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The Influence of CAP^*-Subgroups on the Solvability of Finite Groups

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Abstract. A subgroup H of a group G is said to be a CAP^*-subgroup of a group G if, for any non-Frattini chief factor K/L of G, we have $HK = HL$ or $H \cap K = H \cap L$. In this paper, some new characterizations for finite solvable groups are obtained based on the assumption that some subgroups are CAP^*-subgroups of G.

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1. Introduction

All groups considered in this paper are finite. A subgroup H of a group G is said to have the cover-avoiding property in G if H covers or avoids every chief factor of G, in short, H is a CAP-subgroup of G. There has been much interest in the past in investigating the structure of finite groups when some subgroups have the cover-avoiding property, and many interesting results have been made, for example [3, 7–13, 15, 16, 18, 21, 22].

Our motivation in this paper comes from the following example.

Example 1.1. Let $P = \langle a, b | a^4 = b^4 = [a, b] = 1\rangle$ be a direct product of two cyclic groups of order 4 and $c \in \mathrm{Aut}\,(P)$ such that $a^c = a^2b^3$, $b^c = a^3b$. Then the semidirect product: $K = P \rtimes \langle c\rangle$ is of order $2^4 \times 3$. We set $G = K \times C_2$, a direct product of K and a cyclic group $C_2 = \langle d\rangle$ of order 2.

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An easy proof gives $\Phi(G) = \langle a^2, b^2\rangle$ is a minimal normal subgroup of G, where $\Phi(G)$ is the Frattini subgroup of G. It follows that all chief factors of G are non-Frattini besides $\Phi(G)/1$ and $\Phi(G)C_2/C_2$. We consider the subgroup $H = \langle a^2\rangle$ of order 2 of G. It is easy to see that H avoids every non-Frattini chief factors of G. However, $H \cap \Phi(G) = H \neq 1 = H \cap 1$ and $H\Phi(G) = \Phi(G) \neq H$. This implies that the chief factor $\Phi(G)/1$ is neither covered nor avoided by H. Thus H is not a CAP-subgroup of G. Hence it is quite natural to ask non-Frattini chief factor how to affect the structure of finite groups. For convenience, we give the following definition.

Definition 1.1. *A subgroup H of a group G is said to be a CAP^*-subgroup of G if, for any non-Frattini chief factor K/L of G, we have $HK = HL$ or $H \cap K = H \cap L$.*

In this paper, some new characterizations for finite solvable groups are obtained based on the assumption that some subgroups are CAP^*-subgroups of G. We write M_G to indicate that the core of a subgroup M in a group G. If M is a maximal subgroup of G and H is a maximal subgroup of M, then we call H a 2-maximal subgroup of G.

2. Basic definitions and preliminary results

Let K and L be normal subgroups of a group G with $K \leq L$. Then K/L is called a normal factor of G. A subgroup H of G is said to cover K/L if $HK = HL$. On the other hand, if $H \cap K = H \cap L$, then H is said to avoid K/L. If K/L is a chief factor of G and $K/L \leq \Phi(G/L)$ (respectively $K/L \nleq \Phi(G/L)$), then K/L is said to be a Frattini (respectively non-Frattini) chief factor of G.

Lemma 2.1. [6, Chapter A, 9.9] *Let K/L be a chief factor of a group G. If N is a normal subgroup of G contained in L, then K/L is a Frattini chief factor of G if and only if $(K/N)/(L/N)$ is a Frattini chief factor of G/N.*

Lemma 2.2. *Every non-Frattini chief factor of G is avoided by every subgroup of $\Phi(G)$.*

Proof. Let K/L be a non-Frattini chief factor of G and $B \leq \Phi(G)$. Then $BL/L \leq \Phi(G/L)$. It is easy to see that $K/L \cap \Phi(G/L) = \overline{1}$ since K/L is a minimal normal subgroup of G/L. It follows that $(B\cap K)L = L$. Hence $B\cap K = B\cap L$, as desired. ∎

Lemma 2.3. *Let N be a normal subgroup of a group G. If H is a CAP^*-subgroup of G, then:*

(1) *HN/N is a CAP^*-subgroup of G/N;*
(2) *$H \cap N$ is a CAP^*-subgroup of G.*

