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The Edge Steiner Number of a Graph

MICHAEL B. FRONDOZA AND SERGIO R. CANOY, JR.
Department of Mathematics, College of Science and Mathematics,
MSU-Iligan Institute of Technology, 9200 Iligan City, Philippines
michael.frondoza@g.msuiit.edu.ph, serge_canoy@yahoo.com

Abstract. The concepts of edge Steiner set and edge Steiner number of a graph are investigated in this study. A necessary and sufficient condition for a graph G to satisfy $\mathrm{st}_e(G) = |V(G)| - 1$, where $\mathrm{st}_e(G)$ denotes the edge Steiner number of G, is obtained. Edge Steiner sets in the joins of graphs are also studied and the Steiner numbers of these graphs are determined.

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1. Introduction

Given a connected graph G and a nonempty subset W of $V(G)$, a Steiner W-tree is a tree of minimum order that contains W. The sets $S(W)$ and $S_e(W)$ denote, respectively, the sets of all vertices and edges of G that lie on any Steiner W-tree. W is called a vertex Steiner set if $S(W) = V(G)$. If $S_e(W) = E(G)$, then W is said to be an edge Steiner set of G. A vertex (edge) Steiner set of minimum cardinality is called a minimum vertex (edge) Steiner set. The cardinality of a minimum vertex (edge) Steiner set of G is defined as the vertex (edge) Steiner number $\mathrm{st}(G)$ (resp. $\mathrm{st}_e(G)$) of G.

Steiner sets and Steiner numbers have been studied recently in [1, 2, 4]. In [2], the authors characterized the Steiner sets in the join $G+H$ and the composition $G[H]$ of two nontrivial connected graphs G and H. Edge Steiner sets, edge Steiner number, minimal edge Steiner sets, and upper edge Steiner numbers have been extensively studied very recently in [5]. For other terminologies, one may refer to [3].

2. Results

The following remarks are immediate from the definitions of edge Steiner set and edge Steiner number of a graph. The first and the third of these can be found in [5].

Remark 2.1. If G is a connected graph of order $n \geq 2$, then $2 \leq \mathrm{st}_e(G) \leq n$.

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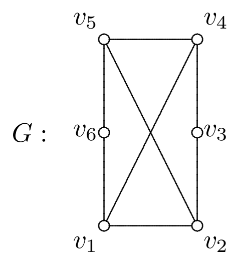
Remark 2.2. Let G be a connected graph of order $n \geq 2$. Then $\mathrm{st}_e(G) < n$ if and only if there exists a proper subset W of $V(G)$ such that $\langle W \rangle$ is disconnected and $S_e(W) = E(G)$.

Remark 2.3. $\mathrm{st}_e(K_n) = n$ for each positive integer $n \geq 2$.

Next, we briefly define the concepts of independent cutset and essential independent cutset in a graph and look at some relationships between these concepts and the concept of edge Steiner set.

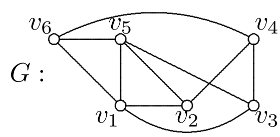
Definition 2.1. *Let G be a connected graph. A subset Y of $V(G)$ is said to be an **independent cutset** (or simply an **ics**) in G if it is independent and $\langle V(G) \setminus Y \rangle$ is disconnected. Y is said to be an **essential independent cutset** (or **eics**) if it is an ics and $\langle (V(G) \setminus Y) \cup \{y\} \rangle$ is connected for every $y \in Y$. An eics of G of maximum cardinality is called a **maximum eics** of G.*

Example 2.1. Consider the graph below.



$S = \{v_2, v_4, v_6\}$ and $R = \{v_4, v_2\}$ are independent cutsets. S is not an *eics* since $\langle V(G) \backslash S \rangle \cup \{v_6\}$ is disconnected. The set R is an *eics* since $\langle V(G) \backslash R \rangle \cup \{v_2\}$ and $\langle V(G) \backslash R \rangle \cup \{v_4\}$ are connected.

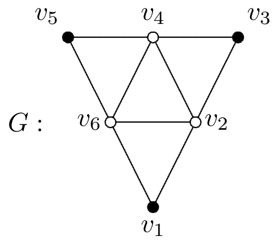
Example 2.2. Consider another graph below.



It can be verified that the sets $\{v_4, v_5\}$, $\{v_4, v_1\}$, $\{v_2, v_3\}$, $\{v_2, v_6\}$, $\{v_3, v_6\}$, $\{v_2, v_3, v_6\}$ are the only essential independent cutsets of G. Thus $U = \{v_2, v_3, v_6\}$ is a *maximum eics* of G.

