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Simple Groups Which are 2-Fold OD-Characterizable

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Abstract. Let G be a finite group and $\mathrm{D}(G)$ be the degree pattern of G. Denote by $h_{\mathrm{OD}}(G)$ the number of isomorphism classes of finite groups H satisfying $(|H|, \mathrm{D}(H)) = (|G|, \mathrm{D}(G))$. A finite group G is called k-*fold* OD-*characterizable* if $h_{\mathrm{OD}}(G) = k$. As the main results of this paper, we prove that each of the following pairs $\{G_1,\ G_2\}$ of groups:

$$\{B_n(q),\ C_n(q)\},\quad n = 2^m > 2,\quad \left|\pi\left(\frac{q^n+1}{2}\right)\right| = 1,\quad q \text{ is odd prime power;}$$

$$\{B_p(3),\ C_p(3)\},\quad \left|\pi\left(\frac{3^p-1}{2}\right)\right| = 1,\quad p \text{ is an odd prime,}$$

$$\{B_3(5),\ C_3(5)\},$$

satisfies $h_{\mathrm{OD}}(G_i) = 2$, $i = 1, 2$. We also prove that, if (1) $n = 2$ and q is any prime power such that $|\pi(q^2 + 1/(2, q-1))| = 1$ or (2) $n = 2^m \geq 2$ and q is a power of 2 such that $|\pi(q^n + 1)| = 1$, then $h_{\mathrm{OD}}(C_n(q)) = h_{\mathrm{OD}}(B_n(q)) = 1$.

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1. Introduction

Let G be a finite group, $\pi(G)$ the set of all prime divisors of its order and $\omega(G)$ be the spectrum of G, that is the set of its element orders. The *Gruenberg-Kegel graph* $\mathrm{GK}(G)$ or *prime graph of* G is a simple graph with vertex set $\pi(G)$ in which two vertices p and q are joined by an edge (and we write $p \sim q$) if and only if $pq \in \omega(G)$. Let $s(G)$ be the number of connected components of $\mathrm{GK}(G)$. The ith connected component is denoted by $\pi_i = \pi_i(G)$ for each i. If $2 \in \pi(G)$, then we assume that $2 \in \pi_1(G)$.

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The classification of finite simple groups with disconnected Gruenberg-Kegel graph was obtained by Williams [21] and Kondrat'ev [11]. An corrected list of these groups can be found in [12].

The *degree* $\deg(p)$ *of a vertex* $p \in \pi(G)$ is the number of edges incident on p. If $\pi(G) = \{p_1, p_2, \ldots, p_k\}$ with $p_1 < p_2 < \cdots < p_k$, then we define

$$\mathrm{D}(G) := \big(\deg(p_1), \deg(p_2), \ldots, \deg(p_k)\big),$$

which is called the *degree pattern of* G.

Given a finite group M, denote by $h_{\mathrm{OD}}(M)$ the number of isomorphism classes of finite groups G such that $|G| = |M|$ and $\mathrm{D}(G) = \mathrm{D}(M)$. In terms of the function h_{OD}, groups M are classified as follows:

Definition 1.1. *A finite group M is called* k-fold OD-characterizable *if* $h_{\mathrm{OD}}(M) = k$. *Usually, a 1-fold OD-characterizable group is simply called* OD-characterizable.

In order to formulate the obtained results, we need some notation and definitions. Throughout the paper, we assume that q is a prime power. We write $L_n(q)$ instead of the projective special linear group $\mathrm{PSL}(n, q)$ and write $U_n(q)$ instead of the projective special unitary group $\mathrm{PSU}(n, q)$. We use $B_n(q)$ and $C_n(q)$ to denote the simple orthogonal and symplectic groups, respectively. (In Atlas [4] notation, these are the groups $O_{2n+1}(q)$ and $S_{2n}(q)$, respectively.)

Table 1 lists finite simple groups which are currently known to be OD-characterizable or 2-fold OD-characterizable.

Table 1. Finite simple groups which are currently known to be OD-characterizable or 2-fold OD-characterizable

M	Conditions on M	$h_{\mathrm{OD}}(M)$	References
A_n	$n = p, p+1, p+2$ (p a prime)	1	[13, 14]
	$n = p+3,\ p \in \pi(100!) \setminus \{7\}$	1	[7, 16, 17, 24]
	$n = 10$	2	[15]
$L_2(q)$	$q \neq 2, 3$	1	[13, 14, 25, 30]
$L_3(q)$	$\lvert\pi(\frac{q^2+q+1}{d})\rvert = 1,\ d = (3, q-1)$	1	[13]
$U_3(q)$	$\lvert\pi(\frac{q^2-q+1}{d})\rvert = 1,\ d = (3, q+1), q > 5$	1	[13]
$L_4(q)$	$q = 5, 7$	1	[1]
$L_3(9)$		1	[27]
$U_3(5)$		1	[29]
$U_4(7)$		1	[1]
$L_n(2)$	$n = p$ or $p+1$, for which $2^p - 1$ is a prime	1	[1]
$R(q)$	$\lvert\pi(q \pm \sqrt{3q} + 1)\rvert = 1,\ q = 3^{2m+1},\ m \geq 1$	1	[13]
$\mathrm{Sz}(q)$	$q = 2^{2n+1} \geq 8$	1	[13, 14]

Continuation of Table 1.

