# A $q$-Analogue of the Meyer-König and Zeller Operators 

Nazim Mahmudov and Pembe Sabancigil<br>Department of Mathematics, Eastern Mediterranean University, Gazimagusa, Mersin 10 Turkey<br>nazim.mahmudov@emu.edu.tr, pembe.sabancigil@emu.edu.tr


#### Abstract

In this paper we introduce a new $q$-analogue of the Meyer-König and Zeller operators ( $M_{n, q}(f ; x)$ ). We estimate the rate of convergence of $M_{n, q}(f ; x)$ by the first and the second modulus of continuity.


2010 Mathematics Subject Classification: 41A35, 41A25, 41A36
Keywords and phrases: Meyer-Konig and Zeller operators, rate of convergence, $q$-calculus.

## 1. Introduction

Philips [24] proposed a new generalization of Bernstein polynomials by

$$
B_{n, q}(f ; x)=\sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right)\left[\begin{array}{c}
n \\
k
\end{array}\right] x^{k} \prod_{s=0}^{n-k-1}\left(1-q^{s} x\right)
$$

where $\left[\begin{array}{l}n \\ k\end{array}\right]$ is defined by (2.1). These were studied widely by a number of authors [2, 3, 5-26]. In [16], Lupaş proposed the $q$-Bernstein polynomials: for each positive integer $n$, and $f \in C[0,1]$, the $q$-Bernstein polynomial of $f$ is

$$
R_{n, q}(f ; x)=\sum_{k=0}^{\infty} f\left(\frac{[k]}{[n]}\right)\left[\begin{array}{c}
n \\
k
\end{array}\right] \frac{q^{\frac{k(k-1)}{2}} x^{k}(1-x)^{n-k}}{\prod_{j=0}^{n-1}\left(1-x+q^{j} x\right)}
$$

The polynomial $B_{n, q}(f ; x)$ and the rational function $R_{n, q}(f ; x)$ have much in common. They reduce to the Bernstein polynomials when we put $q=1$ and share the shape-preserving properties of the Bernstein polynomials when $0<q<1$. Lupaş [16] investigated approximating and shape-preserving properties of $R_{n, q}(f ; x)$. For $q \neq 1$, the operators $R_{n, q}(f ; x)$ are rational functions rather than polynomials.

[^0]In [22], Ostrovska proved that for each $f \in C[0,1]$ and $q \in(0,1)$, the sequence $\left\{R_{n, q}(f ; x)\right\}_{n \geq 1}$ converges uniformly to $R_{\infty, q}(f ; x)$ on $[0,1]$ as $n \rightarrow \infty$, where

$$
R_{\infty, q}(f ; x)= \begin{cases}\sum_{k=0}^{\infty} f\left(1-q^{k}\right) r_{\infty, k}(q ; x), & x \in[0,1), \\ f(1), & x=1,\end{cases}
$$

and

$$
r_{\infty, k}(q ; x):=\frac{q^{\frac{k(k-1)}{2}}(x /(1-x))^{k}}{(1-q)^{k}[k]!\prod_{j=0}^{\infty}\left(1+q^{j} x /(1-x)\right)} .
$$

Cheney and Sharma [3] introduced a slight modification of the MKZ operators [20] and called the operators Bernstein power series. On the other hand, in [26], Trif introduced the $q$-Meyer-König and Zeller operators (or the $q$-MKZ operators for simplicity): for each positive integer $n$, and $f \in C[0,1]$,

$$
\mathbb{M}_{n, q}(f ; x):= \begin{cases}(1-x)_{q}^{n+1} \sum_{k=0}^{\infty} f\left(\frac{[k]}{[n+k]}\right)\left[\begin{array}{c}
n+k \\
k
\end{array}\right] x^{k}, & 0 \leq x<1 \\
f(1), & x=1\end{cases}
$$

Doğru and Duman [6] introduced a new generalization of $q$-MKZ operators and studied statistical approximation properties of such operators. A $q$-generalization of Meyer-König and Zeller operators in several variables was studied by Aktuglu, Ozarslan and Duman [1]. In [13], Wang investigated properties of convergence for the $q$-MKZ operators $\mathbb{M}_{n, q}$. He also gave explicit formulas of Voronovskaya type for the $q$-MKZ operators $\mathbb{M}_{n, q}$ for fixed $0<q<1$. In [10], Govil and Gupta introduced a new type of $q$-integrated Meyer-König-Zeller-Durrmeyer operators, obtained moments for the these operators and estimated the convergence of these integrated $q$-MKZD operators.

Motivating by the work [16], we introduce a new $q$-analogue of the MKZ operators.
Definition 1.1. The linear operator $M_{n, q}: C[0,1] \rightarrow C[0,1]$ defined by
$M_{n, q}(f ; x):= \begin{cases}(1-x)^{n+1} \sum_{k=0}^{\infty} f\left(\frac{[k]}{[n+k]}\right)\left[\begin{array}{c}n+k \\ k\end{array}\right] \frac{q^{\frac{k(k-1)}{2}} x^{k}}{\prod_{j=0}^{n+k}\left(1-x+q^{j} x\right)}, & 0 \leq x<1, \\ f(1), & x=1 .\end{cases}$
is called the q-analoque of the Meyer-König and Zeller operator.
New $q$-MKZ operators $M_{n, q}(f ; x)$ have an advantage of generating positive linear operators for all $q>0$, whereas $q$-MKZ operators $\mathbb{M}_{n, q}(f ; x)$ generate positive linear operators only if $0<q<1$. In this paper, we study the rate of convergence of the new $q$-MKZ operators $M_{n, q}(f ; x)$. We obtain the estimates for the rate of convergence of $M_{n, q}(f ; x)$ by the modulus of continuity of $f$, and the estimates are sharp in the sense of order for Lipschitz continuous functions. Our results demonstrate that the estimates for the rate of convergence of the new $q$-MKZ operators $M_{n, q}(f ; x)$ are essentially different from those for the classical MKZ operators, however they are very similar to those for the $q$-MKZ operators $\mathbb{M}_{n, q}(f ; x)$ in the case $0<q<1$.

