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Notes on Non-Vanishing Elements of Finite Solvable Groups

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Abstract. Let G be a finite solvable group. The element $g \in G$ is said to be a non-vanishing element of G if $\chi(g) \neq 0$ for all $\chi \in \mathrm{Irr}\ (G)$. It is conjectured that all of non-vanishing elements of G lie in its Fitting subgroup $F(G)$. In this note, we prove that this conjecture is true for nilpotent-by-supersolvable groups. Write $\mathscr{V}(G)$ to denote the subgroup generated by all non-vanishing elements of G, and $F_n(G)$ the nth term of the ascending Fitting series. It is proved that $\mathscr{V}(F_n(G)) \leq F_{n-1}(G)$ whenever G is solvable. If this conjecture were not true, then it is proved that the minimal counterexample is a solvable primitive permutation group and the more detailed information is presented. Some other related results are proved.

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1. Introduction

Throughout this note, G always denotes a finite group and $\mathrm{Irr}\ (G)$ denotes the full set of complex irreducible characters of G. Let $\chi \in \mathrm{Irr}\ (G)$. If $g \in G$ satisfies $\chi(g) \neq 0$, then g is said to be a non-vanishing element of χ; further if g is a non-vanishing element for all members of $\mathrm{Irr}\ (G)$, then g is said to be a non-vanishing element of G. In [4], it is conjectured that all non-vanishing elements of finite solvable group G lie in its Fitting subgroup $F(G)$, which is the largest nilpotent normal subgroup of G. This assertion was referred to as Isaacs-Navarro-Wolf Conjecture in [8]. We use $\mathscr{V}(G)$ to denote the subgroup generated by all non-vanishing elements of G, i.e., $\mathscr{V}(G) = \langle g \mid \chi(g) \neq 0,\ \text{all}\ \chi \in \mathrm{Irr}\ (G)\rangle$, which is called the strongly vanishing-off subgroup of G. Expressed in terms of $\mathscr{V}(G)$, this conjecture equivalently asserts that the inequality $\mathscr{V}(G) \leq F(G)$ is true for solvable group G. Some of results are obtained in [4]. For examples, it was proved in [4, Theorem D] that the images of non-vanishing elements modulo $F(G)$ are of 2-power order, which implies that the

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conjecture is true for groups of odd order. It is also proved in [4, Theorem B] that $\mathscr{V}(G)$ lies in the center $Z(F(G))$ of $F(G)$ for supersolvable group G, in particular, if G is nilpotent, then $\mathscr{V}(G) \leq Z(G)$.

The latter result motivates us to consider the following problem: If the solvable group G is not supersolvable, but all of its proper subgroups or all of its proper homomorphic images (quotient groups) are supersolvable, then does the conjecture hold for G? We affirmatively answer the problem, actually we prove a generalized result.

Theorem 1.1. *Assume that G is a nilpotent-by-supersolvable group. Then $\mathscr{V}(G) \leq F(G)$.*

Observe that the groups in the above problem are all nilpotent-by-supersolvable groups, thus the above problem is positively answered. If G is solvable but not nilpotent, then Theorem 2.4 of [4] shows that $\mathscr{V}(G)$ lies in the penultimate of the ascending Fitting series of G. By $F_n(G)$, we denote the nth term of the ascending Fitting series of finite group G. The following result is an improved version of Theorem 2.4 of [4].

Theorem 1.2. *Assume that G is a solvable group but not nilpotent. Then $\mathscr{V}(F_n(G)) \leq F_{n-1}(G)$.*

If this conjecture were false, then the following result shows that the minimal counterexample is a primitive solvable permutation group.

Theorem 1.3. *If Isaacs-Navarro-Wolf Conjecture were not true, then the minimal counterexample G would be a primitive solvable permutation group. Furthermore, $\mathscr{V}(G) = F(G) \rtimes Q$, a semidirect product of 2-group Q acting coprimely and faithfully on elementary abelian group $F(G)$.*

We mention that, for a solvable group G, A. Moretó and T. R. Wolf proved in [8] that $\mathscr{V}(G) \leq F_{10}(G)$, and Yong Yang further proved in [11] that $\mathscr{V}(G) \leq F_8(G)$.