M	Conditions on M	$h_{\mathrm{OD}}(M)$	References
$B_3(3)$		2	[13]
$C_3(3)$		2	[13]
M	A sporadic simple group	1	[13]
M	$\|\pi(M)\| = 4, \quad M \neq A_{10}$	1	[26]
M	$\|M\| \leq 10^8, \quad M \neq A_{10},\ U_4(2)$	1	[23]

It was shown in [13] and [15] that each of the following pairs $\{G_1, G_2\}$ of groups:

$$\{A_{10},\ \mathbb{Z}_3 \times J_2\}, \qquad \{B_3(3),\ C_3(3)\}$$

satisfies $|G_1| = |G_2|$ and $\mathrm{D}(G_1) = \mathrm{D}(G_2)$, and $h_{\mathrm{OD}}(G_i) = 2$, $i = 1, 2$. Until recently, no examples of simple groups M with $h_{\mathrm{OD}}(M) \geq 3$ are known. In [14], we posed the following question:

Problem 1.1. *Is there a simple group which is k-fold* OD*-characterizable for $k \geq 3$?*

If n is a positive integer, then $\pi(n)$ denotes the set of prime divisors of n. Given a finite group G, the order of G can be expressed as a product of some coprime positive integers m_i, $i = 1, 2, \ldots, s(G)$, with $\pi(m_i) = \pi_i$. These integers m_i's are called the *order components of* G. Let $\mathrm{OC}(G) = \{m_1, m_2, \ldots, m_{s(G)}\}$ be the set of order components of G. The order components of simple groups with disconnected prime graphs are obtained in Tables 1–4 in [3].

Given a finite group M, define $h_{\mathrm{OC}}(M)$ to be the number of isomorphism classes of finite groups with the same set $\mathrm{OC}(M)$ of order components. In terms of the function h_{OC}, groups M are classified as follows:

Definition 1.2. *A finite group M is called k-*fold OC-characterizable *if $h_{\mathrm{OC}}(M) = k$. Usually, a 1-fold* OC*-characterizable group is simply called* OC-characterizable*.*

It is clear that $1 \leq h_{\mathrm{OD}}(M) < \infty$ and $1 \leq h_{\mathrm{OC}}(M) < \infty$ for any finite group M. In fact, by Cayley's theorem, for each positive integer n, there are only finitely many distinct types of groups of order n. Evidently, a simple group S with connected prime graph is not OC-characterizable, because $h_{\mathrm{OC}}(S) \geq \nu_{\mathrm{nil}}(|S|) \geq 2$, where $\nu_{\mathrm{nil}}(n)$ denotes the number of isomorphism classes of nilpotent groups of order n.

Table 2. The groups of order 30.

G	$\mathrm{GK}(G)$	$s(G)$	$\mathrm{OC}(G)$	$D(G)$	$h_{\mathrm{OD}}(G)$	$h_{\mathrm{OC}}(G)$
$\mathbb{Z}_{30}$	$2 \sim 3 \sim 5 \sim 2$	1	$\{30\}$	$(2, 2, 2)$	1	3
$\mathbb{Z}_3 \times D_{10}$	$2 \sim 3 \sim 5$	1	$\{30\}$	$(1, 2, 1)$	1	3
$\mathbb{Z}_5 \times D_6$	$2 \sim 5 \sim 3$	1	$\{30\}$	$(1, 1, 2)$	1	3
D_{30}	$2,\ 3 \sim 5$	2	$\{2, 15\}$	$(0, 1, 1)$	1	1

Note that, the values of the functions h_{OD} and h_{OC} may be different. For example, there are only four non-isomorphic groups of order 30, which we list in Table 2. Now, it can be easily seen that $h_{\rm OD}(\mathbb{Z}_{30}) = h_{\rm OD}(\mathbb{Z}_3 \times D_{10}) = h_{\rm OD}(\mathbb{Z}_5 \times D_6) = 1$, while $h_{\rm OC}(\mathbb{Z}_{30}) = h_{\rm OC}(\mathbb{Z}_3 \times D_{10}) = h_{\rm OC}(\mathbb{Z}_5 \times D_6) = 3$.

We recall that a *clique* in a graph is a set of pairwise adjacent vertices. An independence set in a graph is a set of pairwise non-adjacent vertices. Note that the prime graph of a nilpotent finite group is always a clique. Moreover, if S is a simple group with disconnected prime graph, then all connected components $\pi_i(S)$ for $2 \leq i \leq s(S)$ are clique, for instance, see [11, 18, 21].

The purpose of this paper is to prove the following theorems.