Proof. (1) Let $\overline{G} = G/N$ and $\overline{K}/\overline{L}$ be a non-Frattini chief factor of $\overline{G}$. It is easy to see that K/L is a non-Frattini chief factor of G by Lemma 2.1. Then H covers or avoids K/L by hypotheses. If $HK = HL$, then $HN/N \cdot K/N = HN/N \cdot L/N$ and so HN/N covers $\overline{K}/\overline{L}$. If $H \cap K = H \cap L$, then $HN \cap K = HN \cap L$, which implies that HN/N avoids $\overline{K}/\overline{L}$, as desired.
(2) Let K/L be a non-Frattini chief factor of G. If one of H and N avoids K/L, then $H \cap N$ avoids K/L. Hence we may assume that both H and N cover K/L.

We use the induction on the length of any piece of chief series of G below N. If N is a minimal normal subgroup of G and $N \not\leq \Phi(G)$, then $HN = H$ or $H \cap N = 1$ by hypotheses. It follows that $H \cap N$ is a CAP^*-subgroup of G. If $N \leq \Phi(G)$, then $H \cap N$ is a CAP^*-subgroup of G by Lemma 2.2. Now, we let D be a minimal normal subgroup of G contained in N. By (1), HD/D is a CAP^*-subgroup of G/D. It follows that $HD/D \cap N/D = (H \cap N)D/D$ is a CAP^*-subgroup of G/D by induction.

Suppose that $N \cap L \neq 1$. Then, there exists a minimal normal subgroup D of G such that $D \leq N \cap L$. By the above paragraph, $(H \cap N)D/D$ either covers or avoids $(K/D)\big/(L/D)$. If covers, then $(H \cap N)K = (H \cap N)L$, as desired. If avoids, then $(H \cap N \cap K)D = (H \cap N \cap L)D$ and so $H \cap N \cap K = H \cap N \cap L$ by comparing the orders, as desired. Hence we can consider that $N \cap L = 1$.

By our assumption, $N \cap K \neq 1$. Let M be a minimal normal subgroup of G with $M \leq N \cap K$. It is clear that $M \cap K \neq M \cap L$ and so $MK = ML = K$. This means that $(H \cap N)K = (H \cap N)ML$. If $M \leq \Phi(G)$, then $K/L = ML/L \leq \Phi(G)L/L \leq \Phi(G/L)$. However, this contradicts the fact that K/L is a non-Frattini chief factor of G. It follows that $M/1$ is the non-Frattini chief factor of G. By hypotheses, H covers or avoids M. On the one hand, if $HM = H$, then $(H \cap N)K = (H \cap N)ML = (H \cap N)L$, as desired. On the other hand, if $H \cap M = 1$, then $H \cap N \cap M = 1$, and an easy calculation gives

$$|(H \cap N)K| = \frac{|H \cap N||K|}{|H \cap N \cap K|} = |H \cap N||ML| = |(H \cap N)ML|.$$

It follows that $H \cap N \cap K = 1$. This implies that $H \cap N$ avoids K/L, which completes our proof. ∎

Remark 2.1. If H is a CAP^*-subgroup of a group G and $H \leq M$, then it does not necessarily follow that H is a CAP^*-subgroup of M. This means that the CAP^*-subgroups are not inherited in intermediate subgroups.

Example 2.1. The Example 1.3 of [2] shows that the CAP^*-subgroups are not inherited in intermediate subgroups.

Even if G is solvable, the CAP^*-subgroups are not inherited in intermediate subgroups. For example, let $G = A_4 \times A_4$, a direct product of two Alternating groups on four letters. We write $V_4 \times V_4$ as $\langle x, y\rangle \times \langle a, b\rangle$ with generators x, y, a and b of order 2. Let $H = \{(1,1), (x,a), (y,b), (xy,ab)\}$, then H covers or avoids every non-Frattini chief factor of G, which implies that H is a CAP^*-subgroup of G. Put $M = A_4 \times V_4$. If $K = V_4 \times \langle a\rangle$ and $L = 1 \times \langle a\rangle$, then K/L is a non-Frattini chief factor of M. Since $H \cap K = \{(1,1), (x,a)\} \neq \{(1,1)\} = H \cap L$, we can see that

$$|HK| = \frac{|H||K|}{|H \cap K|} = 16 \quad \text{and} \quad |HL| = \frac{|H||L|}{|H \cap L|} = 8.$$

Therefore, K/L is neither covered nor avoided by H. In particular, H is not a CAP^*-subgroup of M.