In general, Isaacs-Navarro-Wolf Conjecture is not true for non-solvable groups. For examples, $\mathscr{V}(A_5) = 1$, but $\mathscr{V}(A_7) = A_7$. In fact, using GAP (see [10]), one may still verify $\mathscr{V}(A_{11}) = A_{11}$. Here A_n denote alternating groups of degree n.

2. Preliminaries

The following list some basic properties of $\mathscr{V}(G)$.

Proposition 2.1. *Assume that G is a finite solvable group and $\mathscr{V}(G)$ is its strongly vanishing-off subgroup. Then*

1. *$\mathscr{V}(G)$ is a characteristic subgroup of G.*
2. *$\mathscr{V}(G)$ is a proper subgroup of G whenever G is nonabelian.*
3. *If N is a normal subgroup of G, then the preimage of $\mathscr{V}(G/N)$ in G contains $\mathscr{V}(G)$.*

Proof. Let σ be an automorphism of G, then since $\chi^\sigma(\sigma(g)) = \chi(g)$, we get that g is a non-vanishing element of G if and only if $\sigma(g)$ is, part 1 follows. Let n be the Fitting height of G. If $n \geq 2$, then we have that $\mathscr{V}(G)$ lies in the penultimate term of the ascending Fitting series by [4, Theorem 2.4], hence it is a proper subgroup; if $n = 1$, then G is a nilpotent group, Theorem B of [4] shows that $\mathscr{V}(G) \leq Z(G) < G$(since G is nonabelian), yielding part 2. Let $M/N = \mathscr{V}(G/N)$, then for any non-vanishing element g of G, gN is clear a non-vanishing element of G/N and lies in M/N, thus we conclude that $g \in M$ and $\mathscr{V}(G) \leq M$, yielding part 3. ∎

It is easy to see that all of non-vanishing elements of G lie in $F(G)$ if and only if $\mathscr{V}(G) \leq F(G)$. We shall freely use the above facts without reference. The following lemma is quite essential to our work.

Lemma 2.1. *Let $M \geq N$ be normal subgroups of G. If $\theta^M = e\eta$ for $\theta \in \mathrm{Irr}\ (N)$, $\eta \in \mathrm{Irr}\ (M)$ and e a positive integer, then there exist $\chi \in \mathrm{Irr}\ (G)$ such that $\chi(a) = 0$ for all $a \in M - N$.*

Proof. It is immediate that $\eta(a) = 0$ for any $a \in M - N$. Since $\eta \in \mathrm{Irr}\ (M)$, we get that $\eta^g \in \mathrm{Irr}\ (M)$ for any $g \in G$ and $\eta^g(a) = \eta(gag^{-1}) = 0$ for any $a \in M - N$. Observe that, for all $g \in G$, $a^g \in M - N$ if and only if $a \in M - N$. For any $\chi \in \mathrm{Irr}\ (G)$ with $[\chi_M, \eta] \neq 0$, we know that χ_M is a sum of some conjugates of η by elements of G. Thus $\chi(a) = 0$ for all $a \in M - N$. ∎

Corollary 2.1. *Let N be a subnormal subgroup, and assume that $\chi = \theta^G$ is irreducible. If $\chi(a) \neq 0$, then $a \in N$.*

Proof. Using induction on $|G|$. Let $N \leq M \lhd G$, $\chi = \eta^G$ and $\eta = \theta^M$. By the above lemma, we get that $a \in M$ and some conjugate $\eta^t(a) \neq 0$. It is seen that $\eta^t = (\theta^t)^M$. Applying the inductive hypothesis to $|M|$, we conclude that $a \in N$. ∎

Lemma 2.2. *Let G be a non-nilpotent group with $\Phi(G) = 1$, and D be the intersection of all non-normal maximal subgroups. Then $D = Z(G)$.*

Proof. Let M be any non-normal maximal subgroup, it is easy to see that $Z(G) \leq M$ and so $Z(G) \leq D$. Conversely, since $[G, D] \leq G' \cap D \leq \Phi(G) = 1$, it follows that $D \leq Z(G)$. We attain that $D = Z(G)$, as desired. ∎

The following result is a sufficient condition for the existence of a regular orbit, which is a known fact.