Theorem 1.1. *Let r be an odd prime such that $|\pi(\frac{3^r-1}{2})| = 1$. Then, we have*

$$h_{\rm OD}(B_r(3)) = h_{\rm OD}(C_r(3)) = 2.$$

Example 1.1. For $2 < r < 100$, we obtain the following simple groups among $B_r(3)$ and $C_r(3)$:

$$B_3(3),\ C_3(3);\quad B_5(3),\ C_5(3);\quad B_7(3),\ C_7(3);\quad B_{13}(3),\ C_{13}(3).$$

Theorem 1.2. *Let q be a prime power and $n = 2^m \geq 2$. Then we have*

(a) *If q is even, $|\pi(q^n+1)| = 1$ and $(n, q) \neq (2, 2)$, then $h_{\rm OD}(B_n(q)) = h_{\rm OD}(C_n(q)) = 1$.*

(b) *If q is odd, $|\pi((q^n + 1)/2)| = 1$ and $(n, q) \neq (2, 3)$, then*

$$h_{\rm OD}(B_n(q)) = h_{\rm OD}(C_n(q)) = \begin{cases} 2 & \text{if} \quad n \geq 4, \\ 1 & \text{if} \quad n = 2. \end{cases}$$

Example 1.2. Some groups $B_n(q)$ and $C_n(q)$ satisfying the hypothesis of Theorem 1.2 have been computed and as a consequence we have listed the following OD-characterizable or 2-fold OD-characterizable simple groups in Table 3.

Theorem 1.3. *The simple groups $B_3(5)$ and $C_3(5)$ are 2-fold OD-characterizable.*

In fact, the pair $\{B_3(5), C_3(5)\}$ is the first pair of finite simple groups with connected prime graph which are 2-fold OD-characterizable.

We conclude the introduction with notation to be used throughout the paper. The *socle* of a group G is the subgroup generated by the set of all minimal normal subgroups of G; it is denoted by $\mathrm{soc}(G)$. If H is a subgroup of G, then $C_G(H)$ and $N_G(H)$ are, respectively, the centralizer and the normalizer of H in G. If a is a natural number, r is an odd prime and $(r, a) = 1$, then by $e(r, a)$ we denote the multiplicative order of a modulo r, that is the minimal natural number n with $a^n \equiv 1 \pmod r$. If a is odd, we put

$$e(2, a) = \begin{cases} 1 & \text{if } a \equiv 1 \pmod 4, \\ 2 & \text{if } a \equiv -1 \pmod 4. \end{cases}$$

We also define the function $\eta : \mathbb{N} \longrightarrow \mathbb{N}$, as follows

$$\eta(m) = \begin{cases} m & \text{if } m \equiv 1 \pmod 2, \\ \frac{m}{2} & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

Table 3. The simple groups $B_n(q)$ and $C_n(q)$, where $n = 2^m \geq 2$ and $d = (2, q-1)$.

G	$\frac{q^n+1}{d}$	$h_{\rm OD}(G)$
$B_2(2^2)$	17	1
$B_2(2^4)$	257	1
$B_2(2^8)$	65537	1
$B_4(2)$	17	1
$B_4(2^2)$	257	1
$B_4(2^4)$	65537	1
$B_8(2)$	257	1
$B_8(2^2)$	65537	1
$B_{16}(2)$	65537	1
$B_4(3)$, $C_4(3)$	41	2
$B_{16}(3)$, $C_{16}(3)$	p_1	2
$B_{32}(3)$, $C_{32}(3)$	p_2	2
$B_{64}(3)$, $C_{64}(3)$	p_3	2
$B_2(3^2)$	41	1
$B_8(3^2)$, $C_8(3^2)$	p_1	2
$B_{16}(3^2)$, $C_{16}(3^2)$	p_2	2
$B_{32}(3^2)$, $C_{32}(3^2)$	p_3	2
$B_4(3^4)$, $C_4(3^4)$	p_1	2
$B_8(3^4)$, $C_8(3^4)$	p_2	2
$B_{16}(3^4)$, $C_{16}(3^4)$	p_3	2
$B_2(3^8)$	p_1	1
$B_4(3^8)$, $C_4(3^8)$	p_2	2
$B_8(3^8)$, $C_8(3^8)$	p_3	2
$B_2(3^{16})$	p_2	1
$B_4(3^{16})$, $C_4(3^{16})$	p_3	2
$B_2(3^{32})$	p_3	1
$B_2(5)$	13	1

