BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **35**(1) (2012), 25–38

Existence of Periodic Solutions of $p(t)$-Laplacian Systems

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Abstract. In this paper, by using the least action principle in critical point theory, we obtain some existence theorems of periodic solutions for $p(t)$-Laplacian system

$$\begin{cases} \dfrac{d}{dt}(|\dot{u}(t)|^{p(t)-2}\dot{u}(t)) = \nabla F(t,u(t)) & \text{a.e. } t \in [0,T] \\ u(0)-u(T) = \dot{u}(0)-\dot{u}(T) = 0, \end{cases}$$

which generalize some existence theorems.

2010 Mathematics Subject Classification: 34C25, 35A15

Keywords and phrases: Periodic solutions, $p(t)$-Laplacian systems, variational methods, least action principle.

1. Introduction

This paper is concerned with the existence of periodic solutions for the following $p(t)$-*Laplacian* system

$$\text{(1.1)} \qquad \begin{cases} \dfrac{d}{dt}(|\dot{u}(t)|^{p(t)-2}\dot{u}(t)) = \nabla F(t,u(t)) & \text{a.e. } t \in [0,T] \\ u(0)-u(T) = \dot{u}(0)-\dot{u}(T) = 0, \end{cases}$$

where $p(t) \in C([0,T], R^+)$, $p^+ = \max_{0\le t\le T} p(t)$, $p(t) = p(t+T)$ for all $t \in R^1$, $T > 0$, $F : [0,T] \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ satisfies the following assumption:

(A) $F(t,x)$ is measurable in t for every $x \in \mathbb{R}^{\mathbb{N}}$ and continuously differentiable in x for a.e. $t \in [0,T]$, and there exist $a \in C(\mathbb{R}^+,\mathbb{R}^+), b \in L^1(0,T;\mathbb{R}^+)$, such that

$$|F(t,x)| \le a(|x|)b(t), \quad |\nabla F(t,x)| \le a(|x|)b(t)$$

for all $x \in R^N$ and a.e. $t \in [0,T]$.

The $p(t)$-Laplacian system can be applied to describe the physical phenomena with "pointwise different properties" which arose from the nonlinear elasticity theory (see [12]). System (1.1) has been studied by Fan in a series of papers (see [2–4]).

Communicated by Norhashidah Hj. Mohd. Ali.
Received: November 13, 2009; *Revised:* February 21, 2010.

The $p(t)$-Laplacian system possesses more complicated nonlinearity than that of the p-Laplacian, for example, it is not homogeneous, this causes many troubles, and some classical theories and methods, such as the theory of Sobolev spaces, are not applicable.

If $p = 2$, system (1.1) reduces to

$$\begin{cases} \ddot{u}(t) = \nabla F(t, u(t)) \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases} \tag{1.2}$$

the corresponding functional ψ on H_T^1 given by

$$\psi(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt + \int_0^T F(t, u(t))dt$$

is a continuously differentiable and weakly lower semi-continuous on H_T^1 (see [5]), where

$$H_T^1 = \big\{u : [0, T] \to \mathbb{R}^{\mathbb{N}},\ u \text{ is absolutely continuous,} \\ u(0) = u(T), \dot{u} \in L^2(0, T; \mathbb{R}^{\mathbb{N}})\big\}$$

is a Hilbert space with a norm defined by

$$\|u\|_{H_T^1} = \left(\int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt\right)^{1/2}$$

for $u \in H_T^1$.

Considerable attention has been paid to the periodic solutions for system (1.2) in recent years. It has been proved by the least action principle that system (1.2) has at least one solution which minimizes φ on H_T^1 (see [5, 6–8, 10]). When $F(t, \cdot)$ is convex for a.e. $t \in [0, T]$, Mawhin-Willem (see [4]) have proved the existence of solutions for system (1.2) which minimizes φ on H_T^1. For non-convex potential cases, by using the least action principle and the minimax method, the existence of solutions which minimize φ on H_T^1 has been researched by many people, for example, (see [5–10]). Inspired and motivated by the results due to Mawhin (see [5]) and Tang (see [6–7]) , in this paper, we start to consider system (1.1). Some new solvability conditions are obtained by using the least action principle, and the results in this paper develop and generalize some corresponding results.

2. Preliminaries

In this section, we recall some known results in critical point theory, and the properties of space $W_T^{1,p(t)}$ are listed for the convenience of readers.

