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On Lower Semi-Continuity of Interval-Valued Multihomomorphisms

S. PIANSKOOL, P. UDOMKAVANICH AND P. YOUNGKHONG
Department of Mathematics and Computer Science, Faculty of Science
Chulalongkorn University, Bangkok 10330, Thailand
sajee.p@chula.ac.th, pattanee.u@chula.ac.th, pyoungkhong@yahoo.com

Abstract. It is well known that if f is a continuous homomorphism on $(\mathbb{R}, +)$, then there exists a constant $c \in \mathbb{R}$ such that $f(x) = cx$ for all $x \in \mathbb{R}$. Termwuttipong *et al.* extended this result to interval-valued multifunctions on $\mathbb{R}$. They proved that an interval-valued multifunction f on $\mathbb{R}$ is an upper semi-continuous multihomomorphism on $(\mathbb{R}, +)$ if and only if f has one of the following forms : $f(x) = \{cx\}, f(x) = \mathbb{R}, f(x) = (0, \infty), f(x) = (-\infty, 0), f(x) = [\,cx, \infty), f(x) = (-\infty, cx\,]$ where c is a constant in $\mathbb{R}$. In this paper, we extend the above well known result by considering lower semi-continuity. It is shown that an interval-valued multifunction f on $\mathbb{R}$ is a lower semi-continuous multihomomorphism on $(\mathbb{R}, +)$ if and only if f is one of the following: $f(x) = \{cx\}, f(x) = \mathbb{R}, f(x) = (cx, \infty), f(x) = (-\infty, cx), f(x) = [\,cx, \infty), f(x) = (-\infty, cx\,]$ where c is a constant in $\mathbb{R}$.

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1. Introduction

By a *multifunction* from a nonempty set X into a nonempty set Y we mean $f : X \to \mathcal{P}(Y) \smallsetminus \{\emptyset\}$ where $\mathcal{P}(Y)$ is the power set of Y. A *multifunction on* X is a multifunction from X into itself.

A multifunction f from a group G into a group G' is a *multihomomorphism* if

$$f(xy) = f(x)f(y) \; (= \{st \mid s \in f(x) \text{ and } t \in f(y)\}) \text{ for all } x, y \in G.$$

The concept of multi-valued endomorphisms appeared in [1, p.176] is more general than the one given here. In [5], multihomomorphisms between cyclic groups were characterized. These results were used in [2] in order to characterize surjective multihomomorphisms between cyclic groups. In [6], some necessary conditions of multihomomorphisms from any group into groups of real numbers under the usual addition and multiplication were given.

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A multifunction from a topological space X into a topological space Y is said to be *upper semi-continuous* at $a \in X$ if for any open set V in Y such that $f(a) \subseteq V$, there exists an open set U in X such that $a \in U$ and $f(U) \subseteq V$. Such a multifunction f is called *lower semi-continuous* at $a \in X$ if for any open set V in Y such that $V \cap f(a) \neq \emptyset$, there exists an open set U in X such that $a \in U$ and $f(x) \cap V \neq \emptyset$ for all $x \in U$. If f is both upper and lower semi-continuous at $a \in X$, f is called *continuous* at a. If f is upper semi-continuous [lower semi-continuous, continuous] at every point of X, we call f *upper semi-continuous* [*lower semi-continuous, continuous*] *on* X. (See [3, p. 261]). The upper semi-continuity, lower semi-continuity and continuity of a single-valued functions coincide obviously.

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{Q}$ the set of rational numbers and $\mathbb{N}$ the set of natural numbers (positive integers). By an *interval-valued multifunction* on $\mathbb{R}$ we mean a multifunction f on $\mathbb{R}$ such that $f(x)$ is an interval in $\mathbb{R}$. Evidently, interval-valued multihomomorphisms on $(\mathbb{R},+)$ generalize homomorphisms on $(\mathbb{R},+)$. It is well known that if f is a continuous homomorphism on $(\mathbb{R},+)$, then there is a constant $c \in \mathbb{R}$ such that $f(x) = cx$ for all $x \in \mathbb{R}$. This result was extended in [4] to interval-valued multifunctions on $\mathbb{R}$ as follows:

Theorem 1.1. [4] *Let f be an interval-valued multifunction on $\mathbb{R}$. Then f is an upper semi-continuous multihomomorphism on $(\mathbb{R},+)$ if and only if f is one of the following:*