## 2. Some auxilary results

Before introducing the operators, we mention some basic definitions of $q$ calculus.
Let $q>0$. For any $n \in N \cup\{0\}$, the $q$-integer $[n]=[n]_{q}$ is defined by

$$
[n]:=1+q+\ldots+q^{n-1}, \quad[0]:=0
$$

and the $q$-factorial $[n]!=[n]_{q}!$ by

$$
[n]!:=[1][2] \ldots[n], \quad[0]!:=1 .
$$

For integers $0 \leq k \leq n$, the $q$-binomial is defined by

$$
\left[\begin{array}{c}
n  \tag{2.1}\\
k
\end{array}\right]:=\frac{[n]!}{[k]![n-k]!} .
$$

Also, we use the following standard notations:

$$
\begin{aligned}
(1-z)_{q}^{n} & :=\prod_{j=0}^{n-1}\left(1-q^{j} z\right), \quad(1-z)_{q}^{\infty}:=\prod_{j=0}^{\infty}\left(1-q^{j} z\right), \\
m_{n, k}(q ; x) & :=\left[\begin{array}{c}
n+k \\
k
\end{array}\right] \frac{q^{k(k-1) / 2}(x /(1-x))^{k}}{\prod_{j=0}^{n+k}\left(1+q^{j}(x /(1-x))\right)}, \\
m_{\infty, k}(q ; x) & :=r_{\infty, k}(q ; x)=\frac{q^{\frac{k(k-1)}{2}}(x /(1-x))^{k}}{(1-q)^{k}[k]!\prod_{j=0}^{\infty}\left(1+q^{j} x /(1-x)\right)} .
\end{aligned}
$$

It will be convenient to use for $x \in[0,1)$ the substitution

$$
u:=\frac{x}{1-x}, \quad u \in[0, \infty)
$$

We may express $m_{n, k}(q ; x), m_{\infty, k}(q ; x)$ for $x \in[0,1)$ as follows:

$$
\begin{aligned}
& m_{n, k}(q ; x)=\left[\begin{array}{c}
n+k \\
k
\end{array}\right] \frac{q^{k(k-1) / 2} u^{k}}{(1+u)_{q}^{n+k+1}}=: b_{n, k}(q ; u), \\
& m_{\infty, k}(q ; x)=\frac{q^{\frac{k(k-1)}{2}} u^{k}}{(1-q)^{k}[k]!(1+u)_{q}^{\infty}}=: b_{\infty, k}(q ; u) .
\end{aligned}
$$

Clearly

$$
b_{n, k}(q ; u)=m_{n, k}\left(q ; \frac{u}{1+u}\right)
$$

and

$$
M_{n, q}(f ; x)=M_{n, q}\left(f ; \frac{u}{1+u}\right)=\sum_{k=0}^{\infty} f\left(\frac{[k]}{[n+k]}\right) b_{n, k}(q ; u) .
$$

For $f \in C[0,1], t>0$, the modulus of continuity $\omega(f ; t)$ and the second modulus of smoothness $\omega_{2}(f ; t)$ of $f$ are defined by

$$
\begin{aligned}
\omega(f ; t) & =\sup _{|x-y| \leq t}|f(x)-f(y)| \\
\omega_{2}(f ; t) & =\sup _{0 \leq h \leq t} \sup _{0 \leq x \leq 1-2 h}|f(x+2 h)-2 f(x+h)+f(x)| .
\end{aligned}
$$

Lemma 2.1. The following are true:

$$
\begin{align*}
M_{n, q}(1 ; x) & =1, \quad M_{n, q}(t ; x)=x,  \tag{2.2}\\
0 & \leq M_{n, q}\left(t^{2} ; x\right)-x^{2} \leq \frac{1}{[n+1]} x+\frac{q^{2} x^{2}}{1-x+q x}-x^{2} \\
& \leq \frac{1}{[n+1]} x-\frac{x^{2}(1-x)(1-q)}{1-x+q x},  \tag{2.3}\\
M_{\infty, q}\left(t^{2} ; x\right) & =(1-q) x+\frac{q^{2} x^{2}}{1-x+q x} . \tag{2.4}
\end{align*}
$$

Proof. First we prove that $M_{n, q}(1 ; x)=\sum_{k=0}^{\infty} m_{n, k}(q ; x)=1$. Indeed if $g_{n}(u)=$ $1 /(1+u)_{q}^{n+1}$, then by the $q$-Taylor formula

$$
\begin{aligned}
1=g_{n}(0) & =\sum_{k=0}^{\infty} \frac{(-1)^{k} u^{k} q^{k(k-1) / 2}}{[k]!} D_{q}^{(k)} g_{n}(u) \\
& =\sum_{k=0}^{\infty} \frac{(-1)^{k} u^{k} q^{k(k-1) / 2}}{[k]!} \frac{(-1)^{k}[n+1] \ldots[n+k]}{(1+u)_{q}^{n+k+1}} \\
& =\sum_{k=0}^{\infty}\left[\begin{array}{c}
n+k \\
k
\end{array}\right] \frac{q^{k(k-1) / 2} u^{k}}{(1+u)_{q}^{n+k+1}}=\sum_{k=0}^{\infty} b_{n, k}(q ; u)=\sum_{k=0}^{\infty} m_{n, k}(q ; x) .
\end{aligned}
$$

By using the above identity, the second equality of (2.2) folllows:

$$
\begin{aligned}
M_{n, q}(t ; x) & =M_{n, q}\left(t ; \frac{u}{1+u}\right)=\sum_{k=0}^{\infty} \frac{[k]}{[n+k]}\left[\begin{array}{c}
n+k \\
k
\end{array}\right] \frac{q^{k(k-1) / 2} u^{k}}{(1+u)_{q}^{n+k+1}} \\
& =\frac{u}{1+u} \sum_{k=1}^{\infty}\left[\begin{array}{c}
n+k-1 \\
k-1
\end{array}\right] \frac{q^{(k-1)(k-2) / 2}(q u)^{k-1}}{(1+q u)_{q}^{n+k}} \\
& =\frac{u}{1+u} \sum_{k=0}^{\infty} b_{n, k}(q ; q u)=\frac{u}{1+u}=x .
\end{aligned}
$$