G	$\frac{q^n+1}{d}$	$h_{\rm OD}(G)$
$B_4(5)$, $C_4(5)$	313	2
$B_2(5^2)$	313	1
$B_2(7)$	5^2	1
$B_4(7)$, $C_4(7)$	1201	2
$B_2(7^2)$	1201	1
$B_2(11)$	61	1
$B_4(11)$, $C_4(11)$	7321	2
$B_2(11^2)$	7321	1
$B_4(13)$, $C_4(13)$	14281	2
$B_8(13)$, $C_8(13)$	p_4	2
$B_2(13^2)$	14281	1
$B_4(13^2)$, $C_4(13^2)$	p_4	2
$B_2(13^4)$	p_4	1
$B_4(17)$, $C_4(17)$	41761	2
$B_2(17^2)$	41761	1
$B_2(19)$	181	1
$B_4(23)$, $C_4(23)$	139921	2
$B_2(23^2)$	139921	1
$B_2(29)$	421	1
$B_4(29)$, $C_4(29)$	353641	2
$B_{16}(29)$, $C_{16}(29)$	p_5	2
$B_2(29^2)$	353641	1
$B_8(29^2)$, $C_8(29^2)$	p_5	2
$B_4(29^4)$, $C_4(29^4)$	p_5	2
$B_2(29^8)$	p_5	1
$B_2(41)$	29^2	1
$B_{16}(41)$, $C_{16}(41)$	p_6	2

Continuation of Table 3.

G	$\frac{q^n+1}{d}$	$h_{\mathrm{OD}}(G)$	G	$\frac{q^n+1}{d}$	$h_{\mathrm{OD}}(G)$
$B_8(41^2)$, $C_8(41^2)$	p_6	2	$B_2(61)$	1861	1
$B_4(41^4)$, $C_4(41^4)$	p_6	2	$B_4(61)$, $C_4(61)$	6922921	2
$B_2(41^8)$	p_6	1	$B_2(61^2)$	6922921	1
$B_8(43)$, $C_8(43)$	p_7	2	$B_2(71)$	2521	1
$B_4(43^2)$, $C_4(43^2)$	p_7	2	$B_4(71)$, $C_4(71)$	12705841	2
$B_2(43^4)$	p_7	1	$B_2(71^2)$	12705841	1
$B_8(47)$, $C_8(47)$	p_8	2	$B_4(73)$, $C_4(73)$	14199121	2
$B_4(47^2)$, $C_4(47^2)$	p_8	2	$B_{16}(73)$, $C_{16}(73)$	p_{10}	2
$B_2(47^4)$	p_8	1	$B_2(73^2)$	14199121	1
$B_8(53)$, $C_8(53)$	p_9	2	$B_8(73^2)$, $C_8(73^2)$	p_{10}	2
$B_4(53^2)$, $C_4(53^2)$	p_9	2	$B_4(73^4)$, $C_4(73^4)$	p_{10}	2
$B_2(53^4)$	p_9	1	$B_2(73^8)$	p_{10}	1
$B_2(59)$	1741	1	$B_2(79)$	3121	1

$p_1 = 21523361$,
$p_2 = 926510094425921$
$p_3 = 1716841910146256242328924544641$
$p_4 = 407865361$
$p_5 = 1251232368401736743937 61$
$p_6 = 3187951545732652717321 6321$
$p_7 = 5844100138801$
$p_8 = 11905643330881$
$p_9 = 31129845205681$
$p_{10} = 3251889399089047855210614172 81$

2. Preliminary results

The following lemma is a consequence of Zsigmondy's theorem (see [30]).

Lemma 2.1. *Let a be a natural number greater than 1. Then for every natural number n there exists a prime r with $e(r, a) = n$ but for the cases $(n, a) \in \{(1, 2), (1, 3), (6, 2)\}$*

A prime r with $e(r, a) = n$ is called a *primitive prime divisor* of $a^n - 1$. By Lemma 2.1, such a prime exists except for the cases mentioned in the lemma. Given a natural number a, we denote by $R_n(a)$ the set of all primitive prime divisors of $a^n - 1$ and by $r_n(a)$ any element of $R_n(a)$. By our definition, we have $\pi(a - 1) = R_1(a)$ but for the following sole exception, namely, $2 \notin R_1(a)$ if $e(2, a) = 2$. In this case, we assume that $2 \in R_2(a)$.

From [2, Theorems 11.3.2 and 14.5.2], we have the following lemma.

Lemma 2.2. *The following isomorphisms hold:*

(1) $B_n(q) \cong P\Omega_{2n+1}(q) \cong O_{2n+1}(q)$,
(2) $C_n(q) \cong PSp_{2n}(q) \cong S_{2n}(q)$,

(3) $B_2(3) \cong {}^2A_4(2^2)$, $B_n(2^m) \cong C_n(2^m)$, $B_2(q) \cong C_2(q)$.

In what follows, we concentrate on the simple groups $B_n(q)$ and $C_n(q)$, where $n \geq 2$. Note that, if $n = 1$, then we have

$$B_1(q) \cong C_1(q) \cong L_2(q).$$

Also, in the case that $n \geq 3$ and q is an odd prime power, we have $B_n(q) \not\cong C_n(q)$ (see [8]).