Lemma 2.3. *Let A be an abelian group and assume that U is a completely reducible and faithful FA-module where F is a finite field of order p. Then A has regular orbits on U and $\mathrm{Irr}\ (U)$, respectively.*

Sketch of proof. By Proposition 0.20 of [7], we get that p does not divide $|A|$. By using routine arguments, the desired result may follow from [4, Lemma 3.1] and [12, Lemma 1]. Or see the proof of Theorem 18.1 of [7]. ∎

3. Proofs of main results

Proof of Theorem 1.1. Use induction on $|G|$, the order of G. By induction, if $\Phi(G) \neq 1$, we get that $\mathscr{V}(G/\Phi(G)) \leq F(G/\Phi(G)) = F(G)/\Phi(G)$, thus $\mathscr{V}(G) \leq F(G)$. Hence we may assume that $\Phi(G) = 1$. Likewise, we may also assume that $Z(G) = 1$. If G is nilpotent, the result is trivial. Otherwise, let $H_1, H_2, \cdots, H_n$ be non-normal maximal subgroups of G and $K_i = \mathrm{Core}_G(H_i)$, the intersections of all conjugates of H_i in G. By Lemma 2.2, we may take n minimal such that $\cap_{i=1}^n K_i = 1$. Assume that $n > 1$ and $N = \cap_{i=1}^{n-1} K_i$. Then there exists an injective homomorphism τ from G into $G/N \times G/K_n$, defined by $g \mapsto (gN, gK_n)$. By induction, we have

$$\mathscr{V}(G/N) \leq F(G/N) \quad \text{and} \quad \mathscr{V}(G/K_n) \leq F(G/K_n).$$

Since also

$$\mathscr{V}(G) \cong \tau(\mathscr{V}(G)) \leq \mathscr{V}(G/N) \times \mathscr{V}(G/K_n) \leq F(G/N) \times F(G/K_n)$$

which is nilpotent. Hence $\mathscr{V}(G)$ is nilpotent and $\mathscr{V}(G) \leq F(G)$.

Now we assume that $n = 1$, thus G primitively permutes the collection of cosets of H_1 in G. We get that G is a solvable primitive permutation group and Galois' Satz. II.3.2 [3] shows that $G = F(G) \rtimes H_1$, a semidirect product of H_1 acting on $F(G)$, and $F(G)$ is the unique minimal normal subgroup of G. It is obvious that $F(G)$ is an elementary abelian p-group. Also G is nilpotent-by-supersolvable, we conclude that H_1 is supersolvable. Using Theorem B of [4] (see the above Introduction), we get that $\mathscr{V}(G/F(G)) \leq Z(F(G/F(G)))$. Then $\mathscr{V}(G) = F(G) \rtimes H_2$, the semidirect product of abelian group H_2 acting faithfully on elementary abelian group $F(G)$. Thus $F(G)$ is a completely reducible and faithful FH_2-module, where F is a finite field of order p. By Lemma 2.3, H_2 has a regular orbit on Irr $(F(G))$, i.e., there exists a character of $F(G)$ inducing irreducibly to $\mathscr{V}(G)$. Lemma 2.1 shows that for each element $x \in \mathscr{V}(G) - F(G)$, there exits $\chi \in$ Irr (G) such that $\chi(x) = 0$. This fact forces that $\mathscr{V}(G) = F(G)$, which contradicts the fact that G is a counterexample. The proof is completed. ∎

Corollary 3.1. *Suppose that solvable group G is not supersolvable, but all of its proper subgroups or all of its proper homomorphic images are supersolvable. Then $\mathscr{V}(G) \leq F(G)$.*

Proof. By [2], we know that if G satisfies the hypotheses, then G is a semidirect product of S acting on N. The subgroup N is either a Sylow p-subgroup or an abelian subgroup; and the subgroup S always be supersolvable. Thus G is a nilpotent-by-supersolvable and the desired result follows from the above theorem. ∎

Using Theorem 1.1 of [9] and Theorem 1.1, we may get that if solvable group G has an irreducible character which exactly vanishes on a conjugacy class, then $\mathscr{V}(G) \leq F(G)$. The following is also an application of Theorem 1.1.