Now we estimate $M_{n, q}\left(t^{2} ; x\right)-x^{2}$. We find an upper estimate for $M_{n, q}\left(t^{2} ; x\right)$ :

$$
\begin{aligned}
M_{n, q}\left(t^{2} ; x\right)= & M_{n, q}\left(t^{2} ; \frac{u}{1+u}\right)=\sum_{k=0}^{\infty} \frac{[k]^{2}}{[n+k]^{2}}\left[\begin{array}{c}
n+k \\
k
\end{array}\right] \frac{q^{k(k-1) / 2} u^{k}}{(1+u)_{q}^{n+k+1}} \\
= & \sum_{k=0}^{\infty} \frac{[k]}{[n+k]}\left[\begin{array}{c}
n+k-1 \\
k-1
\end{array}\right] \frac{q^{k(k-1) / 2} u^{k}}{(1+u)_{q}^{n+k+1}} \\
= & \sum_{k=1}^{\infty} \frac{1}{[n+k]}\left[\begin{array}{c}
n+k-1 \\
k-1
\end{array}\right] \frac{q^{k(k-1) / 2} u^{k}}{(1+u)_{q}^{n+k+1}} \\
& +\sum_{k=2}^{\infty} \frac{[k]-1}{[n+k]}[n+k-1] \frac{[n+k-2]!!}{[k-1]![n]!} \frac{q^{k(k-1) / 2} u^{k}}{(1+u)_{q}^{n+k+1}}=: I_{1}+I_{2}
\end{aligned}
$$

It is clear that

$$
I_{1} \leq \frac{1}{[n+1]} \frac{u}{1+u} \sum_{k=0}^{\infty}\left[\begin{array}{c}
n+k \\
k
\end{array}\right] \frac{q^{k(k-1) / 2}(q u)^{k}}{(1+q u)_{q}^{n+k+1}}=\frac{1}{[n+1]} \frac{u}{1+u}=\frac{x}{[n+1]}
$$

and

$$
\begin{aligned}
I_{2} & =q \sum_{k=2}^{\infty} \frac{[n+k-1]}{[n+k]} \frac{[n+k-2]!}{[k-2]![n]!} \frac{q^{k(k-1) / 2} u^{k}}{(1+u)_{q}^{n+k+1}} \\
& =q \sum_{k=2}^{\infty}\left(\frac{[n+k]-q^{n+k-1}}{[n+k]}\right)\left[\begin{array}{c}
n+k-2 \\
k-2
\end{array}\right] \frac{q^{k(k-1) / 2} u^{k}}{(1+u)_{q}^{n+k+1}} \\
& \leq q \sum_{k=0}^{\infty}\left[\begin{array}{c}
n+k \\
k
\end{array}\right] \frac{q^{k(k-1) / 2} q^{2 k+1} u^{k+2}}{(1+u)_{q}^{n+k+3}} \\
& =q \frac{q u^{2}}{(1+u)(1+q u)} \sum_{k=0}^{\infty}\left[\begin{array}{c}
n+k \\
k
\end{array}\right] \frac{q^{k(k-1) / 2}\left(q^{2} u\right)^{k}}{\left(1+q^{2} u\right)_{q}^{n+k+1}} \\
& =\frac{q^{2} u^{2}}{(1+u)(1+q u)}=\frac{q^{2} x^{2}}{1-x+q x} .
\end{aligned}
$$

Thus

$$
M_{n, q}\left(t^{2} ; x\right)=I_{1}+I_{2} \leq \frac{x}{[n+1]}+\frac{q^{2} x^{2}}{1-x+q x}
$$

From [5, p. 281] we know that if a positive linear operator $L$ on $C[0,1]$ reproduces linear functions, then $L(f ; x) \geq f(x)$ for any convex function $f$ and for any $x \in[0,1]$. So $M_{n, q}\left(t^{2} ; x\right)-x^{2} \geq 0$.

Finally we prove (2.4):

$$
\begin{aligned}
& M_{\infty, q}\left(t^{2} ; x\right) \\
& =M_{\infty, q}\left(t^{2} ; \frac{u}{1+u}\right)=\sum_{k=0}^{\infty}\left(1-q^{k}\right)^{2} b_{\infty, k}(q ; u) \\
& =\sum_{k=0}^{\infty}(1-q)^{2}[k]^{2} \frac{q^{k(k-1) / 2} u^{k}}{(1-q)^{k}[k]!(1+u)_{q}^{\infty}} \\
& =\sum_{k=0}^{\infty} \frac{[k](q[k-1]+1) q^{k(k-1) / 2} u^{k}}{(1-q)^{k-2}[k]!(1+u)_{q}^{\infty}} \\
& =\frac{1-q}{(1+u)_{q}^{\infty}} \sum_{k=1}^{\infty} \frac{q^{k(k-1) / 2} u^{k}}{(1-q)^{k-1}[k-1]!}+\frac{q}{(1+u)_{q}^{\infty}} \sum_{k=2}^{\infty} \frac{q^{k(k-1) / 2} u^{k}}{(1-q)^{k-2}[k-2]!} \\
& =(1-q) \frac{u}{1+u} \sum_{k=0}^{\infty} b_{\infty, k}(q ; q u)+\frac{q^{2} u^{2}}{(1+u)(1+q u)} \sum_{k=0}^{\infty} b_{\infty, k}\left(q ; q^{2} u\right) \\
& =(1-q) \frac{u}{1+u}+\frac{q^{2} u^{2}}{(1+u)(1+q u)}=(1-q) x+\frac{q^{2} x^{2}}{1-x+q x} .
\end{aligned}
$$

Lemma 2.2. Let $0<q<1, k \geq 0, n \geq 1$. For any $x \in[0,1)$ we have

$$
\left|m_{n, k}(q ; x)-m_{\infty, k}(q ; x)\right| \leq \frac{x}{1-x} \frac{q^{n+k+1}}{1-q} m_{n, k}(q ; x)+\frac{q^{n+1}}{1-q} m_{\infty, k}(q ; x)
$$