Lemma 2.4. *Let N be a normal subgroup of a group G and let H be a CAP^*-subgroup of G. Then HN is a CAP^*-subgroup of G if one of the following holds:*

(1) *$N \leq \Phi(G)$, the Frattini subgroup of G;*

(2) *HN is a maximal subgroup of G.*

Proof. Let K/L be a non-Frattini chief factor of G. If one of H and N covers K/L, then HN covers K/L. Hence we may assume that both H and N avoid K/L. Then KN/LN is a chief factor of G.

Suppose that $N \leq \Phi(G)$. There exists a maximal subgroup M/L of G/L such that $K/L \nleq M/L$, then $KN/LN \nleq M/LN$. This implies that KN/LN is a non-Frattini chief factor of G. By the hypotheses, H covers or avoids KN/LN. We only need to consider that H avoids KN/LN, that is, $(HN \cap K)N = (HN \cap L)N$. It follows from comparing the orders that $HN \cap K = HN \cap L$, as desired.

Let HN be a maximal subgroup of G. If KN/LN is a non-Frattini chief factor of G, then HN covers or avoids K/L. We may assume that $KN/LN \leq \Phi(G/LN)$. If $LN \leq HN$, then $HNK = HNL$. If $LN \nleq HN$, then $HNK = HNL$. Therefore HN covers K/L. ∎

Recall that the normal index of a maximal subgroup M in a group G is defined as the order of a chief factor H/K of G, where H is minimal in the set of normal supplements to M in G. We let $\eta(G : M)$ denote this number.

Lemma 2.5. [4, Lemma 2] *If $N \lhd G$ and M is a maximal subgroup of a finite group G such that $N \subseteq M$, then $\eta(G/N : M/N) = \eta(G : M)$.*

Lemma 2.6. [11, Lemma 2.8] *Let N be a minimal normal subgroup and M a maximal subgroup of a group G. If M is solvable and $M \cap N = 1$, then G is solvable.*

3. Main result

In this section, we study the solvability of a group G when some subgroups are CAP^*-subgroups of G.

Theorem 3.1. *Let G be a finite group. Then the following statements are equivalent:*

(1) *G is solvable;*
(2) *Every Hall subgroup of G is a CAP^*-subgroup of G;*
(3) *Every Sylow subgroup of G is a CAP^*-subgroup of G;*
(4) *Every maximal subgroup of G is a CAP^*-subgroup of G.*

Proof. (1)$\Longrightarrow$ (2) Let H be a Hall subgroup of G and K/L be a non-Frattini chief factor of G. Since G is solvable, K/L is an elementary abelian p-group for some prime p. If H is a p'-group, then $H \cap K = H \cap L$. Otherwise, $HK = HL$. It follows that H is a CAP^*-subgroup of G.

(1)$\Longrightarrow$ (4) Let M be a maximal subgroup of G and K/L be a non-Frattini chief factor of G. If $L \nleq M$ or $K \leq M$, then $MK = ML$. If $L \leq M$ and $K \nleq M$, then $M/L \cap K/L$ is a normal subgroup of G/L since K/L is an elementary abelian group. By the minimal normality of K/L, we can see that $M \cap K = M \cap L$. Hence M is a CAP^*-subgroup of G.

(2)$\Longrightarrow$ (3) Trivial.