Remark 2.4. A connected non-complete graph may have no *eics*.

To see this, consider the graph below.



It can be verified that G has no eics.

Theorem 2.1. *Let G be a connected non-complete graph of order $n \geq 2$. If Y is an essential independent cutset of G, then $V(G) \setminus Y$ is an edge Steiner set of G.*

Proof. Let $W = V(G) \setminus Y$, where Y is an eics. Then $\langle W \rangle$ is disconnected. Let $y \in Y$. Since Y is an eics, $\langle W \cup \{y\} \rangle$ is connected. Hence every spanning tree of $\langle W \cup \{y\} \rangle$ is a Steiner W-tree. This implies that $E(\langle W \cup \{y\} \rangle) \subseteq S_e(W)$. Since Y is independent, it follows that $E(G) = \cup_{y \in Y} E(\langle W \cup \{y\} \rangle) \subseteq S_e(W)$. Thus W is an edge Steiner set of G. ∎

The following result is immediate from Theorem 2.1.

Corollary 2.1. *Let G be a connected non-complete graph of order $n \geq 2$. If G has an essential independent cutset, then $\mathrm{st}_e(G) \leq n - r$, where $r = \max\{|Y| : Y \text{ is an eics in } G\}$.*

Remark 2.5. The converse of Theorem 2.1 is not true.

To see this, consider again the graph in Example 2.4. The graph G has no eics and $W = \{v_1, v_3, v_5\}$ is a minimum edge Steiner set of G. Thus $\mathrm{st}_e(G) = 3 \neq 6$.

Lemma 2.1. *Let G be a connected graph and v a cut-vertex of G. If $W \subseteq V(G)$ and $W \cap H \neq \varnothing$ for every component H of $\langle V(G) \backslash \{v\} \rangle$, then $v \in V(T)$ for every Steiner W-tree T of G.*

Proof. Let v be a cut-vertex of a connected graph G and $W \subseteq V(G)$. Then $\langle V(G) \backslash \{v\} \rangle$ is disconnected. If $v \in W$, then we are done. Suppose that $v \notin W$. Let $Y_1, Y_2, \ldots, Y_k$ be the components of $\langle V(G) \backslash \{v\} \rangle$ and suppose that $V(Y_j) \cap W \neq \varnothing$ for all $j \in I = \{1, 2, \ldots, k\}$. Clearly, $\cup_{j \in I}(V(Y_j) \cap W) = W$; hence $\langle W \rangle = \langle \cup_{j \in I}(V(Y_j) \cap W) \rangle$ is disconnected. Now, let T be a Steiner W-tree of G. Pick $v_1 \in V(Y_1) \cap W$ and $v_2 \in V(Y_2) \cap W$. Since $W \subseteq V(T)$, it follows that $v_1, v_2 \in V(T)$. Hence there is a path in T connecting v_1 and v_2. Clearly, this path contains v. Therefore, $v \in V(T)$. ∎

The next result is found in [5].

Lemma 2.2. *Let G be a connected graph and v a cut-vertex of G. If W is an edge Steiner set of G, then $v \in V(T)$ for every Steiner W-tree T of G.*

Theorem 2.2. *Let v be a cut-vertex of a connected graph G and $W \subseteq V(G)$ with $v \notin W$. Then $W \cup \{v\}$ is an edge Steiner set of G if and only if W is an edge Steiner set of G.*

Proof. Suppose that $W' = W \cup \{v\}$ is an edge Steiner set of G and $e \in E(G)$. Since $S_e(W') = E(G)$, there exists a Steiner W'-tree T_e of G such that $e \in E(T_e)$. Since $W' \cap V(H) \neq \varnothing$ for every component H of $\langle V(G) \backslash \{v\} \rangle$, $W \cap V(H) \neq \varnothing$ for every component H of $\langle V(G) \backslash \{v\} \rangle$. By Lemma 2.1, T_e is also a Steiner W-tree of G. Thus $e \in S_e(W)$, that is, $E(G) \subseteq S_e(W)$. Hence $E(G) = S_e(W)$. This implies that W is also an edge Steiner set of G.

Conversely, assume that W is an edge Steiner set of G and let $e \in E(G)$. Since $S_e(W) = E(G)$ it follows that there exists a Steiner W-tree T_e such that $e \in E(T_e)$.

From Lemma 2.2, $v \in V(T_e)$. This implies that T_e is also a Steiner $(W \cup \{v\})$-tree of G. Thus $e \in S_e(W \cup \{v\})$, that is, $E(G) \subseteq S_e(W \cup \{v\})$. Consequently, $E(G) = S_e(W \cup \{v\})$. Therefore $W \cup \{v\}$ is an edge Steiner set of G. ∎

The following result is found in [5].