Proof. Standard computations show that

$$
\begin{align*}
& \left|m_{n, k}(q ; x)-m_{\infty, k}(q ; x)\right|=\left|b_{n, k}(q ; u)-b_{\infty, k}(q ; u)\right| \\
& =\left\lvert\,\left[\begin{array}{c}
n+k \\
k
\end{array}\right] \prod_{j=0}^{k-1} \frac{q^{j} u}{1+q^{j} u} \prod_{j=0}^{n}\left(1-\frac{q^{k+j} u}{1+q^{k+j} u}\right)\right. \\
& \left.\quad-\frac{1}{(1-q)^{k}[k]!} \prod_{j=0}^{k-1} \frac{q^{j} u}{1+q^{j} u} \prod_{j=0}^{\infty}\left(1-\frac{q^{k+j} u}{1+q^{k+j} u}\right) \right\rvert\, \\
& =\left\lvert\,\left[\begin{array}{c}
n+k \\
k
\end{array}\right] \prod_{j=0}^{k-1} \frac{q^{j} u}{1+q^{j} u}\left(\prod_{j=0}^{n}\left(1-\frac{q^{k+j} u}{1+q^{k+j} u}\right)-\prod_{j=0}^{\infty}\left(1-\frac{q^{k+j} u}{1+q^{k+j} u}\right)\right)\right. \\
& \left.\quad+\prod_{j=0}^{k-1} \frac{q^{j} u}{1+q^{j} u} \prod_{j=0}^{\infty}\left(1-\frac{q^{k+j} u}{1+q^{k+j} u}\right)\left(\left[\begin{array}{c}
n+k \\
k
\end{array}\right]-\frac{1}{(1-q)^{k}[k]!}\right) \right\rvert\, \\
& \leq b_{n, k}(q ; u)\left|1-\prod_{j=n+k+1}^{\infty}\left(1-\frac{q^{j} u}{1+q^{j} u}\right)\right|+b_{\infty, k}(q ; u)\left|\prod_{j=n+1}^{n+k}\left(1-q^{j}\right)-1\right| . \tag{2.5}
\end{align*}
$$

Now using the inequality

$$
1-\prod_{j=1}^{k}\left(1-a_{j}\right) \leq \sum_{j=1}^{k} a_{j},\left(a_{1}, a_{2}, \ldots, a_{k} \in(0,1), k=1,2, \ldots, \infty\right)
$$

we get from (2.5) that

$$
\begin{aligned}
\left|b_{n, k}(q ; u)-b_{\infty, k}(q ; u)\right| & \leq b_{n, k}(q ; u) \sum_{j=n+k+1}^{\infty} \frac{q^{j} u}{1+q^{j} u}+b_{\infty, k}(q ; u) \sum_{j=n+1}^{n+k} q^{j} \\
& \leq b_{n, k}(q ; u) \frac{u q^{n+k+1}}{1-q}+b_{\infty, k}(q ; u) \frac{q^{n+1}}{1-q}
\end{aligned}
$$

Lemma 2.3. With the definitions of $m_{n, k}(q ; x)$ and $m_{\infty, k}(q ; x)$, we have

$$
\sum_{k=0}^{\infty} q^{k} m_{n, k}(q ; x) \leq 1-x+q^{n} x, \quad \sum_{k=0}^{\infty} q^{k} m_{\infty, k}(q ; x)=1-x .
$$

Proof. Using (2.2), we get

$$
\begin{aligned}
\sum_{k=0}^{\infty} q^{k} m_{n, k}(q ; x) & =\sum_{k=0}^{\infty}\left(q^{n+k}-1\right) \frac{q^{k}-1}{q^{n+k}-1} m_{n, k}(q ; x)+\sum_{k=0}^{\infty} m_{n, k}(q ; x) \\
& \leq\left(q^{n}-1\right) \sum_{k=0}^{\infty} \frac{[k]}{[n+k]} m_{n, k}(q ; x)+\sum_{k=0}^{\infty} m_{n, k}(q ; x) \\
& =\left(q^{n}-1\right) M_{n, q}(t ; x)+1=1-x+q^{n} x .
\end{aligned}
$$

The second identity can be done in a similar way.
Theorem 2.1. If $f:[0,1] \rightarrow R$ is a convex function, then the sequence $\left\{M_{n, q}(f\right.$;
$x)\}_{n \geq 1}$ is nonincreasing in $n$ for each $q>0$ and $x \in[0,1]$.

Proof. One can easily obtain

$$
\begin{aligned}
& M_{n, q}(f ; x)-M_{n+1, q}(f ; x) \\
&= M_{n, q}\left(f ; \frac{u}{1+u}\right)-M_{n+1, q}\left(f ; \frac{u}{1+u}\right) \\
&= \sum_{k=0}^{\infty} f\left(\frac{[k]}{[n+k]}\right)\left[\begin{array}{c}
n+k \\
k
\end{array}\right] \frac{q^{k(k-1) / 2} u^{k}}{(1+u)_{q}^{n+k+1}} \\
&-\sum_{k=0}^{\infty} f\left(\frac{[k]}{[n+k+1]}\right)\left[\begin{array}{c}
n+k+1 \\
k
\end{array}\right] \frac{q^{k(k-1) / 2} u^{k}}{(1+u)_{q}^{n+k+1}}\left(1-\frac{q^{n+k+1} u}{1+q^{n+k+1} u}\right) \\
&= \sum_{k=0}^{\infty} f\left(\frac{[k]}{[n+k]}\right)\left[\begin{array}{c}
n+k \\
k
\end{array}\right] \frac{q^{k(k-1) / 2} u^{k}}{(1+u)_{q}^{n+k+1}} \\
&-\sum_{k=0}^{\infty} f\left(\frac{[k]}{[n+k+1]}\right)\left[\begin{array}{c}
n+k+1 \\
k
\end{array}\right] \frac{q^{k(k-1) / 2} u^{k}}{(1+u)_{q}^{n+k+1}} \\
&+\sum_{k=1}^{\infty} q^{n+1} f\left(\frac{[k-1]}{[n+k]}\right)\left[\begin{array}{c}
n+k \\
k-1
\end{array}\right] \frac{q^{k(k-1) / 2} u^{k}}{(1+u)_{q}^{n+k+1}} \\
&= \sum_{k=1}^{\infty}\left(\frac{[n+1]}{[n+k+1]} f\left(\frac{[k]}{[n+k]}\right)+q^{n+1} \frac{[k]}{[n+k+1]} f\left(\frac{[k-1]}{[n+k]}\right)-f\left(\frac{[k]}{[n+k+1]}\right)\right) \\
& \times\left[\begin{array}{c}
n+k+1 \\
k
\end{array}\right] \frac{q^{k(k-1) / 2} u^{k}}{(1+u)_{q}^{n+k+1}}
\end{aligned}
$$

