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A New System of Nonlinear Variational Inclusions with (A_i, η_i)-Accretive Operators in Banach Spaces

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Abstract. In this paper, we introduce and study a new system of nonlinear variational inclusions with (A, η)-accretive operators in Banach spaces. Using the resolvent operator technique associated with (A, η)-accretive operator, we prove the existence and uniqueness of solutions for the system of nonlinear variational inclusions, construct a Mann iterative algorithm with errors for solving the system of nonlinear variational inclusions and discuss the convergence of the iterative sequence generated by the algorithm.

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1. Introduction

It is well known that variational inequalities and variational inclusions have wide applications in mechanics, physics, optimization and control, nonlinear programming, economics and engineering sciences and that various variational inclusions have been intensively studied in recent years. For more details, we refer the reader to [1–12, 14–22, 24] and the references therein.

In 2006, Verma [21, 22] introduced the notions of A-maximal monotonicity and (A, η)-maximal monotonicity for solving nonlinear variational inclusion problems. These notions generalize the general class of maximal monotone set-valued mappings, including the notion of H-maximal monotonicity introduced by Fang and Huang [2]

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in a Hilbert space setting. Very recently, Lan *et al.* [12] have introduced a new concept of (A,η)-accretive operators, which is a generalization of the monotone or accretive operators. They also studied a class of variational inclusions using the resolvent operator associated with (A,η)-accretive operators in Banach spaces.

Inspired and motivated by recent research works in this field, in this paper, we introduce and study a new system of nonlinear variational inclusions with (A,η)-accretive operators in Banach spaces. By using the resolvent operator associated with (A,η)-accretive operator, we construct a Mann iterative algorithm with errors for finding the approximate solutions of the system of nonlinear variational inclusions in Banach spaces. Under certain conditions, we obtain the existence and uniqueness of solution for the system of nonlinear variational inclusions. Furthermore, the convergence result of the iterative sequence generated by the Mann iterative algorithm with errors is presented in this paper. Our result improves, extends and unifies the corresponding results in [1, 3, 7, 11, 17–20].

2. Preliminaries

In what follows, let X be a real Banach space with the dual space X^*, $\langle\cdot,\cdot\rangle$ be the dual pair between X and X^*, and 2^X denote the families of all nonempty subsets of X. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{f^* \in X^* : \langle x, f^*\rangle = \|x\|^q \text{ and } \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in X,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \neq 0$, and J_q is single-valued if X^* is strictly convex. In the sequel, unless otherwise specified, we assume that X is a real Banach space such that J_q is single-valued and H is a Hilbert space. If $X = H$, then J_2 becomes the identity mapping of H.

We recollect and introduce the following concepts and lemmas, which will be used in the next section.

Definition 2.1. *Let $N,\eta : X \times X \to X$ and $g : X \to X$ be mappings.*

(1) *g is said to be r-strongly accretive if there exists a constant $r > 0$ such that*

$$\langle g(u) - g(v), J_q(u - v)\rangle \geq r \parallel u - v \parallel^q, \quad \forall u, v \in X;$$

(2) *g is said to be s-Lipschitz continuous if there exists a constant $s > 0$ such that*

$$\|g(u) - g(v)\| \leq s \parallel u - v \parallel, \quad \forall u, v \in X;$$

(3) *g is said to be r-strongly accretive with respect to the first argument of N if there exists a constant $r > 0$ such that*

$$\langle N(g(u), x) - N(g(v), x), J_q(u - v)\rangle \geq r\|u - v\|^q, \quad \forall u, v, x \in X;$$

(4) *g is said to be (r,η)-strongly accretive if there exists a constant $r > 0$ such that*

$$\langle g(u) - g(v), J_q(\eta(u, v))\rangle \geq r\|u - v\|^q, \quad \forall u, v \in X;$$

(5) *N is said to be s-Lipschitz continuous in the first argument if there exists a constant $s > 0$ such that*

$$\|N(u, x) - N(v, x)\| \leq s \parallel u - v \parallel, \quad \forall u, v, x \in X.$$

Similarly, we can define the Lipschitz continuity of N in the second argument and the strong accretivity of g with respect to the second argument of N.