Corollary 3.2. *Suppose that G is a solvable group with $cd(G) = \{1, m, n\}$ where m and n are relatively prime. Then the conjecture is true for G.*

Proof. By Carrison's theorem 12.21 of [5], we know that the Fitting height $h(G)$ of a solvable group G is not greater than $|cd(G)|$. In our situation, we get that $h(G) \leq 3$. Since also $(m, n) = 1$ and $mn \notin cd(G)$, it follows that G is not nilpotent and so $2 \leq h(G) \leq 3$. If $h(G) = 2$, then G is nilpotent-by-nilpotent, yielding the result; Otherwise $h(G) = 3$, by Lemma 3.1(a) of [6], we know that $G/F(G)$ is supersolvable, then G is a nilpotent-by-supersolvable group, the desired result may follows from Theorem 1.1. ∎

It is proved in [4, Theorem B] that $\mathscr{V}(G) \leq Z(G)$ for nilpotent group G. The next theorem is one of its generalized versions.

Theorem 3.1. *Let G be a finite group and $Z_n(G)$ be the final term of the upper central series of G. Suppose that $Z(G)$ is the center of G. Then for any $g \in Z_n(G) - Z(G)$, there exist $\chi \in \mathrm{Irr}\ (G)$ such that $\chi(g) = 0$.*

Proof. It is sufficient to prove that the claim is true for any $g \in Z_{i+1}(G) - Z_i(G)$. Furthermore we only need to focus on the case $i = 1$ and then apply the special case to each of the quotient groups $G/Z_i(G), i = 1, 2, \cdots, n-1$. Let $Z_1(G) = Z(G)$ and $g \in Z_2(G) - Z(G)$. Suppose that $\mu_1, \mu_2, \cdots, \mu_t$ are all of irreducible characters of $Z(G)$. It is well-known that the intersection of the kernels of all these characters is trivial. Pick $g \in Z_2(G) - Z(G)$, then there is an $h \in G$ such that $1 \neq y = [g, h] \in Z(G)$ and $\mu_s(y) \neq 1$ for some $1 \leq s \leq t$. We may take $\chi \in \mathrm{Irr}\ (G)$ lying over μ_s, then $\chi(g) = \chi(g^h) = \chi(gy) = \mu_s(y)\chi(g)$, which forces $\chi(g) = 0$, as desired. ∎

The following is Theorem 1.2, which may also be regarded as a consequence of Theorem 1.1.

Proof of Theorem 1.2. Let $1 \leq F(G) = F_1(G) \leq F_2(G) \leq \cdots \leq F_m(G) = G$ be the ascending Fitting series, that is, for each positive integer n, $F_n(G)/F_{n-1}(G) = F(G/F_{n-1}(G))$, the largest nilpotent normal subgroup of $G/F_{n-1}(G)$.

Let

$$U_{n-1} = F(F_n(G)/F_{n-2}(G))/\Phi(F_n(G)/F_{n-2}(G)),$$

then Gaschütz's theorem 1.12 of [7] shows that U_{n-1} is a completely reducible and faithful $F_n(G)/F_{n-1}(G)$-module (of possibly mixed characteristic). Since $A_{n-1} = \mathscr{V}(F_n(G))F_{n-1}(G)/F_{n-1}(G)$ lies in the center of $F_n(G)/F_{n-1}(G)$, U_{n-1} is also a completely reducible and faithful A_{n-1}-module (of possibly mixed characteristic) by Clifford's theorem.