(i) *There is a constant $c \in \mathbb{R}$ such that $f(x) = \{cx\}$ for all $x \in \mathbb{R}$.*
(ii) *$f(x) = \mathbb{R}$ for all $x \in \mathbb{R}$.*
(iii) *$f(x) = (0,\infty)$ for all $x \in \mathbb{R}$.*
(iv) *$f(x) = (-\infty,0)$ for all $x \in \mathbb{R}$.*
(v) *There is a constant $c \in \mathbb{R}$ such that $f(x) = [\,cx,\infty)$ for all $x \in \mathbb{R}$.*
(vi) *There is a constant $c \in \mathbb{R}$ such that $f(x) = (-\infty, cx\,]$ for all $x \in \mathbb{R}$.*

This result motivates us to extend the above well known result by considering lower semi-continuous interval-valued multihomomorphisms on $\mathbb{R}$. We characterize lower semi-continuous interval-valued multihomomorphisms on $(\mathbb{R},+)$. This characterization indicates that every upper semi-continuous multihomomorphism on $(\mathbb{R},+)$ is continuous.

The following results given in [4] are needed.

Lemma 1.1. [4] *If f is an interval-valued multihomomorphism on $(\mathbb{R},+)$, then $f(0)$ is one of the following: $\{0\}, \mathbb{R}, (0,\infty), (-\infty,0), [\,0,\infty), (-\infty,0\,]$.*

Lemma 1.2. [4] *If f is an interval-valued multihomomorphism on $(\mathbb{R},+)$, then for every $x \in \mathbb{R}, f(x)$ and $f(0)$ are intervals in $\mathbb{R}$ of the same form, that is,*

$$f(x) = \begin{cases} \{y\} & \text{if } f(0) = \{0\}, \\ \mathbb{R} & \text{if } f(0) = \mathbb{R}, \\ (y,\infty) & \text{if } f(0) = (0,\infty), \\ (-\infty,y) & \text{if } f(0) = (-\infty,0), \\ [\,y,\infty) & \text{if } f(0) = [\,0,\infty), \\ (-\infty,y\,] & \text{if } f(0) = (-\infty,0\,], \end{cases}$$

for some $y \in \mathbb{R}$.

Lemma 1.3. [4] *Let f be an interval-valued multihomomorphism on $(\mathbb{R},+)$. If $x,y \in \mathbb{R}$ are such that $f(x) = (y,\infty),(-\infty,y),[\,y,\infty)$ or $(-\infty,y\,]$, then $f(-x) = (-y,\infty),(-\infty,-y),[-y,\infty)$ or $(-\infty,-y\,]$, respectively.*

Lemma 1.4. [4] *If f is an interval-valued multihomomorphism on $(\mathbb{R},+)$, then for all $x \in \mathbb{R}$ and $m,n \in \mathbb{N}$, $f\big((m/n)x\big) = (m/n)f(x)$.*

2. Main results

The following lemmas are also needed to obtain the main result.

Lemma 2.1. *Let f be an interval-valued multihomomorphism on $(\mathbb{R},+)$ and $c \in \mathbb{R}$.*

(i) *If $f(1) = (c,\infty)$, then for all $q \in \mathbb{Q}$, $f(q) = (cq,\infty)$.*
(ii) *If $f(1) = (-\infty,c)$, then for all $q \in \mathbb{Q}$, $f(q) = (-\infty,cq)$.*
(iii) *If $f(1) = [\,c,\infty)$, then for all $q \in \mathbb{Q}$, $f(q) = [\,cq,\infty)$.*
(iv) *If $f(1) = (-\infty,c\,]$, then for all $q \in \mathbb{Q}$, $f(q) = (-\infty,cq\,]$.*

Proof. (i) By Lemma 1.2, $f(0) = (0,\infty) = (c0,\infty)$. If $q \in \mathbb{Q}$ is such that $q > 0$, then by Lemma 1.3 and Lemma 1.4,

$$f(q) = f(q1) = qf(1) = q(c,\infty) = (cq,\infty),$$
$$f(-q) = f(q(-1)) = qf(-1) = q(-c,\infty) = (-cq,\infty) = (c(-q),\infty).$$

Therefore (i) is proved. The results (ii)–(iv) can be proved analogously. ∎

Lemma 2.2. *Let f be an interval-valued multihomomorphism on $(\mathbb{R},+)$. If f is lower semi-continuous at 0, then f is lower semi-continuous on $\mathbb{R}$.*

Proof. If $f(0) = \{0\}$, by Lemma 1.2, f is a homomorphism which is continuous at 0. It follows obviously that f is continuous on $\mathbb{R}$. Also, if $f(0) = \mathbb{R}$, then by Lemma 1.2, $f(x) = \mathbb{R}$ for all $x \in \mathbb{R}$ and hence f is lower semi-continuous on $\mathbb{R}$.