Definition 2.2. *A single-valued mapping $\eta : X \times X \to X$ is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that*

$$\|\eta(u, v)\| \leq \tau \|u - v\|, \quad \forall u, v \in X.$$

Definition 2.3. *Let $\eta : X \times X \to X$ and $A, H : X \to X$ and $M : X \to 2^X$ mappings. Then the multi-valued mapping $M : X \to 2^X$ is said to be*

(1) *accretive if*

$$\langle x - y, J_q(u - v)\rangle \geq 0, \quad \forall u, v \in X,\ x \in Mu,\ y \in Mv;$$

(2) *η-accretive if*

$$\langle x - y, J_q(\eta(u, v))\rangle \geq 0, \quad \forall u, v \in X,\ x \in Mu,\ y \in Mv;$$

(3) *strictly η-accretive if M is η-accretive and equality holds if and only if $x = y$;*

(4) *(α, η)-strongly accretive if there exists a constant $\alpha > 0$ satisfying*

$$\langle x - y, J_q(\eta(u, v))\rangle \geq \alpha \|u - v\|^q, \quad \forall u, v \in X,\ x \in Mu,\ y \in Mv;$$

(5) *(m, η)-relaxed accretive if if there exists a constant $m > 0$ satisfying*

$$\langle x - y, J_q(\eta(u, v))\rangle \geq -m \|u - v\|^q, \quad \forall u, v \in X,\ x \in Mu,\ y \in Mv.$$

(6) *(A, η)-accretive if M is (m, η)-relaxed accretive and $(A + \rho M)(X) = X$ for every $\rho > 0$.*

Remark 2.1. For appropriate and suitable choices of m, A, η and X, it is easy to see that Definition 2.3 (6) includes the definitions of monotone and accretive operators (see [12]) as special cases.

It is easy to see that $(A + \rho M)^{-1}$ is a single-valued operator if $M : X \to 2^X$ is (A, η)-accretive operator and $A : X \to X$ is (r, η)-strongly accretive. Based on this fact, we can define the resolvent operator $R^{A,\eta}_{M,\rho}$ associated with an (A, η)-accretive operator M as follows:

Definition 2.4. *Let X be a Banach space, $A : X \to X$ be (r, η)-strongly accretive and $M : X \to 2^X$ be (A, η)-accretive. For any fixed $\rho > 0$, the mapping $R^{A,\eta}_{M,\rho} : X \to X$ defined by*

$$R^{A,\eta}_{M,\rho}(x) = (A + \rho M)^{-1}(x), \quad \forall x \in X,$$

is said to be resolvent operator of M.

Remark 2.2. The resolvent operators associated with (A, η)-accretive operators include as special cases the corresponding resolvent operators associated with (H, η)-monotone operators [5], H-monotone operators [3], generalized m-accretive operators [9], maximal η-monotone operators [1], A-monotone operators [20], the classical m-accretive and maximal monotone operators.

Lemma 2.1. [13] *Let $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$ be nonnegative sequences satisfying*

$$\alpha_{n+1} \leq (1 - \delta_n)\alpha_n + \delta_n \beta_n + \gamma_n, \quad \forall n \geq 0,$$

where $\{\delta_n\}_{n\geq 0} \subset [0, 1]$, $\sum_{n=0}^{\infty} \delta_n = +\infty$, $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \gamma_n < +\infty$. Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.2. [12] *Let $\eta : X \times X \to X$ be τ-Lipschitz continuous, $A : X \to X$ be (r, η)-strongly accretive and $M : X \to 2^X$ be (A, η)-accretive. Then the resolvent operator $R^{A,\eta}_{M,\rho} : X \to X$ is $\frac{\tau^{q-1}}{r-\rho m}$-Lipschitz continuous, that is,*

$$\|R^{A,\eta}_{M,\rho}(x) - R^{A,\eta}_{M,\rho}(y)\| \le \frac{\tau^{q-1}}{r - \rho m}\|x - y\|, \quad \forall x, y \in X,$$

where $\rho \in (0, \frac{r}{m})$.

Lemma 2.3. [23] *Let X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that*

$$\|x + y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q, \quad \forall x, y \in X.$$

3. A system of nonlinear variational inclusions and a Mann iterative algorithm

In this section, we introduce a new system of nonlinear variational inclusions with (A_i, η_i)-accretive operators and construct a new iterative algorithm for solving the system of nonlinear variational inclusions in Banach spaces.