Definition 2.1. [2] *Let* $p(t) \in C([0, T], R^+)$ *with* $p^- = \min_{0 \le t \le T} p(t) > 1$. *Define*

$$L^{p(t)}([0, T], R^N) = \left\{u \in L^1([0, T], R^N); \int_0^T |u|^{p(t)} dt < \infty\right\}$$

with the norm

$$|u|_{p(t)} := \inf\left\{\lambda > 0; \int_0^T \left|\frac{u}{\lambda}\right|^{p(t)} dt \le 1\right\}.$$

For $u \in L^1_{loc}([0,T], R^N)$, let u' denotes the weak derivative of u, if $u' \in L^1_{loc}([0,T], R^N)$ and satisfies

$$\int_0^T u'\phi dt = -\int_0^T u\phi' dt, \quad \forall \phi \in C_0^\infty([0,T], R^N).$$

Define

$$W^{1,p(t)}([0,T], R^N) = \{u \in L^{p(t)}([0,T], R^N);\ u' \in L^{p(t)}([0,T], R^N)\}$$

with the norm $\|u\|_{W^{1,p(t)}} := |u|_{p(t)} + |u'|_{p(t)}$.

Remark 2.1. If $p(t) = p$, where $p \in [1, \infty)$ is a constant, by the definition of $|u|_{p(t)}$, it is easy to get $|u|_p = (\int_0^T |u(t)|^p dt)^{1/p}$, which is the same with the usual norm in space L^p.

The space $L^{p(t)}$ is a generalized Lebesgue space, and the space $W^{1,p(t)}$ is a generalized Sobolev space. Because most of the following Lemmas have appeared in [2–5], we omit their proofs.

Lemma 2.1. [3] *$L^{p(t)}$ and $W^{1,p(t)}$ are both Banach spaces with the norms defined above, when $p^- > 1$, they are reflexive.*

Definition 2.2. [5]

$$C_T^\infty = C_T^\infty(R, R^N) = \{u \in C^\infty(R, R^N) : u \text{ is } T\text{-periodic}\}$$

with the norm $\|u\|_\infty := \max_{t \in [0,T]} |u(t)|$.

For a constant $p \in [1, \infty)$, using another conception of weak derivative which is called T-weak derivative, Mawhin and Willem gave the definition of the space $W_T^{1,p}$ by the following way.

Definition 2.3. [5] *Let $u \in L^1([0,T], R^N)$ and $v \in L^1([0,T], R^N)$, if*

$$\int_0^T v\phi dt = -\int_0^T u\phi' dt, \quad \forall \phi \in C_T^\infty$$

then v is called a T-weak derivative of u and is denoted by $\dot{u}$.

Definition 2.4. [5] *Define*

$$W_T^{1,p}([0,T], R^N) = \{u \in L^p([0,T], R^N); \dot{u} \in L^p([0,T], R^N)\}$$

with the norm $\|u\|_{W_T^{1,p}} = (|u|_p^p + |\dot{u}|_p^p)^{1/p}$.

Definition 2.5. [2] *Define*

$$W_T^{1,p(t)}([0,T], R^N) = \{u \in L^{p(t)}([0,T], R^N);\ \dot{u} \in L^{p(t)}([0,T], R^N)\}$$

and $H_T^{1,p(t)}([0,T], R^N)$ to be the closure of C_T^∞ in $W^{1,p(t)}([0,T], R^N)$.

Remark 2.2. By Definition 2.5, if $u \in W_T^{1,p(t)}([0,T], R^N)$, it is easy to conclude that $u \in W_T^{1,p^-}([0,T], R^N)$.

By Definition 2.1 and 2.2, we conclude that, for $u \in L^1([0,T], R^N)$, the weak derivative u' and the T-weak derivative $\dot{u}$ are two different conceptions (for details see [4]). Although the two derivatives are distinct, we have

Lemma 2.2. [2]

(i) *$C_T^\infty([0,T],R^N)$ is dense in $W^{1,p(t)}([0,T],R^N)$,*

(ii) $W_T^{1,p(t)}([0,T],R^N)=H_T^{1,p(t)}([0,T],R^N):=\{u\in W^{1,p(t)}([0,T],R^N);u(0)=u(T)\}$,

(iii) *If $u\in H_T^{1,1}$, then the derivative u' is also the T-weak derivative $\dot u$, i.e. $u'=\dot u$.*

Remark 2.3. In the following part of article, we use $\|u\|$ instead of $\|u\|_{W_T^{1,p(t)}}$ for convenience without clear indications.

Lemma 2.3. [2] *Assume that $u\in W_T^{1,1}$, then*

(i) *$\int_0^T \dot u dt=0$,*

(ii) *u has its continuous representation, which is still denoted by $u(t)=\int_0^t \dot u(s)ds+u(0)$, $u(0)=u(T)$,*

(iii) *$\dot u$ is the classical derivative of u, if $\dot u\in C([0,T],R^N)$.*

Since every closed linear subspace of a reflexive Banach space is also reflexive, we have

Lemma 2.4. [2] *$H_T^{1,p(t)}([0,T],R^N)$ is a reflexive Banach space if $p^->1$.*

Obviously, there are continuous embeddings $L^{p(t)}\hookrightarrow L^{p^-}$, $W^{1,p(t)}\hookrightarrow W^{1,p^-}$ and $H_T^{1,p(t)}\hookrightarrow H_T^{1,p^-}$. By the classical Sobolev embedding theorem we obtain

Lemma 2.5. [2] *There is a continuous embedding*

$$W^{1,p(t)}(or\ H_T^{1,p(t)})\hookrightarrow C([0,T],R^N),$$

when $p^->1$, the embedding is compact.