By choosing

$$
\begin{aligned}
\alpha & =\frac{[n+1]}{[n+k+1]}, \quad \beta=q^{n+1} \frac{[k]}{[n+k+1]}, \quad \alpha+\beta=1, \\
x_{1} & =\frac{[k]}{[n+k]}, \quad x_{2}=\frac{[k-1]}{[n+k]}
\end{aligned}
$$

we can write

$$
\begin{aligned}
& M_{n, q}(f ; x)-M_{n+1, q}(f ; x) \\
& =\sum_{k=1}^{\infty}\left(\alpha f\left(x_{1}\right)+\beta f\left(x_{2}\right)-f\left(\alpha x_{1}+\beta x_{2}\right)\right)\left[\begin{array}{c}
n+k+1 \\
k
\end{array}\right] \frac{q^{k(k-1) / 2} u^{k}}{(1+u)_{q}^{n+k+1}} .
\end{aligned}
$$

Because of convexity of $f$, we can say that $\left\{M_{n, q}(f ; x)\right\}_{n \geq 1}$ is nonincreasing in $n$.

## 3. Main results

In this section we will discuss approximating properties of the new $q$-MKZ operators. From the definition of the new $q$-MKZ operators $M_{n, q}$ we know that $M_{n, q}$ are positive linear operators for all $q>0$. Hence the moments $M_{n, q}\left(t^{r} ; x\right)(r=0,1,2)$ are of particular importance by the theory of approximation by positive operators. Based on the formulas for $M_{n, q}\left(t^{r} ; x\right)(r=0,1,2)$ we have the following approximation theorem.

Theorem 3.1. Let $0<q_{n}<1$. Then the sequence $\left\{M_{n, q_{n}}(f)\right\}_{n \geq 1}$ converges to $f$ uniformly on $[0,1]$ for each $f \in C[0,1]$ if $\lim _{n \rightarrow \infty} q_{n}=1$.
Proof. Since $M_{n, q_{n}}(f ; x)$, define positive linear operators, the Korovkin theorem implies that $M_{n, q_{n}}(f ; x) \rightrightarrows f(x)$ for any $f \in C[0,1]$ if and only if $M_{n, q_{n}}\left(t^{m} ; x\right) \rightrightarrows x^{m}$ for $x \in[0,1]$ and $m=0,1,2$. For $m=0,1$ this is true for any sequence $\left\{q_{n}\right\}$ due to (2.2).

Suppose that $q_{n} \rightarrow 1$. Then, for any fixed positive integer $k$, we have $[n]_{q_{n}} \geq[k]_{q_{n}}$ when $n \geq k$. Therefore, $\liminf _{n \rightarrow \infty}[n]_{q_{n}} \geq \lim _{n \rightarrow \infty}[k]_{q_{n}}=k$. Since $k$ was chosen arbitrarily, it follows that $[n]_{q_{n}} \rightarrow \infty$. Hence, $\frac{x}{[n+1]_{q_{n}}} \rightrightarrows 0$ for $x \in[0,1]$. At the same time, for $q_{n} \geq 1 / 2$, we have

$$
0 \geq-\frac{x^{2}(1-x)\left(1-q_{n}\right)}{1-x+q_{n} x} \geq-\frac{1}{4} \frac{1-q_{n}}{1-x+q_{n} x} \geq-\frac{1}{4} \frac{1-q_{n}}{1-x / 2} \geq-\frac{1}{2}\left(1-q_{n}\right)
$$

for all $x \in[0,1]$. Therefore,

$$
\frac{x}{[n+1]_{q_{n}}}-\frac{x^{2}(1-x)\left(1-q_{n}\right)}{1-x+q_{n} x} \rightrightarrows 0
$$

for $x \in[0,1]$. It follows from (2.3) that $M_{n, q_{n}}\left(t^{2} ; x\right) \rightrightarrows x^{2}$ for $x \in[0,1]$.
Theorem 3.1 implies that if $0<q<1$ is fixed, $\left\{M_{n, q}(f)\right\}_{n>1}$ may not be approximating for some continuous functions. We will discuss convergence properties for the new $q$-MKZ operators for fixed $0<q<1$. It should be mentioned that the new limit $q$ - MKZ operators, $M_{\infty, q}(f)$, are exactly the same with the limit $q$-Lupas operators, $R_{\infty, q}(f)$, introduced by Ostrovska in [22], namely

$$
M_{\infty, q}(f ; x):= \begin{cases}\sum_{k=0}^{\infty} f\left(1-q^{k}\right) m_{\infty, k}(q ; x), & x \in[0,1) \\ f(1), & x=1\end{cases}
$$

Theorem 3.2. Let $0<q<1$ and $f \in C[0,1]$. Then

$$
\begin{equation*}
\left\|M_{n, q}(f)-M_{\infty, q}(f)\right\| \leq C_{q} \omega\left(f ; q^{n}\right) \tag{3.1}
\end{equation*}
$$

where $C_{q}=\max \{5,2+3 q /(1-q)\}$. The above estimate is sharp in the following sense of order: for each $\alpha, 0<\alpha \leq 1$, there exists a function $f_{\alpha}(x)$ which belongs to the Lipschitz class Lip $\alpha:=\left\{f \in C[0,1]: \omega(f ; t) \ll t^{\alpha}\right\}$ such that

$$
\left\|M_{n, q}(f)-M_{\infty, q}(f)\right\| \gg q^{\alpha n}
$$

Proof. Consider

$$
\Delta(x):=M_{n, q}(f ; x)-M_{\infty, q}(f ; x)
$$

Since $M_{n, q}(f ; x)$ and $M_{\infty, q}(f ; x)$ possess the end point interpolation property, $\Delta(0)=$ $\Delta(1)=0$. For all $x \in(0,1)$ we rewrite $\Delta$ in the following form

$$
\begin{aligned}
\Delta(x)= & \sum_{k=0}^{\infty}\left[f\left(\frac{[k]}{[n+k]}\right)-f\left(1-q^{k}\right)\right] m_{n, k}(q ; x) \\
& +\sum_{k=0}^{\infty}\left[f\left(1-q^{k}\right)-f(1)\right]\left(m_{n, k}(q ; x)-m_{\infty, k}(q ; x)\right)=: I_{1}+I_{2} .
\end{aligned}
$$

