, Tx_{2n+3})\, p(Ax_{2n+2}, Tx_{2n+3})\};\\
&= \Phi\{\, p(y_{2n+1}, y_{2n+2})\, p(y_{2n+2}, y_{2n+3}), p(y_{2n+1}, y_{2n+3})\, p(y_{2n+2}, y_{2n+2}),\\
&\quad p(y_{2n+1}, y_{2n+2})\, p(y_{2n+1}, y_{2n+3}), p(y_{2n+2}, y_{2n+2})\, p(y_{2n+2}, y_{2n+3}),\\
&\quad p(y_{2n+2}, y_{2n+2})\, p(y_{2n+1}, y_{2n+2}), p(y_{2n+2}, y_{2n+3})\, p(y_{2n+1}, y_{2n+3})\,\}
\end{aligned}$$

$$\leq \Phi\{d_{2n+1}\, d_{2n+2}, 0, d_{2n+1}(d_{2n+1} + d_{2n+2}), 0, 0, d_{2n+2}(d_{2n+1} + d_{2n+2})\} \tag{2.5}$$

If $d_{2_n+2} > d_{2n+1}$ then

$$\max\{d_{2n+2}^2, e_{2n+2}^2\} \leq \Phi\{d_{2n+2}^2, 0, 2d_{2n+2}^2, 0, 0, 2d_{2n+2}^2\} < d_{2n+2}^2, \tag{2.6}$$

by (2.3) a contradiction; hence $d_{2n+2} \leq d_{2n+1}$. Similarly we get

$$d_{2n+1} \leq d_{2n}. \tag{2.7}$$

By (2.5) and (2.6)

$$\begin{aligned}
\max\{d_{2n+2}^2, e_{2n+2}^2\} &\leq \Phi\{d_{2n+1}^2, 0, 2d_{2n+1}^2, 0, 0, 2d_{2n+1}^2\}\\
&\leq \psi(d_{2n+1}^2) = \psi\{p^2(y_{2n+1}, y_{2n+2})\}
\end{aligned} \tag{2.8}$$

Similarly we have

$$\begin{aligned}
\max\{d_{2n+1}^2, e_{2n+1}^2\} &\leq \Phi\{d_{2n}^2, 0, 0, 2d_{2n}^2, 2d_{2n}^2, 0\}\\
&\leq \psi\{p^2(y_{2n}, y_{2n+1})\}
\end{aligned} \tag{2.9}$$

So

$$d_n^2 = p^2(y_n, y_{n+1}) \leq \psi\{p^2(y_{n+1}, y_n)\} \leq \ldots \leq \psi^{n-1}\{p^2(y_1, y_2)\} \tag{2.10}$$

and

$$e_n^2 = p^2(y_{n+1}, y_n) \leq \psi\{p^2(y_{n-1}, y_n)\} \leq \ldots \leq \psi^{n-1}\{p^2(y_1, y_2)\} \tag{2.11}$$

hence by Lemma 1.1 and from (2.10) and (2.11) we obtain

$$\lim d_n = e_n = 0. \tag{2.12}$$

Now we prove $\{y_n\}$ is a P-Cauchy sequence. To show $\{y_n\}$ is P-Cauchy it is enough if we show $\{y_{2n}\}$ is P-Cauchy. Suppose $\{y_{2n}\}$ is not a P-Cauchy sequence then there is an $\varepsilon > 0$ such that for each positive integer $2k$ there exist positive integers $2m(k)$ and $2n(k)$ such that for some $p$ in $\mathcal{P}$,

$$p(y_{2n(k)}, y_{2m(k)}) > \varepsilon \text{ for } 2m(k) > 2n(k) > 2k \tag{2.13}$$

and

$$p(y_{2m(k)}, y_{2n(k)}) > \varepsilon \text{ for } 2m(k) > 2n(k) > 2k \tag{2.14}$$

for each positive even integer $2k$, let $2m(k)$ be the least positive even integer exceeding $2n(k)$ and satisfying (2.13); hence $p(y_{2n(k)}, y_{2m(k)-2}) \leq \varepsilon$ then for each even integer $2k$,

$$\begin{aligned} \varepsilon &< p(y_{2n(k)}, y_{2m(k)}) \\ &\leq p(y_{2n(k)}, y_{2m(k)-2}) + (d_{2m(k)-2} + d_{2m(k)-1}) \end{aligned} \tag{2.15}$$

from (2.12) and (2.15), we obtain $\lim p(y_{2n(k)}, y_{2m(k)}) = \varepsilon$. By the triangle inequality

$$p(y_{2n(k)}, y_{2m(k)}) \leq p(y_{2n(k)}, y_{2m(k)-1}) + d_{2m(k)-1}$$
$$p(y_{2n(k)}, y_{2m(k)-1}) \leq p(y_{2n(k)}, y_{2m(k)}) + e_{2m(k)-1};$$