Lemma 2.3. [20] *Let M be one of the simple groups of Lie type, $B_n(q)$ or $C_n(q)$, over a field of characteristic p. Let r, s be odd primes with $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r, q)$ and $l = e(s, q)$, and suppose that $1 \leq \eta(k) \leq \eta(l)$. Then r and s are non-adjacent if and only if $\eta(k) + \eta(l) > n$ and $1/k$ is not an odd natural number.*

Lemma 2.4. [19] *Let M be one of the simple groups of Lie type, $B_n(q)$ or $C_n(q)$, over a field of characteristic p, and let $r \in \pi(M) \setminus \{p\}$ and $k = e(r, q)$. Then r and p are non-adjacent if and only if $\eta(k) > n - 1$.*

Lemma 2.5. [19] *Let M be one of the simple groups of Lie type, $B_n(q)$ or $C_n(q)$, over a field of characteristic p. Let r be an odd prime in $\pi(M) \setminus \{p\}$ and $k = e(r, q)$. Then 2 and r are non-adjacent if and only if $\eta(k) = n$ and one of the following holds:*

(1) *n is odd and $k = (3 - e(2, q))n$.*
(2) *n is even and $k = 2n$.*

Using Lemmas 2.3–2.5, we have the following corollary.

Corollary 2.1. *Assume that $(B, C) = (B_n(q), C_n(q))$. Then the following statements hold.*

(a) *The prime graphs* $\mathrm{GK}(B)$ *and* $\mathrm{GK}(C)$ *coincide* [19, Proposition 7.5].
(b) *$|B| = |C|$ and $\mathrm{D}(B) = \mathrm{D}(C)$. In particular, if $B \not\cong C$, then we have $h_{\mathrm{OD}}(B) = h_{\mathrm{OD}}(C) \geq 2$.*

Since $\mathrm{GK}(B_n(q)) = \mathrm{GK}(C_n(q))$, in Table 4 we consider these groups together and, for brevity, use the symbol $B_n(q)$ in both cases.

Table 4. The connected components of $\mathrm{GK}\big(B_n(q)\big) = \mathrm{GK}\big(C_n(q)\big)$.

Group	Conditions on n	Conditions on q	π_1	π_2
$B_n(q)$	$n = 2^m \geq 2$	none	$\pi\big(q \prod_{i=1}^{n-1}(q^{2i} - 1)\big)$	$\pi(\frac{q^n+1}{(2,q-1)})$
	$n = r$ odd prime	$q = 2, 3$	$\pi\big(q(q^r + 1) \prod_{i=1}^{r-1}(q^{2i} - 1)\big)$	$\pi(\frac{q^r-1}{(2,q-1)})$
	$n \neq 2^m$	$q \neq 2, 3$	$\pi\big(q^{n^2} \prod_{i=1}^{n}(q^{2i} - 1)\big)$	-
	$n \neq r,\ 2^m$	$q = 2, 3$	$\pi\big(q^{n^2} \prod_{i=1}^{n}(q^{2i} - 1)\big)$	-

Corollary 2.2. *Let $M \in \{B_n(q), C_n(q)\}$, where q is a power of a prime p. Then, the following hold for M:*

(1) *If* $n = 2^m \geq 2$, *then* $\deg(2) = \deg(p) = |\pi_1(M)| - 1$.
(2) *If* $n = r$ *is an odd prime and* $q = 3$, *then* $\deg(2) = |\pi_1(M)| - 1$.

Proof. (1) In this case, from Table 4, we have $\pi_1(M) = \pi\big(q\prod_{i=1}^{n-1}(q^{2i} - 1)\big)$ and $\pi_2(M) = \pi\left(q^n + 1/(2, q-1)\right)$. Moreover, by Lemma 2.4, it follows that only primitive prime divisors of $q^{2n} - 1$ are non-adjacent to p. But since

$$R_{2n}(q) \subset \pi\left(\frac{q^n + 1}{(2, q-1)}\right) = \pi_2(G),$$

we deduce that $\deg(p) = |\pi_1(M)| - 1$, as desired. In the sequel, we assume that p is an odd prime. From Lemma 2.4, it is easy to see that $2 \sim p$. Moreover, by Lemma 2.5, we conclude that only primitive prime divisors of $q^{2n} - 1$ are non-adjacent to 2, and similar to the previous case it yields that $\deg(2) = |\pi_1(M)| - 1$.
(2) Again, in this case we have $\pi_1(M) = \pi\big(3(3^r + 1)\big)\prod_{i=1}^{r-1}(3^{2i} - 1)\big)$ and $\pi_2(M) = \pi((3^r - 1)/2)$. Here, by Lemma 2.5, we conclude that only primitive prime divisors of $3^r - 1$ are non-adjacent to 2. Therefore, we obtain $\deg(2) = |\pi_1(M)| - 1$, as desired. ∎

The following corollary is easily obtained from Lemmas 2.3–2.5 and [4]:

Corollary 2.3. *Let* $M \in \{B_3(5), C_3(5)\}$. *The following hold for* M*:*
(1) $\mathrm{D}(M) = (4, 4, 3, 1, 3, 1)$,
(2) $|M| = 2^9 \cdot 3^4 \cdot 5^9 \cdot 7 \cdot 13 \cdot 31$.
(3) $|\mathrm{Out}(M)| = 2$.
(4) *The prime graph of* M *appears as shown in Figure 1.*