Lemma 2.6. *The space $W_T^{1,p(t)}=\tilde W_T^{1,p(t)}\bigoplus R^N$, where*

$$\tilde W_T^{1,p(t)}:=\left\{u\in W_T^{1,p(t)};\int_0^T u(t)dt=0\right\},$$

there exist $C_0>0$, $C_1>0$, if $u\in \tilde W_T^{1,p(t)}$, such that

$$\|u\|_\infty\le 2C_0\left(\int_0^T|\dot u(t)|^{p(t)}dt\right)^{1/p^-}+C_1.$$

Proof. Let $A=\{t\in[0,T]||\dot u(t)|\ge 1\}$. From Remark 2.2, $u\in W_T^{1,p^-}$, from the inequality in classical Sobolev space, there exists a positive constant $C_0>0$, such that

$$\begin{aligned}\|u\|_\infty&\le C\left(\int_0^T|\dot u(t)|^{p^-}dt\right)^{1/p^-}\\&=C\left(\int_A|\dot u(t)|^{p^-}dt+\int_{[0,T]\setminus A}|\dot u(t)|^{p^-}dt\right)^{1/p^-}\end{aligned}$$

$$\le C\left(\int_A |\dot u(t)|^{p(t)}dt + meas[0,T]\backslash A\right)^{1/p^-}$$
$$\le C\left(\int_0^T |\dot u(t)|^{p(t)}dt + T\right)^{1/p^-}$$
$$\le 2C\left(\int_0^T |\dot u(t)|^{p(t)}dt\right)^{1/p^-} + 2CT^{1/p^-},$$

This completes the proof of Lemma 2.6. ∎

Lemma 2.7. [2] *Each of the following two norms is equivalent to the norm in* $W_T^{1,p(t)}$

(i) $|\dot u|_{p(t)} + |u|_q$, $1 \le q \le \infty$;

(ii) $|\dot u|_{p(t)} + |\bar u|$, *where* $\bar u = (1/T)\int_0^T u(t)dt$.

Lemma 2.8. [2] *If we denote* $\rho(u) = \int_0^T |u|^{p(t)}dt$, $\forall u \in L^{p(t)}$, *then*

(i) $|u|_{p(t)} < 1\ (= 1; > 1) \Longleftrightarrow \rho(u) < 1\ (= 1; > 1)$;

(ii) $|u|_{p(t)} > 1 \Longrightarrow |u|_{p(t)}^{p^-} \le \rho(u) \le |u|_{p(t)}^{p^+}$, $|u|_{p(t)} < 1 \Longrightarrow |u|_{p(t)}^{p^+} \le \rho(u) \le |u|_{p(t)}^{p^-}$;

(iii) $|u|_{p(t)} \to 0 \Longleftrightarrow \rho(u) \to 0$; $|u|_{p(t)} \to \infty \Longleftrightarrow \rho(u) \to \infty$.

Proposition 2.1. *In space* $W_T^{1,p(t)}$, $\|u\| \to \infty \Longrightarrow \int_0^T |\dot u|^{p(t)}dt + |\bar u| \to \infty$.

Proof. By Lemma 2.7, there exists a constant $C_2 > 0$ such that

$$\|u\| \le C_2(|\dot u|_{p(t)} + |\bar u|),$$

If $|\dot u|_{p(t)} < 1$, it is easy to get

$$|\dot u|_{p(t)} < \int_0^T |\dot u|^{p(t)}dt + 1. \tag{2.1}$$

If $|\dot u|_{p(t)} \ge 1$, we conclude that

$$\begin{aligned} |\dot u|_{p(t)} \ge 1 \Longrightarrow |\dot u|_{p(t)} &\le \left(\int_0^T |\dot u|^{p(t)}dt\right)^{1/p^-} \\ \Longrightarrow |\dot u|_{p(t)} &\le \int_0^T |\dot u|^{p(t)}dt + 1 \end{aligned} \tag{2.2}$$

by Lemma 2.8, it follows (2.1) and (2.2) that

$$\|u\| \le C_2\left(\int_0^T |\dot u|^{p(t)}dt + 1 + |\bar u|\right),$$

the above inequality implies that

$$\|u\| \to \infty \Longrightarrow \int_0^T |\dot u|^{p(t)}dt + |\bar u| \to \infty.$$ ∎

Lemma 2.9. [2] *If $u_n, u \in L^{p(t)}$ $(n = 1, 2, \cdots)$, then the following statements are equivalent to each other*

(i) $\lim\limits_{n\to\infty} |u_n - u|_{p(t)} = 0$;

(ii) $\lim\limits_{n\to\infty} \rho(u_n - u) = 0$;

(iii) $u_n \to u$ *in measure in* $[0, T]$ *and* $\lim\limits_{n\to\infty} \rho(u_n) = \rho(u)$.