So

$$|p(y_{2n(k)}, y_{2m(k)}) - p(y_{2n(k)}, y_{2m(k)-1})| \leq \max\{d_{2m(k)-1}, e_{2m(k)-1}\}. \tag{2.16}$$

Similarly by triangle inequality

$$\begin{aligned} &|p(y_{2n(k)+1}, y_{2m(k)-1}) - p(y_{2n(k)}, y_{2m(k)}| \\ &\leq \max\{e_{2n(k)} + e_{2m(k)-1}, d_{2n(k)} + d_{2m(k)-1}\} \end{aligned} \tag{2.17}$$

from (2.16) and (2.17) as $k \to \infty$, $\{p(y_{2n(k)}, y_{2m(k)-1})\}$ and $\{p(y_{2n(k)+1}, y_{2m(k)-1})\}$ converge to $\varepsilon$. Similarly if $p(y_{2m(k)}, y_{2n(k)}) > \varepsilon$,

$$\begin{aligned} \lim p(y_{2m(k)}, y_{2n(k)}) &= \lim p(y_{2m(k)-1}, y_{2n(k)+1}) \\ &= \lim p(y_{2m(k)-1}, y_{2n(k)}) = \varepsilon \text{ as } k \to \infty. \end{aligned}$$

By (2.2)

$$\begin{aligned} \varepsilon &< p(y_{2n(k)}, y_{2m(k)}) \\ &\leq p(y_{2n(k)}, y_{2n(k)+1}) + p(y_{2n(k)+1}, y_{2m(k)}) \\ &\leq d_{2n(k)} + \max\{p(y_{2n(k)+1}, y_{2m(k)}), p(y_{2n(k)}, y_{2n(k)+1})\} \\ &= d_{2n(k)} + \max\{p(Tx_{2n(k)+1}, Sx_{2m(k)}), p(Sx_{2m(k)}, Tx_{2n(k)+1})\} \\ &\leq d_{2n(k)} + [\ \Phi\ \{ p(y_{2m(k)-1}, y_{2m(k)})\, p(y_{2n(k)}, y_{2n(k)+1}), \\ &\qquad p(y_{2m(k)-1}, y_{2n(k)+1})\, p(y_{2n(k)}, y_{2m(k)}), \\ &\qquad p(y_{2m(k)-1}, y_{2m(k)})\, p(y_{2m(k)-1}, y_{2n(k)+1}), \\ &\qquad p(y_{2n(k)}, y_{2m(k)})\, p(y_{2n(k)}, y_{2n(k)+1}), \\ &\qquad p(y_{2n(k)}, y_{2m(k)})\, p(y_{2m(k)-1}, y_{2m(k)}), \\ &\qquad p(y_{2n(k)}, y_{2n(k)+1})\, p(y_{2m(k)-1}, y_{2n(k)+1})\ \}\ ]^{\frac{1}{2}} \end{aligned}$$

Since $\Phi$ is upper semi-continuous, as $k \to \infty$ we get that $\varepsilon \leq \{\Phi(0, \varepsilon^2, 0, 0, 0, 0)\}^{\frac{1}{2}} < \varepsilon$ which is a contradiction. Therefore $\{y_n\}$ is P-Cauchy sequence in $X$. Since $X$ is complete there exists a point $z$ in $X$ such that $\lim n \to \infty y_n = z$.

$$\lim n \to \infty Ax_{2n} = \lim n \to \infty Tx_{2n-1} = z$$

and

$$\lim n \to \infty Bx_{2n+1} = \lim n \to \infty Sx_{2n-2} = z.$$

Since $S(X) \subseteq B(X)$, there exist a point $u \in X$ such that $z = Bu$. Then using (2.2),

$$\begin{aligned}
&\max\{p^2(Sx_{2n}, Tu), p^2(Tu, Sx_{2n})\} \\
&\leq \Phi\{p(Ax_{2n}, Sx_{2n})\, p(Bu, Tu), p(Ax_{2n}, Tu)\, p(Bu, Sx_{2n}), \\
&\quad p(Ax_{2n}, Sx_{2n})\, p(Ax_{2n}, Tu), p(Bu, Sx_{2n})\, p(Bu, Tu), \\
&\quad p(Bu, Sx_{2n})\, p(Ax_{2n}, Sx_{2n}), p(Bu, Tu)\, p(Ax_{2n}, Tu)\}
\end{aligned}$$