By Lemma 2.3, there exists $\mu_{n-1} \in \mathrm{Irr}\ (U_{n-1})$ such that μ_{n-1} induces irreducibly to $V(F_n(G))F_{n-1}(G)$. Roughly speaking, there exists a linear character $\mu_{n-1} \in \mathrm{Irr}\ (F_{n-1}(G))$ inducing irreducibly to $F_{n-1}(G)\mathscr{V}(F_n(G))$. By Lemma 2.1, there exists $\chi \in \mathrm{Irr}\ (F_n(G)$ such that $\chi(a) = 0$ for any $a \in F_{n-1}(G)\mathscr{V}(F_n(G)) - F_{n-1}(G)$. Thus all of non-vanishing elements of $F_n(G)$ lie in $F_{n-1}(G)$ and so $\mathscr{V}(F_n(G)) \leq F_{n-1}(G)$. The proof is finished. ∎

Proposition 3.1. *Let M/N be a chief factor of solvable group G, and $C = C_G(M/N)$, and assume that A/C is a normal abelian subgroup of G/C. Then there exist $\chi \in \mathrm{Irr}\ (G)$ such that $\chi(a) = 0$ for any $a \in A - C$.*

Proof. Because M/N is an irreducible faithful G/C-module, we have via Clifford's theorem that M/N is a faithful and completely reducible A/C-module. Lemma 2.3 shows that there is a character of C inducing irreducibly to A, thus Lemma 2.1 shows that there exist $\chi \in \mathrm{Irr}\ (G)$ such that $\chi(a) = 0$ for any $a \in A - C$, as required. ∎

Using the above result, we may further analyze the Isaacs-Navarro-Wolf Conjecture. Assume that the solvable group G has a chief series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n = G,$$

suppose that $C_i = C_G(G_i/G_{i-1})$ and A_i/C_i are normal abelian factors of G.

Let x be a non-vanishing element of G. Then $x \in G - \cup_{i=1}^{n}(A_i - C_i)$. Considering $\cap_{i=1}^{n} C_i = F(G)$, this result seems to be of interesting.

Proof of Theorem 1.3. Assume that G is a minimal counterexample to the conjecture. Applying the similar techniques as in the proof of Theorem 1.1 to G, we may reduce G to the case that G is a solvable primitive permutation group. By [3, Satz II.3.2], it follows that $G = F(G) \rtimes M$, the Fitting subgroup $F(G)$ is the uniquely minimal normal subgroup of G, the complements M to $F(G)$ in G are non-normal maximal subgroups and all of them are conjugate in G.

Because G is a minimal counterexample, we have that $N = \mathscr{V}(\mathscr{V}(G))$ is nilpotent. Then N is exactly the uniquely minimal normal subgroup $F(G)$. It follows that $\mathscr{V}(G/N)$ and $\mathscr{V}(G)/N$ are nilpotent. Thus we may write $\mathscr{V}(G) = P \rtimes Q$ where $P \geq N$ is a normal Sylow p-subgroup and Q is a nilpotent Hall p'-subgroup of $\mathscr{V}(G)$. The conjugation action of Q on P is faithful, because otherwise the kernel, say K, is nontrivial and $K \cap Z(Q) \neq 1$ (since Q is nilpotent). This implies that $Z(\mathscr{V}(G))$ is nontrivial. However this violates the uniqueness of the minimal normal subgroup of G. Because $\Phi(P) \leq \Phi(G) = 1$, it follows that P is elementary abelian. Further we may get that $N \leq P \leq F(G) = N$ and so $N = P$, since $F(G)$ is the uniquely minimal normal subgroup of G.

The images of all non-vanishing elements of G modulo $F(G)$ are of 2-power order by Theorem 4.3 of [4]. Since $\mathscr{V}(G) = F(G) \rtimes Q$ and $(|F(G)|, |Q|) = 1$, it follows that the nilpotent group Q is generated by elements of 2-power order and so it is a 2-group. The proof is completed. ∎

Observe that if $R/F(G) \leq V(G)/F(G)$ be an abelian normal subgroup of $G/F(G)$, then Lemmas 2.1 and 2.3 imply that there are not non-vanishing elements in $R - F(G)$. If G is a solvable quasi-primitive minimally transitive permutation group, then Proposition 2.2 of [1] shows that $\mathscr{V}(G)$ is cycle. If G is a solvable primitive permutation group and $\mathscr{V}(G)$ be metabelian, then it is easy to prove that $\mathscr{V}(G)$ is abelian. All of these facts seem to imply that the minimal counterexamples are impossible.

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