(3)$\Longrightarrow$ (1) Let N be a minimal normal subgroup of G and P a Sylow subgroup of G. Then PN/N is a CAP^*-subgroup of G/N by hypotheses and Lemma 2.3. It follows from the induction that G/N is solvable. Now, let q be a prime dividing the order of N and $Q \in Syl_q(G)$. If N is solvable, then G is solvable. If N is not

solvable, then $N/1$ is a non-Frattini chief factor of G. By the hypotheses, we can see that $Q \cap N = 1$ or $QN = Q$. However, these two cases are impossible. Hence N is solvable and G is as well.
(4)$\Longrightarrow$ (1) Let N be a minimal normal subgroup of G. Then G/N satisfies the hypotheses of the statement (4) and therefore G/N is solvable by induction. Since $N \leq \Phi(G)$ implies that N is solvable. In this case, G is solvable. Hence we may assume that $N/1$ is a non-Frattini chief factor of G. Then there exists a maximal subgroup M of G such that $G = MN$. By hypotheses, M covers or avoids $N/1$. Since $N \nleq M$, we can see that $M \cap N = 1$. It follows that $M \cong G/N$ is solvable. Applying Lemma 2.6, G is solvable. ∎

Theorem 3.2. *A group G is solvable if and only if there exists a maximal subgroup M of G such that M is a solvable CAP^*-subgroup of G.*

Proof. If G is solvable, then every maximal subgroup of G is a CAP^*-subgroup of G by Theorem 3.1, and M is solvable. Now, we prove the sufficiency of the condition. We prove it by induction on $|G|$.

If $M_G \neq 1$, then G/M_G satisfies the hypotheses of our theorem by Lemma 2.3. We can see that G/M_G is solvable by induction and so is G. If $M_G = 1$, then the group G is primitive and then $\Phi(G) = 1$. Let N be a minimal subgroup of G, then $G/N \cong M/(M \cap N)$ is solvable. It is clear that N is a non-Frattini chief factor of G, then $M \cap N = 1$ by hypotheses. Applying Lemma 2.6, G is solvable. ∎

Theorem 3.3. *A group G is solvable if and only if there exists a maximal subgroup M of G such that every Sylow subgroup of M is a CAP^*-subgroup of G.*

Proof. We firstly prove the necessary condition. Suppose that G is solvable. Then M is a CAP^*-subgroup of G by Theorem 3.1. Let P be a Sylow p-subgroup of M where p is a prime dividing the order of M and let K/L be a non-Frattini chief factor of G. If $p \dagger |K/L|$, then $P \cap K = P \cap L$ and so P avoids K/L. So we may assume that K/L is a p-group. If M avoids K/L, then P avoids K/L too. Suppose that M covers K/L. Then we have $|K/L| = |(M \cap L)(P \cap M \cap K) : M \cap L| = |(M \cap L)(P \cap K) : M \cap L| = |P \cap K : P \cap L|$, so $|PK : PL| = 1$, this means that P covers K/L, as desired.

We now prove the converse. Suppose that $M_G \neq 1$. Let N be a minimal normal subgroup of G with $N \leq M$ and let R be a Sylow r-subgroup of M, where r is a prime dividing the order of M. Then, $RN/N \in Syl_r(M/N)$ and RN/N is a CAP^*-subgroup of G/N by Lemma 2.3. This means that G/N satisfies the hypotheses of our theorem. By induction, G/N is solvable. If N is a Frattini chief factor of G, then N is solvable and G is as well. Suppose that $N \nleq \Phi(G)$. By hypotheses, there exists a Sylow subgroup Q of M such that $N \cap Q \neq 1$, this forces that $N \leq Q$. Thus, N is solvable and so is G.

It remains to consider the case where $M_G = 1$ and let T be a minimal normal subgroup of G. It is clear that $T \nleq M$ and therefore $T/1$ is a non-Frattini chief factor of G. Every Sylow subgroup of M covers or avoids $T/1$ by hypotheses. If some Sylow subgroup of M covers $T/1$, then $T \leq M$, in contradiction to our assumption. Therefore, suppose that every Sylow subgroup of M avoids $T/1$. In this case, it is clear that $M \cap T = 1$ and $G/T \cong M$. We can deduce that every Sylow subgroup of

G/T is a CAP^*-subgroup of G/T. If follows from Theorem 3.1 that G/T is solvable and so is M. By Lemma 2.6, G is solvable. ∎

Theorem 3.4. *Let M be a maximal subgroup of a group G. Then M is a CAP^*-subgroup of G if and only if $\eta(G:M) = |G:M|$.*

Proof. A maximal subgroup M of a group G is a CAP^*-subgroup of G if and only if the core-free maximal subgroup $U = M/M_G$ is a CAP^*-subgroup of the primitive group $X = G/M_G$. On the other hand, U is a CAP^*-subgroup of X if and only if U complements any minimal normal subgroup of X. This happens if and only if X is a primitive group of type 1 or 3, or X is a primitive group of type 2 and U is a small maximal subgroup of X.