Definition 2.6. [5] *Let X be a normed space. A minimizing sequence for a function $\varphi : X \to (-\infty, +\infty]$ is a sequence such that*

$$\varphi(u_k) \to \inf \varphi, \ \ as\ k \to \infty.$$

A function $\varphi : X \to (-\infty, +\infty]$ is lower semi-continuous if

$$u_k \rightharpoonup u \Longrightarrow \varliminf_{k\to\infty} \varphi(u_k) \geq \varphi(u).$$

Lemma 2.10. [11] *The functional on $W_T^{1,p(t)}$ given by*

$$\varphi(u) = \int_0^T \frac{1}{p(t)} |\dot{u}(t)|^{p(t)} + \int_0^T F(t, u(t))dt$$

is a continuously differentiable and weakly lower semi-continuous on $W_T^{1,p(t)}$. Moreover, we have

$$\langle \varphi'(u), v \rangle = \int_0^T [(|\dot{u}(t)|^{p(t)-2}\dot{u}(t), \dot{v}(t)) + (\nabla F(t, u(t)), v(t))]dt$$

for all $u, v \in W_T^{1,p(t)}$. It is well known that the critical points of φ correspond to the solutions for system (1.1).

Lemma 2.11. [5] *If ϕ: $X \to R$ is w.l.s.c. on a reflexive Banach space X and has a bounded minimizing sequence, then ϕ has a minimum on X. If ϕ is differentiable, every local minimum (resp. maximum) point satisfies the Euler equation $\phi'(u) = 0$, that is to say, the minimum (resp. maximum) point corresponds to a critical point of ϕ.*

Remark 2.4. By Lemma 2.10 and Lemma 2.11, as long as we get a bounded minimizing sequence of φ on $W_T^{1,p(t)}$, the existence of a critical point for φ is ensured.

Definition 2.7. [6] *A function $G : R^N \to R$ is called γ-quasisubadditive if*

$$G(x + y) \leq \gamma((G(x) + G(y))$$

for all $x, y \in R^N$. We call G subadditive if G is 1-quasisubadditive.

Lemma 2.12. [7] *Assume that F satisfies the assumption (A) and*

$$F(t, x) \to +\infty, \quad as \quad |x| \to \infty$$

uniformly for a.e. $t \in [0, T]$. Then there exist $\beta \in L^1(0, T)$ and a subadditive function $G : R^N \to R$, that is

$$G(x + y) \leq G(x) + G(y)$$

for all x and y in R^N, such that

$$G(x) + \beta(t) \leq F(t, x)$$

for a.e. $t \in [0,T]$ *and all* $x \in R^N$*, and*

$$G(x) \to +\infty, \quad as \quad |x| \to \infty$$

and

$$0 \leq G(x) \leq |x| + 1$$

for all $x \in R^N$*.*

3. Main results and proofs of theorems

If φ has a bounded minimizing sequence, φ has a minimum on $W_T^{1,p(t)}$ and system (1.1) is solvable. It remains to find conditions under which φ has a bounded minimizing sequence by Lemma 2.11. When ∇F is bounded by a L^1 function for all $x \in R^N$, we shall see that it suffices to require a coercivity condition on the integral value of potential $F(t,x)$.

Theorem 3.1. *Suppose that* F *satisfies the assumption (A) and that there exists* $g(t) \in L^1(0,T,R^+)$ *such that*

$$|\nabla F(t,x)| \leq g(t)$$

for a.e. $t \in [0,T]$ *and all* $x \in R^N$*, and if*

$$\int_0^T F(t,x)dt \to +\infty \quad as \quad |x| \to \infty, \tag{3.1}$$

then system (1.1) has at least one solution which minimizes φ *on* $W_T^{1,p(t)}$*.*