We start with estimation of $I_{1}$. Since

$$
\frac{[k]}{[n+k]}-\left(1-q^{k}\right)=\frac{1-q^{k}}{1-q^{n+k}}-\left(1-q^{k}\right)=q^{n+k} \frac{1-q^{k}}{1-q^{n+k}} \leq q^{n+k} \leq q^{n}
$$

we get

$$
\begin{equation*}
\left|I_{1}\right| \leq \omega\left(f ; q^{n}\right) \sum_{k=0}^{\infty} m_{n, k}(q ; x)=\omega\left(f ; q^{n}\right) . \tag{3.2}
\end{equation*}
$$

Then we estimate $I_{2}$. Using the property of the modulus of continuity

$$
\omega(f ; \lambda t) \leq(1+\lambda) \omega(f ; t), \quad \lambda>0
$$

and Lemma 2.2, we get

$$
\begin{aligned}
\left|I_{2}\right| & \leq \sum_{k=0}^{\infty} \omega\left(f ; q^{k}\right)\left|m_{n, k}(q ; x)-m_{\infty, k}(q ; x)\right| \\
& \leq \omega\left(f ; q^{n}\right) \sum_{k=0}^{\infty}\left(1+q^{k-n}\right)\left|m_{n, k}(q ; x)-m_{\infty, k}(q ; x)\right| \\
& \leq 2 \omega\left(f ; q^{n}\right)+\omega\left(f ; q^{n}\right) \frac{1}{q^{n}} \sum_{k=0}^{\infty} q^{k}\left|m_{n, k}(q ; x)-m_{\infty, k}(q ; x)\right| \\
& \leq 2 \omega\left(f ; q^{n}\right)+\omega\left(f ; q^{n}\right) \frac{1}{q^{n}} \sum_{k=0}^{\infty} q^{k}\left(\frac{x}{1-x} \frac{q^{n+k+1}}{1-q} m_{n, k}(q ; x)+\frac{q^{n+1}}{1-q} m_{\infty, k}(q ; x)\right) .
\end{aligned}
$$

If $q^{n} x /(1-x) \leq 1$, then

$$
\begin{aligned}
\left|I_{2}\right| \leq & 2 \omega\left(f ; q^{n}\right) \\
& +\omega\left(f ; q^{n}\right) \frac{1}{q^{n}}\left(\frac{q^{n+1} x}{(1-q)(1-x)} \sum_{k=0}^{\infty} q^{2 k} m_{n, k}(q ; x)+\frac{q^{n+1}}{1-q} \sum_{k=0}^{\infty} q^{k} m_{\infty, k}(q ; x)\right) \\
\leq & 2 \omega\left(f ; q^{n}\right)+\omega\left(f ; q^{n}\right) \frac{1}{q^{n}}\left(\frac{q^{n+1} x}{(1-q)(1-x)} \sum_{k=0}^{\infty} q^{k} m_{n, k}(q ; x)+\frac{q^{n+1}}{1-q}(1-x)\right) \\
\leq & 2 \omega\left(f ; q^{n}\right)+\omega\left(f ; q^{n}\right) \frac{1}{q^{n}}\left(\frac{q^{n+1} x}{(1-q)(1-x)}\left(1-x+q^{n} x\right)+\frac{q^{n+1}}{1-q}(1-x)\right) \\
= & 2 \omega\left(f ; q^{n}\right)+\omega\left(f ; q^{n}\right) \frac{1}{q^{n}}\left(\frac{q^{n+1} x}{1-q}+\frac{q^{n+1} x}{1-q} \frac{q^{n} x}{1-x}+\frac{q^{n+1}(1-x)}{1-q}\right)
\end{aligned}
$$

$$
\begin{equation*}
\leq \omega\left(f ; q^{n}\right)\left(2+\frac{3 q}{1-q}\right) . \tag{3.3}
\end{equation*}
$$

From (3.2) and (3.3), we conclude the desired estimation for $0<x \leq 1 /\left(1+q^{n}\right)$.
Now suppose $x \in\left(1 /\left(1+q^{n}\right), 1\right)$, that is, $1-x<q^{n} /\left(1+q^{n}\right)<q^{n}$. Then we have
$|\Delta(x)|=\left|M_{n, q}(f ; x)-M_{\infty, q}(f ; x)\right|$

$$
\begin{aligned}
& =\left|\sum_{k=0}^{\infty}\left(f\left(\frac{[k]}{[n+k]}\right)-f(1)\right) m_{n, k}(q ; x)-\sum_{k=0}^{\infty}\left(f\left(1-q^{k}\right)-f(1)\right) m_{\infty, k}(q ; x)\right| \\
& \leq \sum_{k=0}^{\infty}\left|f\left(\frac{[k]}{[n+k]}\right)-f(1)\right| m_{n, k}(q ; x)+\sum_{k=0}^{\infty}\left|f\left(1-q^{k}\right)-f(1)\right| m_{\infty, k}(q ; x)
\end{aligned}
$$

since

$$
\left|\frac{[k]}{[n+k]}-1\right|=\left|\frac{1-q^{k}}{1-q^{n+k}}-1\right|=q^{k} \frac{1-q^{n}}{1-q^{n+k}} \leq q^{k}
$$