Corollary 2.2. *Let G be a connected graph and v a cut-vertex of G. If W is a minimum edge Steiner set of G, then $v \notin W$.*

The next three results are also quick consequences of Theorem 2.2.

Corollary 2.3. *Let G be a connected graph of order n and W an edge Steiner set of G. If C is the set of cut-vertices of G, then $W \backslash C$ is an edge Steiner set of G.*

Proof. Let $C = \{v_1, v_2, \ldots, v_k\}$. Clearly, $W \backslash C = W \backslash (W \cap C)$. If $C \cap W = \varnothing$, then $W \backslash C = W$. Hence $W \backslash C$ is an edge Steiner set of G. Assume that $C_o = C \cap W \neq \varnothing$, say $|C_o| = \{y_1, y_2, \ldots, y_m\}$. Since W is an edge Steiner set of G, $Y_1 = W \backslash \{y_1\}$ is also an edge Steiner set of G by Theorem 2.2. Again, by Theorem, 2.2, $Y_2 = Y_1 \backslash \{y_2\}$ is an edge Steiner set of G. Repeating the process for the remaining vertices of C_o, it follows that $Y_m = Y_{m-1} \backslash \{y_m\}$ is an edge Steiner set of G. Therefore $Y_m = Y_1 \backslash \{y_2, y_3, \ldots, y_{m-1}, y_m\} = W \backslash C_o = W \backslash C$ is an edge Steiner set of G. ∎

Corollary 2.4. *Let G be a connected graph and C the set containing all the cut-vertices of G. Then any superset W_o of $V(G) \backslash C$ is an edge Steiner set of G.*

Proof. Let $C_o = W_o \cap C$. If $C_o = \varnothing$, then $W_o = V(G) \backslash C$ is an edge Steiner set by Corollary 2.3. So, suppose $C_o \neq \varnothing$, say $C_o = \{x_1, x_2, \ldots, x_m\}$. Since $x_1 \notin V(G) \backslash C$, it follows from Theorem 2.2 that $Y_1 = (V(G) \backslash C) \cup \{x_1\}$ is also an edge Steiner set of G. Again, since $x_2 \notin Y_1$, $Y_2 = Y_1 \cup \{x_2\}$ is an edge Steiner set of G. Proceeding in this manner, we find that $W_o = Y_m = Y_{m-1} \cup \{x_m\}$ is an edge Steiner set of G. ∎

Corollary 2.5. *If G is a connected graph and q is the number of cut-vertices of G, then $\mathrm{st}_e(G) \leq |V(G)| - q$.*

Proof. Let $C = \{v : v$ is a cut-vertex of $G\}$. From Corollary 2.3 and the fact that $V(G)$ is an edge Steiner set of G, it follows that $V(G) \backslash C$ is an edge Steiner set of G. Hence, if $|C| = q$, then $\mathrm{st}_e(G) \leq |V(G) \backslash C| = |V(G)| - |C| = |V(G)| - q$. ∎

Theorem 2.3. *Let G be a connected graph of order $n \geq 2$. Then $\mathrm{st}_e(G) = n - 1$ if and only if G has a unique cut-vertex v such that $\mathrm{st}_e(\langle V(H) \cup \{v\}\rangle = |V(H)| + 1$ for every component H of $\langle V(G) \backslash \{v\}\rangle$.*

Proof. Let G be a connected graph of order n and $\mathrm{st}_e(G) = n - 1$. Then there exists a vertex $v \in V(G)$ such that $W = V(G) \backslash \{v\}$ is an edge Steiner set of G. Since $\langle W \rangle$ is disconnected, v is a cut-vertex of G. From Corollary 2.5, v is the unique cut-vertex of G. Let $Y_1, Y_2, \ldots, Y_k$ be the components of $\langle V(G) \backslash \{v\}\rangle$. Suppose that $\mathrm{st}_e(\langle V(Y_m) \cup \{v\}\rangle) < |V(Y_m)| + 1$ for some m, where $1 \leq m \leq k$. Let W_{Y_m} be a minimum edge Steiner set of $\langle V(Y_m) \cup \{v\}\rangle$. Then $\langle W_{Y_m}\rangle$ is a disconnected proper subgraph of $\langle V(Y_m) \cup \{v\}\rangle$. Let $W_o = \cup_{i \neq m} V(Y_i)$ and let $W^* = (W_o \cup W_{Y_m})$. Since v is a cut-vertex of $\langle (\cup_{i \neq m} V(Y_i)) \cup \{v\}\rangle$, it follows that $(\cup_{i \neq m} V(Y_i)) \cup \{v\} \backslash \{v\} = \cup_{i \neq m} V(Y_i)$ is an edge Steiner set of $\langle (\cup_{i \neq m} V(Y_i)) \cup \{v\}\rangle$ by Theorem 2.2. Let $A = W_o \cup \{v\}$, $B = V(Y_m) \cup \{v\}$ and $e \in E(G)$. Consider the following cases.