Proof. For $u \in W_T^{1,p(t)}$, we have $u = \bar{u} + \tilde{u}$, where $\bar{u} = (1/T)\int_0^T u(t)dt$ and

$$\begin{aligned}
\varphi(u) &= \int_0^T \frac{1}{p(t)}|\dot{u}(t)|^{p(t)}dt + \int_0^T F(t,\bar{u}(t))dt + \int_0^T (F(t,u(t) - F(t,\bar{u}(t))dt \\
&= \int_0^T \frac{1}{p(t)}|\dot{u}(t)|^{p(t)}dt + \int_0^T F(t,\bar{u}(t))dt + \int_0^T \int_0^1 (\nabla F(t,\bar{u} + s\tilde{u}(t), \tilde{u}(t))dsdt \\
&\geq \int_0^T \frac{1}{p(t)}|\dot{u}(t)|^{p(t)}dt - \left(\int_0^T g(t)dt\right)\|\tilde{u}\|_\infty + \int_0^T F(t,\bar{u})dt \\
&\geq \int_0^T \frac{1}{p(t)}|\dot{u}(t)|^{p(t)}dt - 2C\int_0^T g(t)dt\left(\int_0^T |\dot{u}(t)|^{p(t)}dt\right)^{1/p^-} \\
&\quad + \int_0^T F(t,\bar{u})dt - 2CT^{1/p^-}\int_0^T g(t)dt \\
&\geq (1/p^+)\int_0^T |\dot{u}(t)|^{p(t)}dt + \int_0^T F(t,\bar{u})dt - C_3\left(\int_0^T |\dot{u}(t)|^{p(t)}dt\right)^{1/p^-} - C_4,
\end{aligned} \tag{3.2}$$

for some positive constants C_3, C_4 by Lemma 2.6. Proposition 2.1, (3.1) and (3.2) imply that

$$\varphi(u) \to +\infty \quad \text{as} \quad |u\| \to \infty,$$

and hence every minimizing sequence is bounded, which completes the proof. ∎

Remark 3.1. Theorem 3.1 generalizes Theorem 1.5 in [5], and we use $\bar{u} = (1/T)\int_0^T u(t)dt$ and $\tilde{u} = u - \bar{u}$ without clear indications in the following part of article.

We show system (1.1) is solvable when F is periodic in each variable x_i. Let (e_i) $(1 \le i \le N)$ denote the canonical basis of R^N.

Theorem 3.2. *Assume that F satisfies the condition (A) and there exists $T_i > 0$ such that*

$$F(t, x + T_i e_i) = F(t, x) \quad (1 \le i \le N)$$

for a.e. $t \in [0, T]$ and all $x \in R^N$, then system (1.1) has at least one solution which minimizes φ on $W_T^{1,p(t)}$.

Proof. It follows from the periodicity of F in x that there exists $h \in L^1(0, T)$ such that

$$F(t, x) \ge h(t)$$

for a.e. $t \in [0, T]$ and all $x \in R^N$. Consequently, if $\int_0^T h(t)dt = C_5$,

$$\varphi(u) \ge \int_0^T \frac{1}{p(t)} |\dot{u}(t)|^{p(t)} dt + C_5$$

for all $u \in W_T^{1,p(t)}$. As $inf_{W_T^{1,p(t)}} \varphi < +\infty$, it follows from the above inequality that if (u_k) is a minimizing sequence for φ, there exists $C_6 > 0$ such that

$$\int_0^T |\dot{u}_k(t)|^{p(t)} dt \le C_6 \tag{3.3}$$

for all $k \in N$. On the other hand, it follows from the periodicity of F in x that

$$\varphi(u + T_i e_i) = \varphi(u), \quad 1 \le i \le N$$

for all $u \in W_T^{1,p(t)}$ and hence if (u_k) is a minimizing sequence for φ, $([(\bar{u}_k, e_1) + k_1 T_1 + (\tilde{u}_k, e_1), ..., (\bar{u}_k, e_N) + k_N T_N + (\tilde{u}_k, e_N)])$ is also a minimizing sequence of φ and we can therefore assume that

$$0 \le (\bar{u}_k, e_i) \le T_i, \quad (1 \le i \le N).$$

Consequently by Proposition 2.1 and (3.3), φ admits a bounded minimizing sequence, and the proof is completed. ∎

Remark 3.2. Theorem 3.2 generalizes Theorem 1.6 in [5].

When F is convex in x, it is possible to eliminate the boundedness condition on ∇F in Theorem 3.1 and to deduce an existence theorem for system (1.1).

Theorem 3.3. *Assume that F satisfies the condition (A) and $F(t, \cdot)$ is convex for a.e. $t \in [0, T]$ and*

$$\int_0^T F(t, x)dt \to +\infty \quad as \quad |x| \to \infty, \tag{3.4}$$

then system (1.1) *has at least one solution which minimizes φ on $W_T^{1,p(t)}$.*