In what follows unless other specified, we assume that X_1 and X_2 are two real Banach spaces, $N_1 : X_1 \times X_2 \to X_1$, $N_2 : X_1 \times X_2 \to X_2$, $A_i, g_i, a_i, b_i : X_i \to X_i$, $\eta_i : X_i \times X_i \to X_i$ are mappings, $M_i : X_i \to 2^X_i$ is an (A_i, η_i)-accretive operator and $I_i : X_i \to X_i$ is the identity mapping for $i \in \{1, 2\}$. Given $f_i \in X_i$ for $i \in \{1, 2\}$, we consider the following problem: Find $(u, v) \in X_1 \times X_2$ such that

$$\begin{aligned} &f_1 \in N_1(a_1u, a_2v) + M_1(g_1u), \\ &f_2 \in N_2(b_1u, b_2v) + M_2(g_2v), \end{aligned} \tag{3.1}$$

which is called a *system of nonlinear variational inclusions with (A_i, η_i)-accretive operators* .

Special cases of the problem (3.1) are as follows:

(A) If $f_1 = f_2 = 0$, $g_1 = a_1 = b_1 = I_1$, $g_2 = a_2 = b_2 = I_2$, X_1 and X_2 are real Hilbert spaces, then the problem (3.1) is equivalent to finding $(u, v) \in X_1 \times X_2$ such that

$$\begin{aligned} &0 \in N_1(u, v) + M_1(u), \\ &0 \in N_2(u, v) + M_2(v), \end{aligned} \tag{3.2}$$

which was introduced and studied by Fang-Huang-Thompson [6].

(B) If $X_1 = X_2$, $f_1 = f_2 = 0$, $g_1 = g_2 = g$, $N_1(a_1u, a_2v) = A_1(gu) - A_1(gv) + \rho_1T(v)$, $M_1(g_1u) = \rho_1W_2(gu)$, $N_2(b_1u, b_2v) = A_2(gv) - A_2(gu) + \rho_2T(u)$, $M_2(g_2u) = \rho_2W_2(gv)$, then the problem (3.1) is equivalent to the following problem studied in [12]:

Find $u, v \in X$ such that

$$\begin{aligned} &0 \in A_1(gu) - A_1(gv) + \rho_1(T(v) + W_2(gu)), \\ &0 \in A_2(gv) - A_2(gu) + \rho_2(T(v) + W_2(gv)), \end{aligned} \tag{3.3}$$

where $\rho_i > 0$ is a constant for $i \in \{1, 2\}$. Some special cases of the problem (3.3) were studied by Verma [17–19].

Lemma 3.1. *Let $A_i : X_i \to X_i$ be (r_i, η_i)-strongly accretive and $M_i : X_i \to 2_i^X$ be (A_i, η_i)-accretive for $i \in \{1, 2\}$. Then $(u, v) \in X_1 \times X_2$ is a solution of the problem* (3.1) *if and only if*

$$g_1 u = R_{M_1,\rho_1}^{A_1,\eta_1}[A_1(g_1 u) - \rho_1 N_1(a_1 u, a_2 v) + \rho_1 f_1],$$
$$g_2 v = R_{M_2,\rho_2}^{A_2,\eta_2}[A_2(g_2 v) - \rho_2 N_2(b_1 u, b_2 v) + \rho_2 f_2],$$

where $R_{M_1,\rho_1}^{A_1,\eta_1}$ and $R_{M_2,\rho_2}^{A_2,\eta_2}$ are the resolvent operators of M_1 and M_2, respectively, and $r_i > m_i$ for $i \in \{1, 2\}$.

Proof. The fact directly follows from Definition 2.4. ∎

Based on Lemma 3.1, we suggest the following Mann iterative algorithm with errors for solving the problem (3.1).

Algorithm 3.1. For any given $(u_0, v_0) \in X_1 \times X_2$, compute the sequences $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ by

$$\begin{aligned} u_{n+1} &= (1 - c_n - d_n)u_n + c_n\{u_n - g_1 u_n + R_{M_1,\rho_1}^{A_1,\eta_1}[A_1(g_1 u_n) \\ &\qquad - \rho_1 N_1(a_1 u_n, a_2 v_n) + \rho_1 f_1]\} + d_n e_n, \\ v_{n+1} &= (1 - c_n - d_n)v_n + c_n\{v_n - g_2 v_n + R_{M_2,\rho_2}^{A_2,\eta_2}[A_2(g_2 v_n) \\ &\qquad - \rho_2 N_2(a_1 u_n, a_2 v_n) + \rho_2 f_2]\} + d_n h_n, \end{aligned} \tag{3.4}$$

for all $n \geq 0$, where $\{e_n\}_{n\geq 0}$ and $\{h_n\}_{n\geq 0}$ are bounded sequences in X_1 and X_2, respectively, introduced to take into account possible in inexact computations and the sequences $\{c_n\}_{n\geq 0}$, and $\{d_n\}_{n\geq 0}$ are in $[0, 1]$ satisfying

$$c_n + d_n \leq 1, \quad \forall\ n \geq 0, \quad \sum_{n=0}^{\infty} c_n = +\infty \quad \text{and} \quad \sum_{n=0}^{\infty} d_n < +\infty. \tag{3.5}$$

