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A Note on Graded Components of Local Cohomology Modules at the Earliest Level of Non-Artinianess

CHIA S. LIM
Taylor's Graduate School, Taylor's University
No. 1 Jalan Taylors, Subang Jaya, Selangor Darul Ehsan, Malaysia
chiasien.lim@taylors.edu.my

Abstract. Let M be a finitely generated graded R-module where R is a Noetherian homogeneous ring with local base ring (R_0, m_0) and R_+, the irrelevant ideal of R. Let $a(M) = \sup\{j \in \mathbb{N}_0 | H^i_{R_+}(M)$ is Artinian for all $i < j\}$. We prove that if $a(M) < \infty$ then $H^{a(M)}_{R_+}(M)$ is tame. Some strategies to establish tameness for graded modules in general will be discussed.

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1. Introduction

Throughout the paper, we will assume that $R = \bigoplus_{n\geq 0}^{\infty} R_n$ is a positively graded homogeneous Noetherian ring of finite Krull dimension; $R_+ = \bigoplus_{n>0}^{\infty} R_n$ is the maximal irrelevant ideal and M is a finitely generated graded R-module. We know that the components of the local cohomology of M with respect to R_+, $H^i_{R_+}(M)_n$, are all Noetherian as R_0-modules and vanish for all $n \gg 0$. However, recent work of Chardin, Cutkosky, Herzog, and Srinivasan [6] showed that there are examples where the negative components can vanish in a variety of patterns.

We say that a graded R-module E is tame if $E_n = (0)$ for $n \ll 0$ or $E_n \neq (0)$ for $n \ll 0$. The question on whether all local cohomological modules of M with respect to R_+ is tame was first raised by Brodmann and Hellus in [3]. A survey on positive answers to this conjecture can be found in [1].

Artinianess plays a signficant role in tameness. Not only does the former implies the latter but also recent work in [4, 8, 10] showed that by studying the Artinianess of $H^i_{R_+}(M)$, we can address the minimal generators of $\Gamma_{m_0 R}(H^i_{R_+}(M))_n$ as well as the depth of $H^i_{R_+}(M)_n$ through the work of Kirby in [9].

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Let

$$a(M) = \sup\{j \in \mathbb{N}_0 | H^i_{R_+}(M) \text{ is Artinian for all } i < j\}.$$

Our main result is to show that if $a(M) < \infty$, then $H^{a(M)}_{R_+}(M)$ is tame.

This result is akin to the statement involving the cohomological dimension of M with respect to R_+ [7] which states that $H^{\mathrm{cd}(M,R_+)}_{R_+}(M)$ is tame, where

$$\mathrm{cd}(M, R_+) := \sup\{i \in \mathbb{N}_0 | H^i_{R_+}(M) \neq (0)\}.$$

In fact they proved that $H^{q(M)}_{R_+}(M)$ is tame, where

$$q(M) = \sup\{i \in \mathbb{N}_0 | H^i_{R_+}(M) \text{ is not Artinian}\}.$$

The second half of the article will give some remarks and observations on the techniques to establish tameness for graded R-modules in general.

For any unexplained terminology, the reader can refer to [5].

2. Main result

The heart of the argument rests on the following observation in Lemma 2.1. After that, we will proceed with a proof by induction on $a(M)$.

We note that when $a(M) = \infty$, $H^i_{R_+}(M)$ is Artinian or 0; in particular, $H^i_{R_+}(M)$ is tame for all i. Furthermore, $H^i_{R_+}(M)$ being Artinian implies that $(R_0/m_0) \otimes_{R_0} H^i_{R_+}(M)$, which is isomorphic to $H^i_{R_+}(M)/m_0 H^i_{R_+}(M)$ as R-module, is also Artinian.

Lemma 2.1. [2, 3.3] *Let $S \subset \mathbb{N}_0$, R_0/m_0 be infinite and assume that the R-module $(R_0/m_0) \otimes_{R_0} H^i_{R_+}(M)$ is Artinian for each $i \in S$. Then, there is some $\theta \in \mathbb{Z} \cup \{\infty\}$ and an $M/\Gamma_{R_+}(M)$-regular element $x \in R_1$ such that the multiplication maps $H^i_{R_+}(M)_n \xrightarrow{x} H^i_{R_+}(M)_{n+1}$ is surjective for all $i \in S$ and for all $n < \theta$.*

2.1. Reduction to $|R_0/m_0| = \infty$

We would like to use a standard construction (cf. [5, 15.2.4]) to argue it suffices to prove our main contention assuming that the residue field, R_0/m_0, is infinite. Put $R'_0 = R_0[X]_{m_0 R_0[X]}$, where X is an indeterminate over R. Then, R'_0 is a faithfully flat local R_0-algebra with an infinite residue field. Put

$$R' = R \otimes_{R_0} R'_0, \qquad M' = M \otimes_R R',$$
$$m = m_0 + R_+, \qquad m' = m_0 R_0[x] + (R_+ \otimes_{R_0} R'_0).$$

Then, (R', m') is a positively graded homogeneous and graded local Noetherian ring with R'_0 as its 0-th component. M' is a finitely generated graded R'-module. By ([5], 15.2.2(iv)), for all $i \in \mathbb{N}_0$ and $n \in \mathbb{Z}$,

$$H^i_{R_+}(M)_n \otimes_{R_0} R'_0 \cong H^i_{R_+R'}(M')_n \tag{2.1}$$

as R'_0-modules. Furthermore, R'_0 being faithfully flat over R_0 implies that for every n, $H^i_{R_+}(M)_n = 0$ if and only if $H^i_{R_+R'}(M')_n = 0$. Therefore, $H^i_{R_+}(M)$ is tame if and only if $H^i_{R_+R'}(M')$ is tame.