Proof. By assumption of (3.4), the real function on R^N defined by

$$x \to \int_0^T F(t,x)dt$$

has a minimum at some point $\bar{x}$ for which

$$\int_0^T \nabla F(t,\bar{x})dt = 0. \tag{3.5}$$

Let (u_k) be a minimizing sequence for φ, it follows from (3.5) and the convex property of $F(t,\cdot)$ that

$$\begin{aligned}\varphi(u_k) &\geq \int_0^T \frac{1}{p(t)}|\dot{u}_k(t)|^{p(t)}dt + \int_0^T F(t,\bar{x})dt + \int_0^T (\nabla F(t,\bar{x}), u_k(t)-\bar{x})dt \\ &= \int_0^T \frac{1}{p(t)}|\dot{u}_k|^{p(t)}dt + \int_0^T F(t,\bar{x})dt + \int_0^T (\nabla F(t,\bar{x}), \tilde{u}_k(t))dt,\end{aligned}$$

where $u_k = \bar{u}_k + \tilde{u}_k$. By Lemma 2.6, we have

$$\begin{aligned}\varphi(u_k) &\geq \int_0^T \frac{1}{p(t)}|\dot{u}_k(t)|^{p(t)}dt + \int_0^T F(t,\bar{x})dt - \left(\int_0^T |\nabla F(t,\bar{x})|dt\right)\|\tilde{u}_k\|_\infty \\ &\geq \int_0^T \frac{1}{p(t)}|\dot{u}_k(t)|^{p(t)}dt - C_7\left(\int_0^T |\dot{u}_k(t)|^{p(t)}\right)^{1/p^-} - C_8 \\ &\geq (1/p^+)\int_0^T |\dot{u}_k(t)|^{p(t)}dt - C_7\left(\int_0^T |\dot{u}_k(t)|^{p(t)}\right)^{1/p^-} - C_8,\end{aligned}$$

for some positive constants C_7, C_8. Hence, there exists a constant $C_9 > 0$ such that

$$\int_0^T |\dot{u}_k(t)|^{p(t)}dt \leq C_9. \tag{3.6}$$

By Lemma 2.6, it is easy to get that

$$\|\tilde{u}_k\|_\infty \leq C_{10} \tag{3.7}$$

for some constant $C_{10} > 0$. Now we have, by convexity,

$$\begin{aligned}F(t,\bar{u}_k/2) &= F(t,(1/2)(u_k(t)-\tilde{u}_k(t))) \\ &\leq (1/2)F(t,u_k(t)) + (1/2)F(t,-\tilde{u}_k(t)),\end{aligned}$$

for a.e. $t \in [0,T]$ and all $k \in N$, hence

$$\varphi(u_k) \geq \int_0^T \frac{1}{p(t)}|\dot{u}_k(t)|^{p(t)}dt + 2\int_0^T F(t,\bar{u}_k/2)dt - \int_0^T F(t,-\tilde{u}_k(t))dt.$$

This implies, by (3.7),

$$\varphi(u_k) \geq 2\int_0^T F(t,\bar{u}_k/2)dt - C_{11}$$

for some $C_{11} > 0$ and therefore, by (3.4), $(\bar{u}_k)$ is bounded, which completes the proof. ∎

Remark 3.3. Theorem 3.3 generalizes Theorem 1.7 in [5].

We continue to consider system (1.1) with a potential which is the sum of a function with uniform coercivity and a function with bounded nonlinearity, and get an existence theorem of periodic solution for system (1.1) under suitable conditions.

Theorem 3.4. *Assume that $F = F_1 + F_2$, where F_1 and F_2 satisfy the condition (A) and*

$$F_1(t,x) \to +\infty, \quad \text{as} \quad |x| \to \infty$$

uniformly for a.e. $t \in [0,T]$, and there exist $D_0 \in R$ and $g_1(t) \in L^1(0,T,R^+)$ such that

$$|\nabla F_2(t,x)| \le g_1(t)$$

for a.e. $t \in [0,T]$ and all $x \in R^N$, and

$$\int_0^T F_2(t,x)dt \ge D_0$$

for all $x \in R^N$, then system (1.1) has at least one solution which minimizes φ on $W_T^{1,p(t)}$.

Proof. Let (u_k) be a minimizing sequence of φ on $W_T^{1,p(t)}$, we obtain

$$\begin{aligned}
\varphi(u_k) &\ge \int_0^T \frac{1}{p(t)}|\dot{u}_k(t)|^{p(t)}dt + \int_0^T \beta(t)dt + \int_0^T F_2(t,\bar{u}_k)dt \\
&\quad + \int_0^T \int_0^1 (\nabla F_2(t,\bar{u}_k + s\tilde{u}_k(t), \tilde{u}_k(t))dsdt \\
&\ge \int_0^T \frac{1}{p(t)}|\dot{u}_k(t)|^{p(t)}dt - \int_0^T g_1(t)dt\|\tilde{u}_k\|_\infty + D_0 - D_1 \\
&\ge \int_0^T \frac{1}{p(t)}|\dot{u}_k(t)|^{p(t)}dt - D_2\left(\int_0^T |\dot{u}_k(t)|^{p(t)}dt\right)^{1/p^-} + D_0 - D_1 \\
&\ge (1/p^+)\int_0^T |\dot{u}_k(t)|^{p(t)}dt - D_2\left(\int_0^T |\dot{u}_k(t)|^{p(t)}dt\right)^{1/p^-} + D_0 - D_1,
\end{aligned}$$