4. Existence of solutions and convergence of a Mann iterative algorithm with errors

In this section, we prove the existence of solutions for the problem (3.1) and the convergence of iterative sequences generated by Algorithm 3.1. For each $i \in \{1, 2\}$, let X_i be an q-uniformly smooth Banach space and c_q^i be the constant in Lemma 2.3 with respect to X_i.

Theorem 4.1. *For $i \in \{1, 2\}$, let X_i be an q-uniformly smooth Banach space, $\eta_i : X_i \times X_i \to X_i$ be τ_i-Lipschitz continuous, a_i, b_i and $g_i : X_i \to X_i$ be λ_i-Lipschitz continuous, p_i-Lipschitz continuous and s_i-Lipschitz continuous, respectively, g_i be ξ_i-strongly accretive, $N_i : X_1 \times X_2 \to X_i$ be σ_i-Lipschitz continuous in the first argument, ν_i-Lipschitz continuous in the second argument, a_1 be α-strongly accretive with respect to the first argument of N_1, b_2 be δ-strongly accretive with respect to the second argument of N_2, $A_i : X_i \to X_i$ be (r_i, η_i)-strongly accretive and β_i-Lipschitz continuous, respectively, $A_i g_i$ be t_i-strongly accretive, $M_i : X_i \to 2^{X_i}$ be (A_i, η_i)-accretive and (m_i, η_i)-relaxed accretive with $r_i > m_i$. If there exist constants $\rho_i \in (0, \frac{r_i}{m_i})$, $i \in \{1, 2\}$, such that*

$$\begin{aligned} 0 < \theta = \max\{&\theta_1 + L_1(\theta_2 + \theta_3) + L_2\rho_2\delta_2 p_1, \theta_4 + L_2(\theta_5 + \theta_6) + L_1\rho_1\nu_1\lambda_2\} \\ &< 1, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
\theta_1 &= (1 - q\xi_1 + c_q^1 s_1^q)^{\frac{1}{q}}, & \theta_2 &= (1 - qt_1 + c_q^1 \beta_1^q s_1^q)^{\frac{1}{q}},\\
\theta_3 &= (1 - q\rho_1\alpha + c_q^1 \rho_1 \sigma_1^q \lambda_1^q)^{\frac{1}{q}}, & \theta_4 &= (1 - q\xi_2 + c_q^2 s_2^q)^{\frac{1}{q}},\\
\theta_5 &= (1 - qt_2 + c_q^2 \beta_2^q s_2^q)^{\frac{1}{q}}, & \theta_6 &= (1 - q\rho_2\delta + c_q^2 \rho_2 \nu_2^q p_2^q)^{\frac{1}{q}},\\
L_1 &= \frac{\tau_1^{q-1}}{r_1 - \rho_1 m_1}, & L_2 &= \frac{\tau_2^{q-1}}{r_2 - \rho_2 m_2},
\end{aligned}$$

then the problem (3.1) *admits a unique solution* $(u, v) \in X_1 \times X_2$ *and the sequences* $\{u_n\}_{n\geq 0}$ *and* $\{v_n\}_{n\geq 0}$ *defined by Algorithm 3.1 converge strongly to* u *and* v*, respectively.*

Proof. First, we prove that the problem (3.1) has a unique solution $(u, v) \in X_1 \times X_2$.

For $i \in \{1, 2\}$, define $T_{\rho_i} : X_1 \times X_2 \to X_i$ by

$$\begin{aligned}
T_{\rho_1}(x, y) &= x - g_1 x + R_{M_1,\rho_1}^{A_1,\eta_1}[A_1(g_1 x) - \rho_1 N_1(a_1 x, a_2 y) + \rho_1 f_1],\\
T_{\rho_2}(x, y) &= y - g_2 y + R_{M_2,\rho_2}^{A_2,\eta_2}[A_2(g_2 y) - \rho_2 N_2(b_1 x, b_2 y) + \rho_2 f_2]
\end{aligned} \tag{4.2}$$

for all $(x, y) \in X_1 \times X_2$.