Next, assume that $f(0) = (0,\infty)$. Let $x \in \mathbb{R}$. By Lemma 1.2, $f(x) = (y,\infty)$ for some $y \in \mathbb{R}$. By Lemma 1.3, $f(-x) = (-y,\infty)$. Let V be an open set in $\mathbb{R}$ such that $V \cap (y,\infty) \neq \emptyset$. Then $V - y$ is an open set in $\mathbb{R}$ and $(V-y) \cap (0,\infty) \neq \emptyset$. Since f is lower semi-continuous at 0, there exists an open set U in $\mathbb{R}$ such that $0 \in U$ and $f(z) \cap (V-y) \neq \emptyset$ for all $z \in U$. Thus $U+x$ is an open set in $\mathbb{R}$ such that $x \in U+x$. Claim that $f(w) \cap V \neq \emptyset$ for all $w \in U+x$. Let $w \in U+x$. Then $w-x \in U$, so $f(w-x) \cap (V-y) \neq \emptyset$. Let $a \in f(w-x) \cap (V-y)$. Then $a+y \in V$. It follows from

$$a \in f(w-x) = f(w) + f(-x) = f(w) + (-y,\infty)$$

that

$$a + y \in f(w) + (0,\infty) = f(w) + f(0) = f(w).$$

Therefore $a+y \in f(w) \cap V$. Hence $f(w) \cap V \neq \emptyset$. This shows that f is lower semi-continuous at x.

By a similar argument, we can show that f is lower semi-continuous on $\mathbb{R}$ for the cases that $f(0) = (-\infty,0)$, $f(0) = [\,0,\infty)$ and $f(0) = (-\infty,0\,]$. ∎

Lemma 2.3. *For $c \in \mathbb{R}$, the following interval-valued multifunctions are lower semi-continuous multihomomorphisms on $(\mathbb{R},+)$.*

(i) *$f(x) = (cx,\infty)$ for all $x \in \mathbb{R}$.*

(ii) $f(x) = (-\infty, cx)$ *for all* $x \in \mathbb{R}$.
(iii) $f(x) = [\,cx, \infty)$ *for all* $x \in \mathbb{R}$.
(iv) $f(x) = (-\infty, cx\,]$ *for all* $x \in \mathbb{R}$.

Proof. Evidently, f in (i)–(iv) is a multihomomorphism on $(\mathbb{R}, +)$ and if $c = 0$, then it is lower semi-continuous on $\mathbb{R}$. Assume that $c \neq 0$.
(i)We have that $f(0) = (c0, \infty) = (0, \infty)$. Let V be an open set such that $V \cap (0, \infty) \neq \emptyset$. Let $a \in V \cap (0, \infty)$. Then $(a-\epsilon, a+\epsilon) \subseteq V$ for some $\epsilon > 0$. Thus $(-\epsilon, \epsilon) \subseteq V - a$. Let $x \in (-\epsilon/|c|, \epsilon/|c|)$. It follows that $cx \in (-\epsilon, \epsilon)$, and so $(cx, \infty) \cap (-\epsilon, \epsilon) \neq \emptyset$. Hence $(cx, \infty) \cap (V - a) \neq \emptyset$. Let $b \in (cx, \infty) \cap (V - a)$. Then $b + a \in V$ and $b + a \in (cx, \infty) + (0, \infty) = (cx, \infty)$. Therefore $b + a \in f(x) \cap V$. This shows that $f(x) \cap V \neq \emptyset$ for all $x \in (-\epsilon/|c|, \epsilon/|c|)$. Thus f is lower semi-continuous at 0. It follows from Lemma 2.2 that f is lower semi-continuous on $\mathbb{R}$. The results (ii)–(iv) can be proved analogously. ∎

Theorem 2.1. *Let f be an interval-valued multifunction on $\mathbb{R}$. Then f is a lower semi-continuous multihomomorphism on $(\mathbb{R}, +)$ if and only if f is one of the following:*