for some constants $D_1, D_2 > 0$ by Lemma 2.6. Hence, there exists a constant $D_3 > 0$ such that

$$\int_0^T |\dot{u}_k(t)|^{p(t)}dt \le D_3.$$

By Lemma 2.6, we have

$$\|\tilde{u}_k\|_\infty \le D_4$$

for some constant $D_4 > 0$. Moreover, we have

$$\begin{aligned}
\varphi(u_k) &\ge \int_0^T G(u_k(t))dt + \int_0^T \beta(t)dt + \int_0^T F_2(t,\bar{u}_k)dt \\
&\quad + \int_0^T \int_0^1 (\nabla F_2(t,\bar{u}_k + s\tilde{u}_k(t), \tilde{u}_k(t))dsdt
\end{aligned}$$

$$\geq TG(\bar{u}_k) - \int_0^T G(-\tilde{u}_k(t))dt - \int_0^T g_1(t)dt\|\tilde{u}_k\|_\infty + D_0 - D_1$$
$$\geq TG(\bar{u}_k) - D_5,$$

for some constant $D_5 > 0$. By the coercivity of G, we obtain

$$|\bar{u}_k| \leq D_6,$$

for some constant $D_6 > 0$. Therefore, (u_k) is bounded in $W_T^{1,p(t)}$, which completes the proof. ∎

Remark 3.4. Theorem 3.4 generalizes Theorem 1.1 in [7].

Corollary 3.1. *Assume that $F(t,x)$ satisfies the assumption (A) and*

$$F(t,x) \to +\infty, \quad \text{as} \quad |x| \to \infty$$

uniformly for a.e. $t \in [0,T]$, then system (1.1) has at least one solution which minimizes φ on $W_T^{1,p(t)}$.

We consider system (1.1) with γ-quasisubadditive potential and obtain two existence theorems of periodic solutions for system (1.1) under suitable conditions.

Theorem 3.5. *Assume that $F = F_1 + F_2$, where F_1 and F_2 satisfy the condition (A) and $F_1(t,\cdot)$ is subadditive for a.e. $t \in [0,T]$ and there exists $g_2(t) \in L^1(0,T,R^+)$ such that*

$$|\nabla F_2(t,x)| \leq g_2(t)$$

for all a.e. $t \in [0,T]$ and $x \in R^N$, and if

$$\int_0^T F(t,x)dt \to +\infty \quad \text{as} \quad |x| \to \infty, \tag{3.8}$$

then system (1.1) has at least one solution which minimizes φ on $W_T^{1,p(t)}$.

Proof. By the subadditivity and assumption (A), we obtain

$$F_1(t,x) \leq ([|x|]+1)F_1(t,x/([|x|]+1)) \leq (|x|+1)ab(t)$$

for a.e. $t \in [0,T]$ and all $x \in R^N$, where $[\cdot]$ is the Gaussian function and

$$a = \max_{0 \leq s \leq 1} a(s).$$

Now write $u = \bar{u} + \tilde{u}$, by Lemma 2.6, we get

$$\begin{aligned}
\varphi(u) &= \int_0^T \frac{1}{p(t)}|\dot{u}(t)|^{p(t)}dt + \int_0^T F_1(t,u(t))dt + \int_0^T F_2(t,u(t))dt. \\
&\geq \int_0^T \frac{1}{p(t)}|\dot{u}(t)|^{p(t)}dt + \int_0^T F_1(t,\bar{u})dt - \int_0^T F_1(t,-\tilde{u}(t))dt \\
&\quad + \int_0^T F_2(t,\bar{u})dt + \int_0^T \int_0^1 (\nabla F_2(t,\bar{u}+s\tilde{u}(t)),\tilde{u}(t))dsdt \\
&\geq \int_0^T \frac{1}{p(t)}|\dot{u}(t)|^{p(t)}dt + \int_0^T F(t,\bar{u})dt - a\int_0^T b(t)dt(\|\tilde{u}\|_\infty + 1)
\end{aligned}$$

$$-\int_0^T g_2(t)dt\|\tilde{u}\|_\infty$$

$$\geq (1/p^+)\int_0^T |\dot{u}(t)|^{p(t)}dt - E_0\left(\int_0^T |\dot{u}(t)|^{p(t)}dt\right)^{1/p^-} + \int_0^T F(t,\bar{u})dt - E_1,$$

for some constants $E_0 > 0$, $E_1 > 0$. It follows from the Proposition 2.1 and (3.8) that

$$\varphi(u) \to +\infty \quad \text{as} \quad |u\| \to \infty,$$

and hence every minimizing sequence is bounded, which completes the proof. ∎

Remark 3.5. Theorem 3.5 generalizes Theorem 1 in [7].