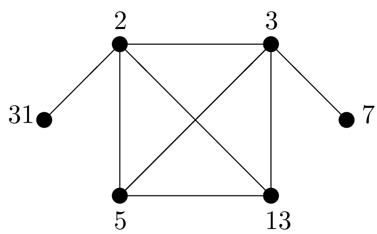

Figure 1. $\mathrm{GK}(B_3(5)) = \mathrm{GK}(C_3(5))$

Lemma 2.6. [6, 9, 10] *Let* M *be one of the finite simple groups of Lie type,* $B_n(q)$ *or* $C_n(q)$, *over a field of characteristic* p *and order* q. *Then*
(a) *If* $n = r$ *be an odd prime and* $q = 3$, *then* $h_{\mathrm{OC}}(M) = 2$.
(b) *If* $n = 2$ *and* $q > 5$, *then* $h_{\mathrm{OC}}(M) = 1$.
(c) *If* $n = 2^m \geq 4$, *then*

$$h_{\mathrm{OC}}(M) = \begin{cases} 2 & \text{if } p > 2, \\ 1 & \text{if } p = 2. \end{cases}$$

Lemma 2.7. [22] *Let* $S = P_1 \times P_2 \times \cdots \times P_t$, *where* P_i*'s are isomorphic non-Abelian simple groups. Then*

$$\mathrm{Aut}(S) \cong \Big(\mathrm{Aut}(P_1) \times \mathrm{Aut}(P_2) \times \cdots \times \mathrm{Aut}(P_t)\Big) \rtimes S_t.$$

In particular, $|\mathrm{Aut}(S)| = \prod_{i=1}^{t} |\mathrm{Aut}(P_i)| \cdot t!$.

Lemma 2.8. [17] *Let S be a simple group such that $\pi(S) \subseteq \{2,3,5,7,13,31\}$. Then S is isomorphic to one of the simple groups listed in Table 5.*

Table 5. The simple groups S with $\pi(S) \subseteq \{2,3,5,7,13,31\}$.

S	$\lvert S\rvert$	$\lvert\mathrm{Out}(S)\rvert$	S	$\lvert S\rvert$	$\lvert\mathrm{Out}(S)\rvert$
A_5	$2^2 \cdot 3 \cdot 5$	2	$L_2(2^6)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	6
A_6	$2^3 \cdot 3^2 \cdot 5$	4	$L_3(3)$	$2^4 \cdot 3^3 \cdot 13$	2
$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	2	$L_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 13$	4
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$B_3(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	$O_8^+(3)$	$2^{12} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13$	24
A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2	$G_2(3)$	$2^6 \cdot 3^6 \cdot 7 \cdot 13$	2
A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2	$C_3(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2
$B_3(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1	$L_3(3^2)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$	4
$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	6	$L_2(3^3)$	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	6
$L_3(2^2)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	12	$U_4(5)$	$2^7 \cdot 3^4 \cdot 5^6 \cdot 7 \cdot 13$	4
$L_2(2^3)$	$2^3 \cdot 3^2 \cdot 7$	3	$B_2(5)$	$2^6 \cdot 3^2 \cdot 5^4 \cdot 13$	2
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	$L_2(5^2)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	4
$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	8	$L_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	2
$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	6	$L_5(2)$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$	2
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	$L_6(2)$	$2^{15} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31$	2
$B_2(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2	$L_3(5)$	$2^5 \cdot 3 \cdot 5^3 \cdot 31$	2
$L_2(7^2)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	4	$L_4(5)$	$2^7 \cdot 3^2 \cdot 5^6 \cdot 13 \cdot 31$	8
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2	$B_3(5)$	$2^9 \cdot 3^4 \cdot 5^9 \cdot 7 \cdot 13 \cdot 31$	2
${}^3D_4(2)$	$2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$	3	$C_3(5)$	$2^9 \cdot 3^4 \cdot 5^9 \cdot 7 \cdot 13 \cdot 31$	2
${}^2F_4(2)'$	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$	2	$O_8^+(5)$	$2^{12} \cdot 3^5 \cdot 5^{12} \cdot 7 \cdot 13^2 \cdot 31$	24
$U_3(2^2)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	4	$G_2(5)$	$2^6 \cdot 3^3 \cdot 5^6 \cdot 7 \cdot 31$	1
$G_2(2^2)$	$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$	2	$L_3(5^2)$	$2^7 \cdot 3^2 \cdot 5^6 \cdot 7 \cdot 13 \cdot 31$	12
$B_2(2^3)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$	6	$L_2(5^3)$	$2^2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 31$	6
$\mathrm{Sz}(2^3)$	$2^6 \cdot 5 \cdot 7 \cdot 13$	3	$L_2(31)$	$2^5 \cdot 3 \cdot 5 \cdot 31$	2