(i) *There is a constant* $c \in \mathbb{R}$ *such that* $f(x) = \{cx\}$ *for all* $x \in \mathbb{R}$.
(ii) $f(x) = \mathbb{R}$ *for all* $x \in \mathbb{R}$.
(iii) *There is a constant* $c \in \mathbb{R}$ *such that* $f(x) = (cx, \infty)$ *for all* $x \in \mathbb{R}$.
(iv) *There is a constant* $c \in \mathbb{R}$ *such that* $f(x) = (-\infty, cx)$ *for all* $x \in \mathbb{R}$.
(v) *There is a constant* $c \in \mathbb{R}$ *such that* $f(x) = [\,cx, \infty)$ *for all* $x \in \mathbb{R}$.
(vi) *There is a constant* $c \in \mathbb{R}$ *such that* $f(x) = (-\infty, cx\,]$ *for all* $x \in \mathbb{R}$.

Proof. Assume that f is a lower semi-continuous multihomomorphism on $(\mathbb{R}, +)$. By Lemma 1.1, $f(0)$ is one of $\{0\}, \mathbb{R}, (0, \infty), (-\infty, 0), [\,0, \infty)$ and $(-\infty, 0\,]$.

Case 1: $f(0) = \{0\}$. It follows from Lemma 1.2 that f is a continuous homomorphism on $(\mathbb{R}, +)$. Hence f satisfies (i).

Case 2: $f(0) = \mathbb{R}$. By Lemma 1.2, f satisfies (ii).

Case 3: $f(0) = (0, \infty)$. From Lemma 1.2, $f(1) = (c, \infty)$ for some $c \in \mathbb{R}$. By Lemma 2.1(i),

$$f(q) = (cq, \infty) \quad \text{for all} \quad q \in \mathbb{Q}. \tag{2.1}$$

Let $x \in \mathbb{R}$. By Lemma 1.2, $f(x) = (y, \infty)$ for some $y \in \mathbb{R}$. Then for each $n \in \mathbb{N}$, $\big(y - 1/n, y + 1/n\big)$ is an open set such that $f(x) \cap (y - 1/n, y + 1/n) \neq \emptyset$. Since f is lower semi-continuous at x, we deduce that

$$\begin{array}{c} \text{for every } n \in \mathbb{N}, \text{ there is a } \delta_n > 0 \text{ such that } \delta_n < 1/n \text{ and} \\ f(z) \cap (y - 1/n, y + 1/n) \neq \emptyset \text{ for all } z \in (x - \delta_n, x + \delta_n). \end{array} \tag{2.2}$$

For each $n \in \mathbb{N}$, let $q_n \in \mathbb{Q}$ be such that $q_n \in (x - \delta_n, x + \delta_n)$. From (2.1), $f(q_n) = (cq_n, \infty)$ for all $n \in \mathbb{N}$. It follows from (2.2) that

$$(cq_n, \infty) \cap \big(y - \frac{1}{n}, y + \frac{1}{n}\big) \neq \emptyset$$

for all $n \in \mathbb{N}$. This implies that $cq_n < y + 1/n$ for all $n \in \mathbb{N}$. Hence

$$cq_n - y < \frac{1}{n} \text{ for all } n \in \mathbb{N}. \tag{2.3}$$

Let $n \in \mathbb{N}$. Since $q_n \in (x - \delta_n, x + \delta_n)$, we have $-\delta_n < x - q_n < \delta_n$, so $2x - q_n \in (x - \delta_n, x + \delta_n)$. By (2.2), $f\big(2x - q_n\big) \cap \big(y - 1/n, y + 1/n\big) \neq \emptyset$. But

$$\begin{aligned} f(2x - q_n) &= f(2x) + f(-q_n) \\ &= 2f(x) + f(-q_n) \quad \text{(from Lemma 1.4)} \\ &= 2(y, \infty) + (-cq_n, \infty) \quad \text{(from (2.1) and Lemma 1.3)} \\ &= (2y - cq_n, \infty), \end{aligned}$$

so $(2y - cq_n, \infty) \cap \big(y - 1/n, y + 1/n\big) \neq \emptyset$. This implies that $2y - cq_n < y + 1/n$. Thus $y - cq_n < 1/n$. This proves that

$$y - cq_n < \frac{1}{n} \text{ for all } n \in \mathbb{N}. \tag{2.4}$$

It is immediate from (2.3) and (2.4), that $|\, y - cq_n| < 1/n$ for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} cq_n = y$. Since $|\, q_n - x| < \delta_n < 1/n$ for all $n \in \mathbb{N}$, we have $\lim_{n\to\infty} q_n = x$. Consequently,

$$y = \lim_{n\to\infty} cq_n = c \lim_{n\to\infty} q_n = cx.$$

This proves that $f(x) = (cx, \infty)$, as desired.