Case 1: $e \in E(\langle A \rangle)$.

Since W_o is an edge Steiner set of $\langle A \rangle$, there exists a Steiner W_o-tree T_e of $\langle A \rangle$ such that $e \in E(T_e)$. Choose $u \in V(Y_m)$ such that $e' = uv \in E(\langle B \rangle)$. Since W_{Y_m} is an edge Steiner set of $\langle B \rangle$, there exists a Steiner W_{Y_m}-tree T'_e of $\langle B \rangle$ such $e' \in E(T'_e)$. Clearly, $v \in V(T_e) \cap V(T'_e)$. Let $T(e)$ be the tree obtained by gluing T_e and T'_e at vertex v. Then $T(e)$ is a Steiner W^*-tree of G with $e \in E(T(e))$.

Case 2: $e \in E(\langle B \rangle)$.

Let T be a Steiner W_o-tree of $\langle A \rangle$. Since W_{Y_m} is an edge Steiner set, there exists a Steiner W_{Y_m}-tree T_e with $e \in E(T_e)$. Consider the following subcases:

Subcase 1: $v \in W_{Y_m}$.

Then $v \in V(T_e)$. Let $T(e)$ be the tree obtained by gluing T_e and T at the vertex v. Then $T(e)$ is a Steiner W^*-tree of G with $e \in E(T(e))$.

Subcase 2: $v \notin W_{Y_m}$.

Extend (if necessary) T_e to a tree T_{uv} ($u \in V(Y_m)$) of minimum order such that $v \in V(T_{uv})$. Let $T(e)$ be the tree obtained by gluing T_{uv} and T at the vertex v. Then $T(e)$ is a Steiner W^*-tree of G with $e \in E(T(e))$.

In any case, $S_e(W^*) = E(G)$. Consequently, W^* is an edge Steiner set of G. By Corollary 2.3, $W^* \backslash \{v\}$ is also an edge Steiner set of G. If $v \in W_{Y_m}$, then $v \in W^*$ and $n-1 = \mathrm{st}_e(G) \leq |W^* \backslash \{v\}| = |W^*| - 1 = |W_o| + |W_{Y_m}| - 1 < |W_o| + |V(Y_m)| + 1 - 1 = n-1$, which is a contradiction. If $v \notin W_{Y_m}$, then $\langle W_{Y_m} \rangle$ is a disconnected subgraph of $\langle V_{Y_m} \rangle$. Thus $|W_{Y_m}| \leq |V(Y_m)| - 1$ and $\mathrm{st}_e(G) \leq |W^* \backslash \{v\}| = |W^*| \leq n-2$, contrary to the assumption that $\mathrm{st}_e(G) = n-1$. Therefore, $\mathrm{st}_e(\langle V(H) \cup \{v\} \rangle) = |V(H)| + 1$ for every component H of $G \backslash \{v\}$.

Conversely, assume that there exists a unique cut-vertex v such that for every component H of $G \backslash v$, $\mathrm{st}_e(\langle H \cup \{v\} \rangle) = |V(H)| + 1$. Then by Corollary 2.5, $\mathrm{st}_e(G) \leq n-1$. Suppose that $\mathrm{st}_e(G) < n-1$. Then there exists $W^* \subset V(G)$ such that $S_e(W^*) = E(G)$ and $\mathrm{st}_e(G) = |W^*| < |V(G)| - 1$. By Corollary 2.2, $v \notin W^*$. This implies that there exists a component H of $G \backslash v$ such that $V(H) \cap W^* \subset V(H)$. Let $W_H = V(H) \cap W^*$. Let $e \in E(\langle V(H) \cup \{v\} \rangle)$. Then $e \in E(G)$ and $e \in E(T_i)$ for some Steiner W^*-tree T_i of G. Let T_e be the portion of the tree T_i, where $V(T_e) = V(T_i) \cap (V(H) \cup \{v\})$. Then T_e is a Steiner $(W_H \cup \{v\})$-tree of $\langle V(H) \cup \{v\} \rangle$ and $e \in E(T_e)$. Hence $W_H \cup \{v\}$ is an edge Steiner set of $\langle V(H) \cup \{v\} \rangle$. This implies that $\mathrm{st}_e(\langle V(H) \cup \{v\} \rangle) \leq |W_H \cup \{v\}| < |V(H)| + 1$, contrary to the assumption. ∎

The next result characterizes the edge Steiner sets in a join of two graphs.