and

$$
\omega(f ; \lambda t) \leq(1+\lambda) \omega(f ; t), \quad \lambda>0
$$

we get

$$
\begin{aligned}
|\Delta(x)| & \leq \sum_{k=0}^{\infty} \omega\left(f ; q^{k}\right) m_{n, k}(q ; x)+\sum_{k=0}^{\infty} \omega\left(f ; q^{k}\right) m_{\infty, k}(q ; x) \\
& \leq \sum_{k=0}^{\infty} \omega\left(f ; q^{n}\right)\left(1+\frac{q^{k}}{q^{n}}\right) m_{n, k}(q ; x)+\sum_{k=0}^{\infty} \omega\left(f ; q^{n}\right)\left(1+\frac{q^{k}}{q^{n}}\right) m_{\infty, k}(q ; x) \\
& =2 \omega\left(f ; q^{n}\right)+\frac{\omega\left(f ; q^{n}\right)}{q^{n}} \sum_{k=0}^{\infty} q^{k} m_{n, k}(q ; x)+\frac{\omega\left(f ; q^{n}\right)}{q^{n}} \sum_{k=0}^{\infty} q^{k} m_{\infty, k}(q ; x) \\
& \leq 2 \omega\left(f ; q^{n}\right)+\frac{\omega\left(f ; q^{n}\right)}{q^{n}}\left(1-x+q^{n} x\right)+\frac{\omega\left(f ; q^{n}\right)}{q^{n}}(1-x) \\
& \leq 2 \omega\left(f ; q^{n}\right)+\frac{\omega\left(f ; q^{n}\right)}{q^{n}}\left(q^{n}+q^{n} x\right)+\frac{\omega\left(f ; q^{n}\right)}{q^{n}} q^{n} \\
& \leq 5 \omega\left(f ; q^{n}\right) .
\end{aligned}
$$

Finally we show that (3.1) is sharp. For each $0<\alpha \leq 1$, suppose that $f_{\alpha}$ is a continuous function defined as follows.

$$
f_{\alpha}(x)=\left\{\begin{array}{cc}
0, & 0 \leq x \leq 1-q \\
(x-(1-q))^{\alpha}, & 1-q \leq x \leq 1-q+\frac{q-q^{2}}{2} \\
-\left[\frac{q(1-q)}{2}\right]^{\alpha-1}\left(x-1+q^{2}\right), & 1-q+\frac{q-q^{2}}{2} \leq x \leq 1-q^{2} \\
0, & 1-q^{2} \leq x \leq 1
\end{array}\right.
$$

Then $\omega\left(f_{\alpha} ; t\right) \leq C t^{\alpha}$, and

$$
\begin{aligned}
& \left\|M_{n, q}\left(f_{\alpha} ; x\right)-M_{\infty, q}\left(f_{\alpha} ; x\right)\right\| \\
& =\left(\frac{q^{n+1}(1-q)}{1-q^{n+1}}\right)^{\alpha}\left\|m_{n, 1}(q ; \cdot)\right\| \asymp q^{\alpha n}\left\|m_{n, 1}(q ; \cdot)\right\| \asymp q^{\alpha n} .
\end{aligned}
$$

The proof of Theorem 3.2 is completed.
Remark 3.1. It should be emphasized that Theorem 3.2 cannot be obtained in a way similar to the proof of the Popoviciu Theorem for the classical Bernstein polynomials. It requires different estimation techniques due to the infinite product involved. Also, the proof in the paper is more difficult than the one used for $q$-MKZ operators (see [13]), since the new $q$-analogue of MKZ operators has the singular nature at the point $x=1$ and need a new method.

Theorem 3.3. Let $0<q<1$. Then

$$
\begin{equation*}
\left\|M_{n, q}(f)-M_{\infty, q}(f)\right\| \leq C \omega_{2}\left(f ; \sqrt{q^{n}}\right) \tag{3.4}
\end{equation*}
$$

Furthermore,

$$
\begin{equation*}
\sup _{0<q \leq 1}\left\|M_{n, q}(f)-M_{\infty, q}(f)\right\| \leq C \omega_{2}\left(f ; n^{-1 / 2}\right) \tag{3.5}
\end{equation*}
$$

where $C$ is an absolute constant.
Proof. It is clear that the new $q$-MKZ operators satisfy conditions (A) and (B) of Theorem 2 [12]:
(A) $M_{n, q}(1 ; x)=M_{\infty, q}(1 ; x), M_{n, q}(t ; x)=M_{\infty, q}(t ; x)$ and

$$
\begin{aligned}
0 & \leq \lambda_{n}(x)=M_{n, q}\left(t^{2} ; x\right)-M_{\infty, q}\left(t^{2} ; x\right) \\
& =M_{n, q}\left((t-x)^{2} ; x\right)-M_{\infty, q}\left((t-x)^{2} ; x\right) \\
& \leq \frac{1}{[n+1]} x+\frac{q^{2} x^{2}}{1-x+q x}-(1-q) x-\frac{q^{2} x^{2}}{1-x+q x}=\frac{q^{n+1}}{[n+1]} x \rightarrow 0
\end{aligned}
$$

as $n \rightarrow \infty$.
(B) The sequence $\left\{M_{n, q}(f ; x)\right\}_{n \geq 1}$ is nonincreasing for any convex function $f$ and for any $x \in[0,1]$, see Theorem 2.1.
On the otherhand, from Theorem 3.2 we know that for $q \in(0,1)$,

$$
M_{n, q}(f ; x) \rightarrow M_{\infty, q}(f ; x)
$$

pointwise as $n \rightarrow \infty$. It follows that

$$
\begin{aligned}
\left|M_{n, q}(f ; x)-M_{\infty, q}(f ; x)\right| & \leq C \omega_{2}\left(f ; \sqrt{M_{n, q}\left(t^{2} ; x\right)-M_{\infty, q}\left(t^{2} ; x\right)}\right) \\
& \leq C \omega_{2}\left(f ; \sqrt{q^{n+1}}\right) \leq C \omega_{2}\left(f ; \sqrt{q^{n}}\right)
\end{aligned}
$$

Since

$$
\begin{aligned}
\sup _{0<q<1}\left|M_{n, q}\left(t^{2} ; x\right)-M_{\infty, q}\left(t^{2} ; x\right)\right| & \leq \sup _{0<q<1} \frac{q^{n+1}(1-q)}{1-q^{n+1}}=\sup _{0<q<1} \frac{q^{n+1}}{[n+1]} \\
& =\sup _{0<q<1} \frac{q}{[n+1]_{1 / q}}=\frac{1}{n+1}<\frac{1}{n} \\
\left|M_{n, 1}\left(t^{2} ; x\right)-x^{2}\right| & \leq \frac{1}{n+1}<\frac{1}{n}
\end{aligned}
$$

the inequality (3.5) follows.
Remark 3.2. Results similar to Theorem 3.2 and Theorem 3.3 for $q$-Bernstein polynomials and $q$-MKZ operators were obtained in [28] and [13], respectively. Note that when $f(x)=x^{2}$, for $q \in(0,1)$, we have

$$
\left\|M_{n, q}(f)-M_{\infty, q}(f)\right\| \asymp q^{n} \asymp \omega_{2}\left(f ; \sqrt{q^{n}}\right) .
$$

Hence, the estimate (3.4) is sharp in the following sense: the sequence $\sqrt{q^{n}}$ in (3.4) cannot be replaced by any other sequence decreasing to zero more rapidly as $n \rightarrow \infty$. However, (3.4) is not sharp for the Lipschitz class Lip $\alpha(\alpha \in(0,1])$ in the sense of order. This combining with Theorem 3.2, shows that in the case $0<q<1$ the
modulus of continuity is more appropriate to describe the rate of convergence for the new $q$-MKZ operators than the second modulus of smoothness. This is different from that in the case $q=1$.