Taking limit as $n \to \infty$,

$$\begin{aligned}
&\max\{p^2(z, Tu), p^2(Tu, z)\} \\
&\leq \Phi\{p(z, z)\, p(z, Tu), p(z, Tu)\, p(z, z), p(z, z)\, p(z, Tu), \\
&\quad p(z, z)\, p(z, Tu), p(z, z)\, p(z, z), p(z, Tu)\, p(z, Tu)\} \\
&\leq \Phi\{0, 0, 0, 0, 0, p(z, Tu)\, p(z, Tu)\} \\
&< p(z, Tu)\, p(z, Tu)
\end{aligned}$$

a contradiction. Thus $Tu = z$. Therefore $Tu = z = Bu$. Similarly, since $T(X) \subseteq A(X)$ there exist a point $v \in X$, such that $z = Av$. Then using (2.2),

$$\begin{aligned}
&\max\{p^2(Sv, Tx_{2n+1}), p^2(Tx_{2n+1}, Sv)\} \\
&\leq \Phi\{p(Av, Sv)\, p(Bx_{2n+1}, Tx_{2n+1}), p(Av, Tx_{2n+1})\, p(Bx_{2n+1}, Sv), \\
&\quad p(Av, Sv)\, p(Av, Tx_{2n+1}), p(Bx_{2n+1}, Sv)\, p(Bx_{2n+1}, Tx_{2n+1}), \\
&\quad p(Bx_{2n+1}, Sv)\, p(Av, Sv), p(Bx_{2n+1}, Tx_{2n+1})\, p(Av, Tx_{2n+1})\}
\end{aligned}$$

Taking limit as $n \to \infty$,

$$\begin{aligned}
&\max\{p^2(Sv, z), p^2(z, Sv)\} \\
&\leq \Phi\{p(z, Sv)\, p(z, z), p(z, z)\, p(z, Sv), \\
&\quad p(z, Sv)\, p(z, z), p(z, Sv)\, p(z, z), \\
&\quad p(z, Sv)\, p(z, Sv), p(z, z)\, p(z, z)\} \\
&\leq \Phi\{0, 0, 0, 0, p(z, Sv)\, p(z, Sv), 0\} \\
&< p(z, Sv)\, p(z, Sv)
\end{aligned}$$

a contradiction. Thus $z = Sv$. Therefore $z = Sv = Av$. Hence, $z = Bu = Tu = Av = Sv$. Since the pair of mappings $B$ and $T$ are weakly compatible, then $BTu = TBu$ i.e. $Bz = Tz$. Now we show that $z$ is a fixed point of $T$. If $Tz \neq z$ , then by (2.2)

$$\begin{aligned}
&\max\{p^2(Sx_{2n}, Tz), p^2(Tz, Sx_{2n})\} \\
&\leq \Phi\{p(Ax_{2n}, Sx_{2n})\, p(Bz, Tz), p(Ax_{2n}, Tz)\, p(Bz, Sx_{2n}), \\
&\quad p(Ax_{2n}, Sx_{2n})\, p(Ax_{2n}, Tz), p(Bz, Sx_{2n})\, p(Bz, Tz), \\
&\quad p(Bz, Sx_{2n})\, p(Ax_{2n}, Sx_{2n}), p(Bz, Tz)\, p(Ax_{2n}, Tz)\}
\end{aligned}$$

Taking limit as $n \to \infty$,

$$\max\{p^2(z, Tz), p^2(Tz, z)\}$$

$$\begin{aligned}
&\leq \Phi \{ p(z,z)\, p(Bz,Tz), p(z,Tz)\, p(Bz,z), \\
&\qquad p(z,z)\, p(z,Tz), p(Bz,z)\, p(Bz,Tz), \\
&\qquad p(Bz,z)\, p(z,z), p(Bz,Tz)\, p(z,Tz) \} \\
&\leq \Phi\{0,0,0,0,0, p(z,Tz)\, p(z,Tz)\} \\
&< p(z,Tz)\, p(z,Tz)
\end{aligned}$$

a contradiction. Thus $Tz = z$. Therefore $Tz = z = Bz$. Similarly we prove that $Sz = z = Az$. Hence $Az = Bz = Sz = Tz = z$; thus $z$ is a common fixed point of $A, B, S$, and $T$. Uniqueness follows trivially. Therefore $z$ is a unique common fixed point of $A, B, S$, and $T$. ∎

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