In fact, $\eta(G:M)$ is the order of any minimal normal subgroup of the primitive group G/M_G.

If G/M_G is a monolithic primitive group [1, Definition 1.1.8], then $\eta(G:M) = |Soc(G/M_G)|$. In this case, M/M_G is a CAP^*-subgroup of the monolithic primitive group G/M_G if and only if $(M/M_G)\cap Soc(G/M_G) = 1$. By order considerations and Lemma 2.5, this implies that $\eta(G:M) = |Soc(G/M_G)| = |G/M_G : M/M_G| = |G:M|$.

Suppose that G/M_G is a primitive group of type 3. If X is a primitive group of type 3, then then $Soc(X)$ has exactly two minimal normal subgroups N_1, N_2 and $Soc(X) = N_1N_2$, $N_1 \cong N_2$ and N_1 is non-abelian. In this case, if U is a core-free maximal subgroup of X, then, by [1, Theorem 1.1.7] $X = UN_i$ and $U \cap N_i = 1$ for $i = 1, 2$. This is to say that $\eta(X:U) = |N_i| = |X:U|$. Hence, if G/M_G is a primitive group of type 3, then M/M_G is always a CAP^*-subgroup of G/M_G and $\eta(G:M) = |G/M_G : M/M_G| = |G:M|$. ∎

Corollary 3.1. [5, 2.5] *A finite group G is solvable if and only if $\eta(G:M) = |G:M|$ for every maximal subgroup M of G.*

Let H be a normal subgroup of a group G and p a prime. We define the following families of subgroups:

$$\begin{aligned}
\mathcal{F}(G) &= \{M | M \lessdot G\} \\
\mathcal{F}_{pc}(G) &= \{M | M \in \mathcal{F}(G),\ |G:M|_p = 1 \text{ and } |G:M| \text{ is composite}\} \\
\mathcal{F}^{pcn}(G) &= \{M | M \in \mathcal{F}(G),\ N_G(P) \leq M \text{ for a Sylow } p\text{-subgroup } P \text{ of } G, \\
&\qquad M \text{ is non-nilpotent and } |G:M| \text{ is composite}\} \\
\mathcal{F}_h(G) &= \{M | M \in \mathcal{F}(G) \text{ and } G = MH\} \\
\mathcal{L}(G) &= \{M | M \in \mathcal{F}(G) \text{ and } M \text{ is not normal in } G\}
\end{aligned}$$

Theorem 3.5. *Let H be a normal subgroup of a group G and p the largest prime dividing the order of G. If every maximal subgroup M of G in $\mathcal{F}_{pc}(G) \cap \mathcal{F}_h(G)$ is a CAP^*-subgroup of G, then H is solvable.*

Proof. If $\mathcal{F}_{pc}(G) \cap \mathcal{F}_h(G) = \emptyset$, then we claim that H is solvable. In fact, if $\mathcal{F}_{pc}(G) = \emptyset$, by [17, Theorem 8], G is solvable and so is H. If $\mathcal{F}_{pc}(G) \neq \emptyset$, then H is contained in every maximal subgroup M of G in $\mathcal{F}_{pc}(G)$. Applying [17, Theorem 8] again, H is solvable. Now we may assume that $\mathcal{F}_{pc}(G) \cap \mathcal{F}_h(G)$ is not empty set.

Let N be a minimal normal subgroup of G, and let M/N be a maximal subgroup of G/N with $M/N \in\mathcal{F}_{pc}(G/N) \cap\mathcal{F}_h(G/N)$. Then $M \in\mathcal{F}_{pc}(G) \cap\mathcal{F}_h(G)$. Furthermore, M/N is a CAP^*-subgroup of G/N by Lemma 2.3. It is clear that G/N satisfies the hypotheses of the theorem for the normal subgroup HN/N and so HN/N is solvable by induction. If $N \not\leq H$, then $H \cong HN/N$ is solvable, as desired. Hence we may assume that $N \leq H$ and N is a non-Frattini chief factor of G.