3. Proof of theorems

Proof of Theorem 1.1. Let p be an odd prime such that $|\pi\big((3^p-1)/2\big)| = 1$, and let M be one of the finite simple groups of Lie type $B_p(3)$ or $C_p(3)$. Assume that G is a finite group such that $|G| = |M|$ and $\mathrm{D}(G) = \mathrm{D}(M)$. We recall that $s(M) = 2$ and $\pi(M) = \pi_1(M) \cup \pi\Big((3^p-1)/2\Big)$. By our hypothesis, it is easy to see that

$$\pi_2(G) = \pi_2(M) = \pi\Big(\frac{3^p-1}{2}\Big) \quad \text{and} \quad \pi(G) = \pi_1(M) \cup \pi\Big(\frac{3^p-1}{2}\Big).$$

Moreover, it follows from Corollary 2.2(2) that $\deg(2) = |\pi_1(M)| - 1$, and so $s(G) = 2$ and $\pi_1(G) = \pi_1(M)$. Therefore, we deduce that $\mathrm{OC}(G) = \mathrm{OC}(M)$. Hence $h_{\mathrm{OD}}(M) \leq h_{\mathrm{OC}}(M)$. Now, from Corollary 2.1(b) and Lemma 2.6, we conclude that $h_{\mathrm{OD}}(G) = 2$, as desired. ∎

Proof of Theorem 1.2. Let q be a power of a prime and $n = 2^m \geq 2$. Suppose $|\pi(q^n + 1/(2, q-1))| = 1$ and $(n, q) \notin \{(2,3), (2,4), (2,5)\}$. Let M be one of the finite simple groups of Lie type $B_n(q)$ or $C_n(q)$, and let G be a finite group such that $|G| = |M|$ and $\mathrm{D}(G) = \mathrm{D}(M)$. Similar arguments as proof of Theorem 1.1, show that $s(M) = 2$ and $\pi(M) = \pi_1(M) \cup \pi\,(q^n + 1/(2, q-1))$. In addition, it is easy to see that

$$\pi_2(G) = \pi_2(M) = \pi\left(\frac{q^n+1}{(2,q-1)}\right) \quad \text{and} \quad \pi(G) = \pi_1(M) \cup \pi\Big(\frac{q^n+1}{(2,q-1)}\Big).$$

Furthermore, it follows from Corollary 2.2(1) that $\deg(2) = |\pi_1(M)| - 1$, and so $s(G) = 2$ and $\pi_1(G) = \pi_1(M)$. Now, we conclude that $\mathrm{OC}(G) = \mathrm{OC}(M)$, and hence $h_{\mathrm{OD}}(M) \leq h_{\mathrm{OC}}(M)$. Suppose first that q is even. Then by Lemma 2.6, we have $h_{\mathrm{OC}}(M) = 1$, which implies that $h_{\mathrm{OD}}(M) = 1$. Suppose next that q is odd. Again, by Lemma 2.6, we see that for $n = 2$, $h_{\mathrm{OC}}(M) = 1$ and for $n \geq 4$, $h_{\mathrm{OC}}(M) = 2$. Now, it is easy to see that in both cases we have $h_{\mathrm{OD}}(M) = h_{\mathrm{OC}}(M)$, as required.

Now, assume that $(n, q) \in \{(2,4), (2,5)\}$. In both cases, we have $|M| < 10^8$ and by a result in [23], we conclude that $h_{\mathrm{OD}}(M) = 1$. ∎

Proof of Theorem 1.3. Let $M \in \{B_3(5), C_3(5)\}$. Suppose G is a finite group, such that

$$|G| = |M| = 2^9 \cdot 3^4 \cdot 5^9 \cdot 7 \cdot 13 \cdot 31 \quad \text{and} \quad \mathrm{D}(G) = \mathrm{D}(M) = (4,4,3,1,3,1).$$

We have to show that G is isomorphic to $B_3(5)$ or $C_3(5)$. It is evident that the prime graph of G is connected, since $\deg(2) = \deg(3) = 4$. Moreover, by hypothesis, we immediately conclude that the only possibilities for the prime graph $\mathrm{GK}(G)$ of G are:

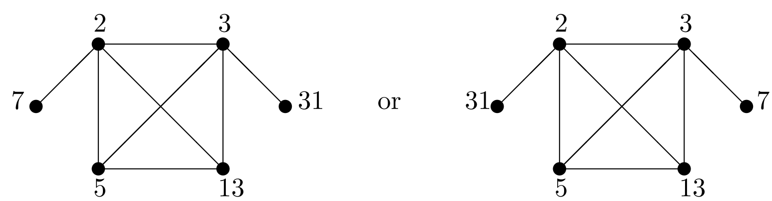

Figure 2. Prime graph $\mathrm{GK}(G)$ of G.

Therefore, we conclude that $\{2,3,5,6,7,10,13,15,26,39,65\} \subseteq \omega(G)$, and the subsets $\{5,7,31\}$ and $\{7,13,31\}$ of vertices are independent sets of $\mathrm{GK}(G)$. In the sequel, we break up the proof into a sequence of lemmas. Let K be the maximal normal solvable subgroup of G.