Theorem 2.4. *Let G and H be graphs of orders n and m, respectively, such that none of them is the empty graph. Then $W \subseteq V(G+H)$ is an edge Steiner set of G if and only if $W = V(G+H)$.*

Proof. Suppose that W is an edge Steiner set of $G+H$. Let $W_1 = W \cap V(G)$ and $W_2 = W \cap V(H)$. If $W_1 = \varnothing$, then $W = W_2 \subseteq V(H)$. Since W is a Steiner set of $V(G+H)$, the graph $\langle W \rangle$ induced by W must be disconnected. Let $v \in V(G)$. Then

$\langle W \cup \{v\} \rangle$ is a connected subgraph of $G+H$. This implies that every Steiner W-tree of $G+H$ has exactly $|W|+1$ vertices. Since G is not an empty graph, there exist $x, y \in V(G)$ such that $xy \in E(G+H)$. Clearly, this edge cannot be in any Steiner W-tree of $G+H$. This contradicts our assumption that W is an edge Steiner set of $G+H$. Therefore $W_1 \neq \varnothing$. Similarly, $W_2 \neq \varnothing$. Consequently, $\langle W \rangle$ is a connected subgraph of $G+H$ and so any Steiner W-tree of $G+H$, therefore, has $|W|$ vertices. Since W is an edge Steiner set of $G+H$, it follows that $W = V(G+H)$.

The converse is clear. ∎

An immediate consequence of the Theorem 2.4 is the following result.

Corollary 2.6. *Let G and H be graphs of orders n and m, respectively, such that none of them is the empty graph. Then* $\mathrm{st}_e(G+H) = n+m$.

Theorem 2.5. *Let G and H be graphs of orders n and m, respectively, such that $G+H$ is not a star, and at least one of them is the empty graph. Then $W \subseteq V(G+H)$ is an edge Steiner set of G if and only if either*

(i) *$W = V(G+H)$;*
(ii) *$W = V(G)$, G is disconnected, and $H = \overline{K}_m$; or*
(iii) *$W = V(H)$, H is disconnected, and $G = \overline{K}_n$.*

Proof. Suppose that W is an edge Steiner set of $G+H$. Suppose $W \neq V(G+H)$. Then $\langle W \rangle$ is disconnected and so $W \subseteq V(G)$ or $W \subseteq V(H)$. Furthermore, any Steiner W-tree of $G+H$ will have $|W|+1$ vertices. Assume that $W \subseteq V(G)$ and suppose that $W \neq V(G)$. Pick $v \in V(G) \backslash W$ and $u \in V(H)$. Then none of the Steiner W-trees of $G+H$ can contain $uv \in E(G+H)$, contrary to our assumption of W. Thus $W = V(G)$. In this case, $H = \overline{K}_m$; otherwise, there exist $a, b \in V(H)$ such that $ab \in E(G+H)$. However, none of the possible Stener W-trees can contain the edge ab, contradicting again our assumption. Similarly, if $W \subseteq V(H)$, then $W = V(H)$, H is disconnected, and $G = \overline{K}_n$.

The converse can easily be proved. ∎

The following result is a direct consequence of Theorem 2.5.

Corollary 2.7. *Let G and H be graphs of orders n and m, respectively, such that $G+H$ is not a star, and at least one of them is the empty graph. Then*

$$\mathrm{st}_e(G+H) = \begin{cases} n, & \text{if } G \text{ is disconnected, } G \neq \overline{K}_n, \\ & \quad \text{and } H = \overline{K}_m \\ m, & \text{if } H \text{ is disconnected, } H \neq \overline{K}_m, \\ & \quad \text{and } G = \overline{K}_n \\ \min\{n, m\}, & \text{if } G = K_{m,n} \\ n+m, & \text{otherwise.} \end{cases}$$

Corollary 2.8. *Let n and m be positive integers.*

(a) $\mathrm{st}_e(\overline{K}_n + P_m) = n+m \quad (m \geq 2)$
(b) $\mathrm{st}_e(\overline{K}_n + C_m) = n+m \quad (m \geq 3)$
(c) $\mathrm{st}_e(K_{n_1,n_2,\cdots,n_k}) = \sum_{i=1}^{k} n_i$, *where* $k \geq 3$.

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