Suppose that N is non-solvable. Let q be the largest prime dividing the order of N and Q a Sylow q-subgroup of N. Then $G = N_G(Q)N$ by the Frattini argument. So there exists a maximal subgroup M of G which contains $N_G(Q)$, but $N \not\leq M$. By hypotheses, $p \geq q$. If $p > q$, it is clear that $|G : M|_p = |N : M\cap N|_p$=1. If $p = q$, then $N_G(Q)$ contains a Sylow p-subgroup of G. Thus we conclude that $|G : M|_p = 1$ in these two cases. If $|G : M| = r$ for some prime r, then, since $M_G = 1$, we have that G is isomorphic to a subgroup of the symmetric group S_r on r letters. This implies that $|G|\big|r!$ and so $|N|\big|r!$, in contradiction to that q is the largest prime in $\pi(N)$. Hence we conclude that $M \in\mathcal{F}_{pc}(G) \cap\mathcal{F}_h(G)$. By the hypotheses, M is a CAP^*-subgroup of G and $N \not\leq \Phi(G)$, we have that $MN = M$ or $M \cap N = 1$. But these two situations are clearly impossible as $N_G(Q)$ is contained in M and $N \not\leq M$, a contradiction. This shows that N is solvable and therefore H is solvable. ∎

From Theorem 3.5 we have the following corollary.

Corollary 3.2. *Let p be the largest prime dividing the order of a group G. If every maximal subgroup M of G in $\mathcal{F}_{pc}(G)$ is a CAP^*-subgroup of G, then G is solvable.*

Proof. Let $G = H$ in Theorem 3.5. Then we have the corollary. ∎

In Theorem 3.5, the group G is not necessary solvable.

Example 3.1. Let K, H be the Alternating groups on 5 and 4 letters, respectively and let $G = K \times H$. Suppose that $M = K \times C_3$, where C_3 is a cyclic group of order 3 of H. Then M is a maximal subgroup of G. It is clear that $H \not\leq M$ and $|G : M| = 4$. Thus $M \in\mathcal{F}_{pc}(G) \cap\mathcal{F}_h(G)$ and we can also see that $\mathcal{F}_{pc}(G) \cap\mathcal{F}_h(G)$=$\{M^g | g \in G\}$. Furthermore, it is easy to see that M avoids $(K_4 \times K)/K$ and $K_4/1$, and covers the other non-Frattini chief factors of G, where K_4 is the Klein four group contained in H. That is, M is a CAP^*-subgroup of G. However, G is not solvable.

Theorem 3.6. *Let H be a normal subgroup of a group G and p the largest prime dividing the order of G. If every maximal subgroup M of G in $\mathcal{F}^{pcn}(G) \cap\mathcal{F}_h(G)$ is a CAP^*-subgroup of G, then H is p-solvable.*

Proof. If $\mathcal{F}^{pcn}(G) \cap\mathcal{F}_h(G) = \emptyset$, then we can see that H is p-solvable by [11, Lemma 2.4]. Now, we may assume that $\mathcal{F}^{pcn}(G) \cap\mathcal{F}_h(G) \neq \emptyset$. Let $P \in Syl_p(G)$. If P is normal in G, then G is certainly p-solvable and so is H. So we may assume that $N_G(P) < G$.

Let N be a minimal normal subgroup of G. It is clear that G/N satisfies the hypotheses of the theorem for the normal subgroup HN/N and so HN/N is p-solvable by induction. By a routine argument, we can assume that N is contained in H and N is a non-Frattini chief factor of G.