Put $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$. It follows from Lemma 2.2 that

$$\begin{aligned}
&\|T_{\rho_1}(x_1, y_1) - T_{\rho_1}(x_2, y_2)\|\\
&\leq \|x_1 - x_2 - (g_1 x_1 - g_1 x_2)\|\\
&\quad + \|R_{M_1,\rho_1}^{A_1,\eta_1}[A_1(g_1 x_1) - \rho_1 N_1(a_1 x_1, a_2 y_1) + \rho_1 f_1]\\
&\quad - R_{M_1,\rho_1}^{A_1,\eta_1}[A_1(g_1 x_2) - \rho_1 N_1(a_1 x_2, a_2 y_2) + \rho_1 f_1]\|\\
&\leq \|x_1 - x_2 - (g_1 x_1 - g_1 x_2)\| + L_1\|A_1(g_1 x_1) - A_1(g_1 x_2)\\
&\quad - \rho_1[N_1(a_1 x_1, a_2 y_1) - N_1(a_1 x_2, a_2 y_2)]\|\\
&\leq \|x_1 - x_2 - (g_1 x_1 - g_1 x_2)\| + L_1[\|x_1 - x_2 - (A_1(g_1 x_1) - A_1(g_1 x_2))\|\\
&\quad + \|x_1 - x_2 - \rho_1[N_1(a_1 x_1, a_2 y_1) - N_1(a_1 x_2, a_2 y_2)]\|].
\end{aligned} \tag{4.3}$$

Using Lemma 2.3 and the assumptions, we infer that

$$\begin{aligned}
&\|x_1 - x_2 - (g_1 x_1 - g_1 x_2)\|^q\\
&\leq \|x_1 - x_2\|^q - q\langle g_1 x_1 - g_1 x_2, J_q(x_1 - x_2)\rangle + c_q^1\|g_1 x_1 - g_1 x_2\|^q\\
&\leq \theta_1^q \|x_1 - x_2\|^q,
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
&\|x_1 - x_2 - (A_1(g_1 x_1) - A_1(g_1 x_2)\|^q\\
&\leq \|x_1 - x_2\|^q - q\langle A_1(g_1 x_1) - A_1(g_1 x_2), J_q(x_1 - x_2)\rangle\\
&\quad + c_q^1\|A_1(g_1 x_1) - A_1(g_1 x_2)\|^q\\
&\leq \theta_2^q \|x_1 - x_2\|^q,
\end{aligned} \tag{4.5}$$

$$\begin{aligned}&\|x_1 - x_2 - \rho_1[N_1(a_1x_1, a_2y_1) - N_1(a_1x_2, a_2y_2)]\| \\ &\le \|x_1 - x_2 - \rho_1[N_1(a_1x_1, a_2y_1) - N_1(a_1x_2, a_2y_1)]\| \\ &\quad + \rho_1\|N_1(a_1x_2, a_2y_1) - N_1(a_1x_2, a_2y_2)\|,\end{aligned} \tag{4.6}$$

$$\begin{aligned}&\|x_1 - x_2 - \rho_1[N_1(a_1x_1, a_2y_1) - N_1(a_1x_2, a_2y_1)]\|^q \\ &\le \|x_1 - x_2\|^q - q\rho_1\langle N_1(a_1x_1, a_2y_1) - N_1(a_1x_2, a_2y_1), J_q(x_1 - x_2)\rangle \\ &\quad + c_q^1\rho_1\|N_1(a_1x_1, a_2y_1) - N_1(a_1x_2, a_2y_1)\|^q \\ &\le \theta_3^q\|x_1 - x_2\|^q,\end{aligned} \tag{4.7}$$

$$\|N_1(a_1x_2, a_2y_1) - N_1(a_1x_2, a_2y_2)\| \le \nu_1\lambda_2\|y_1 - y_2\|. \tag{4.8}$$

Combining (4.3)–(4.8), we have

$$\|T_{\rho_1}(x_1, y_1) - T_{\rho_1}(x_2, y_2)\| \le [\theta_1 + L_1(\theta_2 + \theta_3)]\|x_1 - x_2\| + L_1\rho_1\nu_1\lambda_2\|y_1 - y_2\|. \tag{4.9}$$