Theorem 2.1. *Assume $a(M) < \infty$. Then, $H^{a(M)}_{R_+}(M)$ is tame.*

Proof. Since $H^i_{R_+}(M) = (0)$ for all $i < 0$, $a(M) \in \mathbb{N}_0$. For $a(M) = 0$, $H^0_{R_+}(M)$ is a finitely generated R-module and hence $H^0_{R_+}(M)_n = 0$ for $n \ll 0$. We will assume $a(M) > 0$ henceforth.

By the preceeding remark we may assume $|R_0/m_0| = \infty$. Note that we can assume $H^0_{R_+}(M) = 0$. Putting $S = \{0, 1, 2, 3, ..., a(M) - 1\}$. We choose $x \in R_1$ satisfying the properties in Lemma 2.1. The exact sequence:

$$0 \longrightarrow M[-1] \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0 \tag{2.2}$$

yields the following long exact sequence of graded modules:

$$H^{i-1}_{R_+}(M)[-1] \xrightarrow{x} H^{i-1}_{R_+}(M) \longrightarrow H^{i-1}_{R_+}(M/xM) \longrightarrow H^i_{R_+}(M). \tag{2.3}$$

This further induces the following two sequences of graded modules with graded maps:

(2.4)

$$H^{i-1}_{R_+}(M)[-1] \xrightarrow{x} H^{i-1}_{R_+}(M) \longrightarrow H^{i-1}_{R_+}(M/xM) \longrightarrow A^i \longrightarrow 0,$$

where A^i is the submodule of $H^i_{R_+}(M)$ annihilated by x;

$$0 \longrightarrow A^i \longrightarrow H^i_{R_+}(M)[-1] \longrightarrow B \longrightarrow 0, \tag{2.5}$$

where B is isomorphic to a submodule of $H^i_{R_+}(M/xM)$.

For $i < a(M)$, by the choice of x, the multiplication map $H^i_{R_+}(M)_n \xrightarrow{x} H^i_{R_+}(M)_{n+1}$ is surjective for $n \ll 0$. Then, the sequence in (2.4) will imply that $H^{a(M)-1}_{R_+}(M/xM)_n$ is isomorphic to $A^{a(M)}_n$ for $n \ll 0$.

From the sequences in (2.4) and (2.5), $H^i_{R_+}(M)$ is Artinian for all $i < a(M)$ implies that $H^{i-1}_{R_+}(M/xM)$ is Artinian for all $i < a(M) - 1$. Thus we have,

$$a(M) - 1 \leq a\left(M/xM\right).$$

We have two cases under consideration:

$$a(M) - 1 < a\left(M/xM\right) \text{ and } a(M) - 1 = a\left(M/xM\right).$$

For $a(M) - 1 = a(M/xM)$, by induction hypothesis, $H^{a(M)-1}_{R_+}(M/xM)$ is tame. For $a(M) - 1 < a(M/xM)$, by definition of $a(M/xM)$, $H^{a(M)-1}_{R_+}(M/xM)$ is Artinian and hence, tame.

Since $H^{a(M)-1}_{R_+}(M/xM)_n \simeq A^{a(M)}_n$ for $n \ll 0$, $A^{a(M)}$ is tame. Since $A^{a(M)} \subset H^{a(M)}_{R_+}(M)$, it suffices to consider the situation where $A^{a(M)}_n = (0)$ for $n \ll 0$. This would imply that the multiplication map: $H^{a(M)}_{R_+}(M)_n \xrightarrow{x} H^{a(M)}_{R_+}(M)_{n+1}$ is injective for $n \ll 0$. Hence, $H^{a(M)}_{R_+}(M)_n \neq (0)$ implies that $H^{a(M)}_{R_+}(M)_{n+1} \neq (0)$; in particular, $H^{a(M)}_{R_+}(M)$ is tame. ∎

Observations: Let A, B and C be graded R-modules such that B and C are both tame. Assume q_0 is an ideal of R_0 which is m_0-primary. Then, the following statements hold.

(a) If A_n is a finitely generated R_0-module, then A is tame if and only if A/q_0A is tame.
(b) If we have the following exact sequences, $0 \to B \longrightarrow A \longrightarrow C \to 0$, then A is tame.
(c) Let $x \in R_1$. Then A/xA is tame implies that A is tame.

Remark 2.1. On (a), in [2, Eg. 4.1], we see that $H^1_{R_+}(R)/m_0H^1_{R_+}(R)$ is not Artinian even though $H^f_{R_+}(M)$ is always tame (cf. [3, 5.6], where

$$f := \sup\{j \in \mathbb{N}_0 | \text{ for all } i < j, H^j_{R_+}(M) \text{ is finitely generated}\}.$$

With this, we make another observation whose forward direction was shown in [2, Lemma 2.4]:

Fix $i \in \mathbb{N}_0$. We write $M/\Gamma_{m_0R}(M)$ as $\widetilde{M}$. Then,

$$\frac{H^i_{R_+}(\widetilde{M})}{m_0H^i_{R_+}(\widetilde{M})} \text{ is Artinian if and only if } \frac{H^i_{R_+}(M)}{m_0H^i_{R_+}(M)} \text{ is Artinian.}$$

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