Suppose that N is non-p-solvable. Then p is a divisor of the order of N. We know that $N \cap P \in Syl_p(N)$ and $P \cap N$ is not a normal subgroup of N. By Frattini argument, we have that $G = N_G(P \cap N)N$. So there exists a maximal subgroup M of G which contains $N_G(P \cap N)$ and $M \ngeq N$. It is clear that $N_G(P) \leq M$. If $|G : M| = q$ is a prime, then by Sylow's theorem, we have $q = 1 + kp$ and $q \big| |N|$. This contradicts p being the largest prime which divides the order of N. Hence $|G : M|$ must be a composite number. If M is nilpotent, then the Sylow 2-subgroup M_2 of M is not identity by [19, Theorem 10.4.2]. Let $M_{2'}$ be a Hall $2'$-subgroup of M. By [20, Theorem 1], $M_{2'}$ is normal in G and therefore $P \trianglelefteq G$ since P is a characteristic subgroup of $M_{2'}$. It follow that $P \cap N \trianglelefteq G$, a contradiction. Thus, $M \in \mathcal{F}^{\ pcn}\ (G) \cap \mathcal{F}_{\ h}\ (G)$. By the hypotheses, M is a CAP^*-subgroup of G and so $MN = M$ or $M \cap N = 1$. However, these two situations are impossible. This shows that N is p-solvable and therefore H is p-solvable. The proof of the theorem is now complete. ∎

Corollary 3.3. *Let p be the largest prime dividing the order of G. If every maximal subgroup M of G in $\mathcal{F}^{pcn}(G)$ is a CAP^*-subgroup of G, then G is p-solvable.*

In Theorem 3.6, the group G need not be p-solvable as the following example shows.

Example 3.2. Let $H = C_2 \times C_2 \times C_2 \times C_2$ be an elementary abelian group of order 2^4. Then there is a subgroup $M = A_5$ in the automorphism group of H, where A_5 is the Alternating group on 5 letters. Let $G = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_5$ be the corresponding semidirect product. We can deduce that $\mathcal{F}^{pcn}(G) \cap \mathcal{F}_h(G) = \{M^g | g \in G\}$. Furthermore, M^g covers or avoids every non-Frattini chief factor of G. Thus, M^g is a CAP^*-subgroup of G. That is, G satisfies the hypotheses of the Theorem 3.6. However, G is not 5-solvable.

Theorem 3.7. *A group G is solvable if and only if M is a CAP^*-subgroup of G for every maximal subgroup $M \in \mathcal{L}(G)$.*

Proof. If G is solvable, then by Theorem 3.1, we know that every maximal subgroup of G is CAP^*-subgroup of G. We only need prove the sufficient condition.

If $\mathcal{L}(G) = \emptyset$, then, for any maximal subgroup M of G, $M \trianglelefteq G$. So G is nilpotent by [19, Theorem 5.2.4]. We may assume that $\mathcal{L}(G) \neq \emptyset$. If G is a simple group, then every maximal subgroup M of G is contained in $\mathcal{L}(G)$. It follows from Theorem 3.1 that G is solvable. Hence we may assume that G is not a simple group.

Let N be a minimal normal subgroup of G. If $M/N \in \mathcal{L}(G/N)$, then $M \ntrianglelefteq G$. If follows from Lemma 2.3 that M/N is a CAP^*-subgroup of G/N. Thus, G/N satisfies the hypotheses of our theorem. By induction, G/N is solvable. If $N \leq \Phi(G)$, then G is solvable. If N is a non-Frattini chief factor of G, then there exists a maximal subgroup M of G such that $G = MN$. By hypotheses, $M \cap N = 1$ and $M \cong G/N$ is solvable. Hence G is solvable by Lemma 2.6. Thus, our proof is complete. ∎

By using the 2-maximal subgroups, we obtain the following theorem.

Theorem 3.8. *If every 2-maximal subgroup of a group G is a CAP^*-subgroup of G, then G is solvable.*

Proof. Let G be a minimal counterexample. If G is simple, then the trivial subgroup is the unique 2-maximal subgroup of G. This implies that every maximal subgroup is a cyclic group of prime order. In this case G is solvable. Thus G is a non-simple group.

Let N be a minimal normal subgroup of G. If H/N is a 2-maximal subgroup of G/N, then H is 2-maximal in G. By hypotheses, H is a CAP^*-subgroup of G. By Lemma 2.3, H/N is a CAP^*-subgroup of G/N. Hence, by induction G/N is a solvable group.