Similarly, we can prove that

$$\|T_{\rho_2}(x_1, y_1) - T_{\rho_2}(x_2, y_2)\| \le [\theta_4 + L_2(\theta_5 + \theta_6)]\|y_1 - y_2\| + L_2\rho_2\delta_2p_1\|x_1 - x_2\|. \tag{4.10}$$

Define a norm$\|\cdot\|_*$ on $X_1 \times X_2$ by $\|(x, y)\|_* = \|x\| + \|y\|$ for all $(x, y) \in X_1 \times X_2$. It is easy to see that $(X_1 \times X_2, \|\cdot\|_*)$ is a Banach space. Define $Q : X_1 \times X_2 \to X_1 \times X_2$ by

$$\|Q(x, y)\|_* = (T_{\rho_1}(x, y), T_{\rho_2}(x, y)), \quad \forall (x, y) \in X_1 \times X_2.$$

By (4.9) and (4.10), we have

$$\begin{aligned}&\|Q(x_1, y_1) - Q(x_2, y_2)\|_* \\ &= \|T_{\rho_1}(x_1, y_1) - T_{\rho_1}(x_2, y_2)\| + \|T_{\rho_2}(x_1, y_1) - T_{\rho_2}(x_2, y_2)\| \\ &\le \theta(\|x_1 - x_2\| + \|y_1 - y_2\|) \\ &= \theta\|(x_1, y_1) - (x_2, y_2)\|_*.\end{aligned} \tag{4.11}$$

In light of (4.1) and (4.11), we know that $Q : X_1 \times X_2 \to X_1 \times X_2$ is a contraction mapping. Hence Q possesses a unique fixed point $(u, v) \in X_1 \times X_2$. Consequently, Lemma 3.1 ensures that (u, v) is the unique solution of the problem (3.1).

Now we show that $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} v_n = v$. Notice that

$$\begin{aligned}u &= (1 - c_n - d_n)u + c_n\{u - g_1u + R_{M_1,\rho_1}^{A_1,\eta_1}[A_1(g_1u) \\ &\quad - \rho_1N_1(a_1u, a_2v) + \rho_1f_1]\} + d_nu, \\ v &= (1 - c_n - d_n)v + c_n\{v - g_2v + R_{M_2,\rho_2}^{A_2,\eta_2}[A_2(g_2v) \\ &\quad - \rho_2N_2(a_1u, a_2v) + \rho_2f_2]\} + d_nv.\end{aligned} \tag{4.12}$$

Put $E_1 = \sup\{\|e_n - u\| : n \geq 0\}$ and $E_2 = \sup\{\|h_n - v\| : n \geq 0\}$. Using (3.4) and (4.12), we know that

$$\begin{aligned}
&\|u_{n+1} - u\| \\
&\leq (1 - c_n - d_n)\|u_n - u\| + c_n\|u_n - u - (g_1(u_n) - g_1(u))\| \\
&\quad + c_n\|R^{A_1,\eta_1}_{M_1,\rho_1}[A_1(g_1u_n) - \rho_1 N_1(a_1u_n, a_2v_n) + \rho_1 f_1] \\
&\quad - R^{A_1,\eta_1}_{M_1,\rho_1}[A_1(g_1u) - \rho_1 N_1(a_1u, a_2v) + \rho_1 f_1) + \rho_1 f_1]\| + d_n\|e_n - u\| \\
&\leq (1 - c_n - d_n)\|u_n - u\| + c_n[\theta_1 + L_1(\theta_2 + \theta_3)]\|u_n - u\| \\
&\quad + c_n L_1 \rho_1 \nu_1 \lambda_2 \|v_n - v\| + d_n E_1
\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}
&\|u_{n+1} - u\| + \|v_{n+1} - v\| \\
&\leq (1 - c_n - d_n)(\|u_n - u\| + \|v_n - v\|) \\
&\quad + c_n\theta(\|u_n - u\| + \|v_n - v\|) + d_n(E_1 + E_2) \\
&\leq (1 - (1 - \theta)c_n)(\|u_n - u\| + \|v_n - v\|) + d_n(E_1 + E_2)
\end{aligned} \tag{4.14}$$

for all $n \geq 0$. It follows from Lemma 2.1, (3.5), (4.11) and (4.14) that $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} v_n = v$. This completes the proof. ∎

Remark 4.1. Theorem 4.1 improves, extends and unifies the corresponding results in [1, 3, 7, 11, 17–20].

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