Theorem 3.6. *Assume that $F = F_1 + F_2$, where F_1 and F_2 satisfy the condition (A) and $F_1(t,\cdot)$ is γ-quasisubadditive with $\gamma > 1$ for a.e. $t \in [0,T]$ and there exists $E_2 \in R$ and $g_3(t) \in L^1(0,T,R^+)$ such that*

$$|\nabla F_2(t,x)| \leq g_3(t)$$

for a.e. $t \in [0,T]$ and all $x \in R^N$, and

$$\int_0^T F_2(t,x)dt \geq E_2$$

for all $x \in R^N$, and if

$$\int_0^T F(t,x)dt \to +\infty \quad as \quad |x| \to \infty, \tag{3.9}$$

then system (1.1) *has at least one solution which minimizes φ on $W_T^{1,p(t)}$.*

Proof. By the γ-quasisubadditivity of $F_1(t,\cdot)$ with $\gamma > 1$ and the assumption (A), we have

$$\begin{aligned} F_1(t,x) &\geq (\gamma/(1-\gamma))F_1(t,0) \\ &\geq (\gamma/(1-\gamma))a(0)b(t) \end{aligned}$$

for all $x \in R^N$ and a.e. $t \in [0,T]$. Let (u_k) be a minimizing sequence of φ. By Lemma 2.6, we obtain

$$\begin{aligned} \varphi(u_k) &= \int_0^T \frac{1}{p(t)}|\dot{u}_k(t)|^{p(t)}dt + (a(0)\gamma/(1-\gamma))\int_0^T b(t)dt \\ &\quad + \int_0^T F_2(t,\bar{u}_k)dt + \int_0^T\int_0^1 (\nabla F_2(t,\bar{u}_k + s\tilde{u}_k(t),\tilde{u}_k(t))dsdt \\ &\geq \int_0^T \frac{1}{p(t)}|\dot{u}_k(t)|^{p(t)}dt - \int_0^T g_3(t)dt\|\tilde{u}_k\|_\infty + E_2 - E_3 \\ &\geq \int_0^T \frac{1}{p(t)}|\dot{u}_k(t)|^{p(t)}dt - E_4\left(\int_0^T |\dot{u}_k(t)|^{p(t)}dt\right)^{1/p^-} - E_5 \\ &\geq (1/p^+)\int_0^T |\dot{u}_k(t)|^{p(t)}dt - E_4\left(\int_0^T |\dot{u}_k(t)|^{p(t)}dt\right)^{1/p^-} - E_5, \end{aligned}$$

for some positive constants E_3, E_4, E_5. Hence, we have

$$\int_0^T |\dot{u}_k(t)|^{p(t)} dt \le E_6.$$

for some positive constant E_6 by Lemma 2.6, there exists $E_7 > 0$, such that

$$\|\tilde{u}_k\|_\infty \le E_7.$$

Moreover, we have

$$\begin{aligned}
\varphi(u_k) &= (1/\gamma)\int_0^T F_1(t,\bar{u}_k)dt - \int_0^T F_1(t,-\tilde{u}_k)dt + \int_0^T F_2(t,\bar{u}_k)dt \\
&\quad + \int_0^T \int_0^1 (\nabla F_2(t,\bar{u}_k + s\tilde{u}_k(t)), \tilde{u}_k(t))dsdt \\
&\ge (1/\gamma)\int_0^T F_1(t,\bar{u}_k)dt - (\max_{0\le s\le E_7} a(s))\int_0^T b(t)dt \\
&\quad + \left(\int_0^T F_2(t,\bar{u}_k)dt - E_2\right) + E_2 - \int_0^T g_3(t)dt\|\tilde{u}_k\|_\infty \\
&\ge (1/\gamma)\int_0^T F(t,\bar{u}_k)dt - E_8,
\end{aligned}$$

for some positive constant E_8. Hence, one obtains

$$|\bar{u}_k| \le E_9,$$

for some positive constant E_9 by (3.9). Therefore (u_k) is bounded in $W_T^{1,p(t)}$, and hence every minimizing sequence is bounded, which completes the proof. ∎

Remark 3.6. Theorem 3.6 generalizes Theorem 2 in [7].

Corollary 3.2. *Assume that F satisfies the assumption (A) and $F(t,\cdot)$ is γ-quasi-subadditive with $\gamma \ge 1$ for a.e. $t \in [0,T]$, and if*

$$\int_0^T F(t,x)dt \to +\infty \quad as \quad |x| \to \infty,$$

then system (1.1) has at least one solution which minimizes φ on $W_T^{1,p(t)}$.

Acknowledgement. The authors express sincere thanks to Prof. Xianhua Tang for his many years of instructions in critical point theory and Hamiltonian system. We also thank the anonymous referees for invaluable suggestions which led to the improvement of our article.

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