This holds for every minimal normal subgroup of G. By minimality of G we have that G is a monolithic primitive group of type 2. Suppose that N is the minimal normal subgroup of G and there exists a core-free maximal subgroup M of G such that $M \cap N \neq 1$. Let H be any maximal subgroup of M containing $M \cap N$. Since H is a CAP^*-subgroup of G, we have that $H \cap N = 1$. But this is not possible. This is the final contradiction. ∎

Theorem 3.9. *If there is a 2-maximal subgroup H of a group G such that H is a solvable CAP^*-subgroup of G, then G is solvable.*

Proof. We claim that G is not a simple group. In fact, if G is a simple group and H is a 2-maximal CAP^*-subgroup of G, then $H = 1$. This forces the existence of a maximal subgroup M of G such that M is isomorphic to a cyclic group of prime order. If p is the order of M, then it is clear that M is self-normalizing in G. This implies that $N_G(M) = C_G(M) = M$. By the well-known Burnside Theorem [14, IV. 2.6 Satz], the group G has a normal p-complement. But this contradicts our assumption of simplicity of G.

If $H_G \neq 1$, then it is easy to see that G/H_G satisfies the hypotheses of the theorem. An inductive argument shows that G/H_G is solvable and so is G. Suppose that $H_G = 1$.

If $\Phi(G) \neq 1$. Let N be a minimal normal subgroup of G containing in $\Phi(G)$. Then HN is a maximal subgroup of G and HN is a CAP^*-subgroup of G by Lemma 2.4. We can see that HN is solvable. In view of Theorem 3.2, G is solvable. Hence we may assume that $\Phi(G) = 1$. If N is a minimal normal subgroup of G, then $HN = H$ or $H \cap N = 1$. It follows from the above arguments that $H \cap N = 1$.

We claim that $HN < G$. Otherwise, if $HN = G$. By hypotheses, there is a maximal subgroup M of G such that H is a maximal subgroup of M. It is clear that $M = M \cap G = H(M \cap N)$. Noticing that $M \cap N$ is normal in M and $(M \cap N) \cap H \leq N \cap H = 1$, we see that $M \cap N$ is a minimal normal subgroup of M. Applying Lemma 2.6, M is solvable and so $M \cap N$ is an elementary p-group for some prime p. Let $P = M \cap N$, then we can see that $M \leq N_G(P)$. If $N_G(P) = G$, then $N = P$ by the minimality of N, this implies that $G = HN \leq M$, a contradiction. Hence $N_G(P) = M$. It follows that $N_N(P) = P = C_N(P)$. By the Burnside theorem, we see that N is p-nilpotent. However, because N is a minimal normal subgroup of G, N is a p-group and $N = P \leq M = N_G(P) < G$, which contradicts $HN = G$.

We claim that HN is a maximal subgroup of G. In fact, since H is a 2-maximal subgroup of G, there is a maximal subgroup M of G such that H is a maximal subgroup of M. If $N \nleq M$, then $G = MN$ and $HN < MN$. Let K be a subgroup of G with $HN \leq K \leq MN = G$. Then $K = N(K \cap M)$ and $H \leq K \cap M \leq M$. Noticing that H is a maximal subgroup of M, we have that $K \cap M = H$ or M.

It follows that $K = HN$ or $K = MN = G$. Hence HN is a maximal subgroup of G. If $N \leq M$, then $HN \leq M$. Since H is a maximal subgroup of M, we see that $HN = M$ is also a maximal subgroup of G. This implies that N is a minimal normal subgroup of HN since H is a 2-maximal subgroup of G. By Lemma 2.6, HN is solvable. Applying Lemma 2.4 and Theorem 3.2, G is solvable. Thus, the proof is complete. ∎

Theorem 3.9 is not true for r-maximal subgroup of a group G when $r \geq 3$.

Example 3.3. Let $G = S_n$, the symmetric group on n letters with $n \geq 5$, and $N = A_n$, the alternating group on n letters. If we set $H = \langle (13) \rangle$, then H is a r-maximal subgroup with $r \geq n-2$ since we have the following series of subgroups of G: $H \lessdot S_3 \lessdot S_4 \lessdot S_5 \lessdot \cdots \lessdot S_n$. It is easy to see that H is a solvable CAP^*-subgroup of G, but G is not solvable.

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