Case 4 : $f(0) = (-\infty, 0)$. By Lemma 1.2, $f(1) = (-\infty, c)$ for some $c \in \mathbb{R}$. It follows from Lemma 2.1(ii) that

$$f(q) = (-\infty, cq) \text{ for all } q \in \mathbb{Q}. \tag{2.5}$$

If $x \in \mathbb{R}$, then by Lemma 1.2, $f(x) = (-\infty, y)$ for some $y \in \mathbb{R}$. Then $f(x) \cap (y - 1/n, y + 1/n) \neq \emptyset$ for all $n \in \mathbb{N}$. Since f is lower-semi-continuous we have that

$$\begin{aligned} &\text{for every } n \in \mathbb{N}, \text{ there is a } \delta_n > 0 \text{ such that } \delta_n < 1/n \text{ and} \\ &f(z) \cap (y - 1/n, y + 1/n) \neq \emptyset \text{ for all } z \in (x - \delta_n, x + \delta_n). \end{aligned} \tag{2.6}$$

For each $n \in \mathbb{N}$, let $q_n \in \mathbb{Q} \cap (x - \delta_n, x + \delta_n)$. From (2.5), $f(q_n) = (-\infty, cq_n)$ for all $n \in \mathbb{N}$ and from (2.6), we have

$$(-\infty, cq_n) \cap \left(y - \frac{1}{n}, y + \frac{1}{n}\right) \neq \emptyset \text{ for all } n \in \mathbb{N}.$$

It follows that $y - 1/n < cq_n$ for all $n \in \mathbb{N}$. Thus

$$y - cq_n < \frac{1}{n} \text{ for all } n \in \mathbb{N}. \tag{2.7}$$

If $n \in \mathbb{N}$, then $2x - q_n \in (x - \delta_n, x + \delta_n)$. By (2.6), $f(2x - q_n) \cap \big(y - 1/n, y + 1/n\big) \neq \emptyset$. From (2.5), Lemma 1.3 and Lemma 1.4, we have

$$f(2x - q_n) = 2f(x) + f(-q_n) = 2(-\infty, y) + (-\infty, -cq_n) = (-\infty, 2y - cq_n).$$

This implies that $y - 1/n < 2y - cq_n$, and so $cq_n - y < 1/n$. Hence

$$cq_n - y < \frac{1}{n} \text{ for all } n \in \mathbb{N}. \tag{2.8}$$

It follows from (2.7) and (2.8) that $\lim_{n\to\infty} cq_n = y$. Also, from choosing q_n, we have that $\lim_{n\to\infty} q_n = x$. Consequently, $y = cx$ and therefore $f(x) = (-\infty, cx)$, as desired.

Case 5 : $f(0) = [0, \infty)$. By Lemma 1.2, $f(1) = [c, \infty)$ for some $c \in \mathbb{R}$. We can show similarly to Case 4 by Lemma 2.1(iii), Lemma 1.3 and Lemma 1.4 that $f(x) = [cx, -\infty)$ for all $x \in \mathbb{R}$.

Case 6 : $f(0) = (-\infty, 0]$. By Lemma 1.2, $f(1) = (-\infty, c]$ for some $c \in \mathbb{R}$. It can be shown similarly to Case 5 by Lemma 2.1(iv), Lemma 1.3 and Lemma 1.4 that $f(x) = (-\infty, cx]$ for all $x \in \mathbb{R}$.

Conversely, assume that f satisfies one of (i)–(iv). If f satisfies (i), then f is a continuous homomorphism. Evidently, if f satisfies (ii), then f is a lower semi-continuous multihomomorphism on $(\mathbb{R}, +)$. By Lemma 2.3, f is a lower semi-continuous multihomomorphism on $(\mathbb{R}, +)$ if f satisfies one of (iii)–(vi). Hence the proof is completed. ∎

The following results follow directly from Theorem 1.1 and Theorem 2.1.

Corollary 2.1. *Every upper semi-continuous interval-valued multihomomorphism is continuous.*

Corollary 2.2. *All the lower semi-continuous interval-valued multihomomorphisms which are not continuous are the multifunctions of the following forms:*

$$f(x) = (cx, \infty) \text{ and } g(x) = (-\infty, cx)$$

where c is a nonzero constant in $\mathbb{R}$.

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