Lemma 3.1. *K is a $\{2,3,5\}$-group. In particular, G is non-solvable.*

Proof. First, we show that K is a $31'$-group. Assume the contrary and let 31 divides the order of K. In this case K possesses an element x of order 31. We set $C := C_G(x)$ and $N := N_G(\langle x \rangle)$. By the structure of $\mathrm{D}(G)$, it follows that C is a $\{p, 31\}$-group where $p \in \{2, 3\}$. Now using (N/C)-Theorem the factor group N/C is embedded in $\mathrm{Aut}(\langle x \rangle) \cong \mathbb{Z}_{30}$. Hence, N is a $\{2, 3, 5, 31\}$-group. Now, by Frattini argument $G = KN$. This implies that $\{7, 13\} \subseteq \pi(K)$. Since K is solvable, it possesses a Hall $\{7, 13\}$-subgroup L of order $7 \cdot 13$. Clearly L is cyclic and hence $7 \sim 13$, which is a contradiction.

Next, we show that K is a p'-group for $p \in \{7, 13\}$. Let $p \in \pi(K)$, $K_p \in \mathrm{Syl}_p(K)$ and $N = N_G(K_p)$. Again, by Frattini argument $G = KN$ and hence 31 divides the order of N. Let L be a subgroup of N of order 31. Since L normalizes K_p, G contains a subgroup of order $31 \cdot p$ and this leads to a contradiction as before, since $p \nmid 31 - 1$. Therefore K is a $\{2, 3, 5\}$-group.

In addition, since $K \neq G$, it follows that G is non-solvable. This completes the proof. ∎

Lemma 3.2. *The factor group G/K is an almost simple group. In fact, $S \leq G/K \leq \mathrm{Aut}(S)$ where $S \in \{B_3(5), C_3(5)\}$.*

Proof. Let $H := G/K$ and $S := \mathrm{soc}(H)$. Evidently, $S = P_1 \times P_2 \times \cdots \times P_m$, where P_i's are non-Abelian simple groups. This implies that $Z(S) = 1$, or equivalently $C_H(S) \cap S = 1$. But then $C_H(S) = 1$, since otherwise $C_H(S)$ would contain minimal normal subgroups of H disjoint from S, which is a contradiction. Consequently, we get

$$G/K \cong \frac{N_H(S)}{C_H(S)} \hookrightarrow \mathrm{Aut}(S).$$

In what follows, we will show that $m = 1$ and $P_1 \cong B_3(5)$ or $C_3(5)$.

Suppose that $m \geq 2$. In this case, it is easy to see that $\{7, 31\} \cap \pi(S) = \emptyset$, since otherwise $\deg(7) \geq 2$ or $\deg(31) \geq 2$, which is a contradiction. Hence, for every i we have $\max \pi(P_i) = 13$. On the other hand, by Lemma 3.1, we observe that $31 \in \pi(H) \subseteq \pi(\mathrm{Aut}(S))$. Thus, we may assume that 31 divides the order of $\mathrm{Out}(S)$. But

$$\mathrm{Out}(S) = \mathrm{Out}(S_1) \times \cdots \times \mathrm{Out}(S_r),$$

where the groups S_j are direct products of isomorphic P_i's such that

$$S \cong S_1 \times \cdots \times S_r.$$

Therefore, for some j, 31 divides the order of an outer automorphism group of a direct product S_j of t isomorphic simple groups P_i. Since $\max \pi(P_i) = 13$, it follows that $|\mathrm{Out}(P_i)|$ is not divisible by 31, see [17, Table 4]. Now, by Lemma 2.7, we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^t \cdot t!$. Therefore, $t \geq 31$ and so 2^{62} must divide the order of G, which is a contradiction. Therefore $m = 1$ and $S = P_1$.

Now, from Lemma 3.1, we easily conclude that

$$|S| = 2^a \cdot 3^b \cdot 5^c \cdot 7 \cdot 13 \cdot 31,$$

where $2 \leq a \leq 9$, $0 \leq b \leq 4$ and $0 \leq c \leq 9$. Using collected results contained in Table 5, we deduce that $S \cong B_3(5), C_3(5)$ or $L_3(5^2)$. If $S \cong L_3(5^2)$, then $7 \cdot 31 \in \omega(S)$ (see [5]), which is a contradiction. This completes the proof. ∎

Lemma 3.3. *G is isomorphic to $B_3(5)$ or $C_3(5)$.*

Proof. By Lemma 3.2, $M \leq G/K \leq \mathrm{Aut}(M)$, which implies that $G/K \cong M$ or $\mathrm{Aut}(M)$. In the case that $G/K \cong M$, by order consideration we deduce that $|K| = 1$ and $G \cong M$, as desired. In the latter case, we have $|K| = 2$ and so $K \leq Z(G)$. But then, we obtain $\deg(2) = 5$, which is a contradiction. This proves the lemma and the theorem. ∎

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