Remark 3.3. The constant $C$ in (3.4) is an absolute constant and does not depend on $q$, however the constant $C_{q}$ in (3.1) depends on $q$, and tends to $+\infty$ as $q \uparrow 1$. Hence, (3.4) does not follow from (3.1).

## References

[1] H. Aktuglu, M. A. Özarslan and O. Duman, Matrix summability methods on the approximation of multivariate $q$-mkz operators, Bull. Malays. Math. Sci. Soc. (2) 34 (2011), no. 3, 465-474.
[2] A. Aral and V. Gupta, The $q$-derivative and applications to $q$-Szász Mirakyan operators, Calcolo 43 (2006), no. 3, 151-170.
[3] E. W. Cheney and A. Sharma, Bernstein power series, Canad. J. Math. 16 (1964), 241-252.
[4] M.-M. Derriennic, Modified Bernstein polynomials and Jacobi polynomials in $q$-calculus, Rend. Circ. Mat. Palermo (2) Suppl. No. 76 (2005), 269-290.
[5] R. A. DeVore and G. G. Lorentz, Constructive Approximation, Grundlehren der Mathematischen Wissenschaften, 303, Springer, Berlin, 1993.
[6] O. Doğru and O. Duman, Statistical approximation of Meyer-König and Zeller operators based on $q$-integers, Publ. Math. Debrecen 68 (2006), no. 1-2, 199-214.
[7] O. Doğru and V. Gupta, Korovkin-type approximation properties of bivariate $q$-Meyer-König and Zeller operators, Calcolo 43 (2006), no. 1, 51-63.
[8] Z. Finta and V. Gupta, Approximation by $q$-Durrmeyer operators, J. Appl. Math. Comput. 29 (2009), no. 1-2, 401-415.
[9] T. N. T. Goodman, H. Oruç and G. M. Phillips, Convexity and generalized Bernstein polynomials, Proc. Edinburgh Math. Soc. (2) 42 (1999), no. 1, 179-190.
[10] N. K. Govil and V. Gupta, Convergence of $q$-Meyer-König-Zeller-Durrmeyer operators, Adv. Stud. Contemp. Math. (Kyungshang) 19 (2009), no. 1, 97-108.
[11] V. Gupta, Some approximation properties of $q$-Durrmeyer operators, Appl. Math. Comput. 197 (2008), no. 1, 172-178.
[12] W. Heping, Korovkin-type theorem and application, J. Approx. Theory 132 (2005), no. 2, 258-264.
[13] W. Heping, Properties of convergence for the $q$-Meyer-König and Zeller operators, J. Math. Anal. Appl. 335 (2007), no. 2, 1360-1373.
[14] A. Il'inskii and S. Ostrovska, Convergence of generalized Bernstein polynomials, J. Approx. Theory 116 (2002), no. 1, 100-112.
[15] S. Lewanowicz and P. Woźny, Generalized Bernstein polynomials, BIT 44 (2004), no. 1, 63-78.
[16] A. Lupas, A $q$-analogue of the Bernstein operator, University of Cluj-Napoca, Seminar on Numerical and Statistical Calculus, 9 (1987), 85-92.
[17] N. I. Mahmudov and P. Sabancıgil, $q$-parametric Bleimann Butzer and Hahn operators, J. Inequal. Appl. 2008, Art. ID 816367, 15 pp.
[18] N. I. Mahmudov, Korovkin-type theorems and applications, Cent. Eur. J. Math. 7 (2009), no. 2, 348-356.
[19] N. I. Mahmudov, The moments for $q$-Bernstein operators in the case $0<q<1$, Numer Algorithms, DOI 10.1007/s11075-009-9312-1.
[20] W. Meyer-König and K. Zeller, Bernsteinsche Potenzreihen, Studia Math. 19 (1960), 89-94.
[21] H. Oruç and G. M. Phillips, A generalization of the Bernstein polynomials, Proc. Edinburgh Math. Soc. (2) 42 (1999), no. 2, 403-413.
[22] S. Ostrovska, On the Lupaş $q$-analogue of the Bernstein operator, Rocky Mountain J. Math. 36 (2006), no. 5, 1615-1629.
[23] S. Ostrovska, The first decade of the $q$-Bernstein polynomials: results and perspectives, $J$. Math. Anal. Approx. Theory 2 (2007), no. 1, 35-51.
[24] G. M. Phillips, Bernstein polynomials based on the $q$-integers, Ann. Numer. Math. 4 (1997), no. 1-4, 511-518.
[25] G. M. Phillips, A survey of results on the $q$-Bernstein polynomials, IMA J. Num. Anal. doi:10.1093/imanum/drn088.
[26] T. Trif, Meyer-König and Zeller operators based on the $q$-integers, Rev. Anal. Numér. Théor. Approx. 29 (2000), no. 2, 221-229 (2002).
[27] V. S. Videnskii, On some classes of $q$-parametric positive linear operators, in Selected Topics in Complex Analysis, 213-222, Oper. Theory Adv. Appl., 158 Birkhäuser, Basel, 2005.
[28] H. Wang, Voronovskaya-type formulas and saturation of convergence for $q$-Bernstein polynomials for $0<q<1$, J. Approx. Theory 145 (2007), no. 2, 182-195.


[^0]:    Communicated by Rosihan M. Ali, Dato'.
    Received: November 15, 2009; Revised: March 3, 2010.

