A Production and Repair Model under a Time-Varying Demand Process

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Abstract. In this paper, we propose a model for a production system that satisfies a continuous time-varying demand for a finished product over a known and finite planning horizon by supplying both new and repaired items. New items are fabricated from a single type of raw material procured from external suppliers, while used items are collected from the customers and then repaired to an 'as new' condition before being sold again. For simplicity, we assume that there is no collection of used items during the repair uptime and downtime periods. The problem is to determine a joint policy for raw materials procurement, new items fabrication and used items repair such that the total relevant cost of the model is minimized. We propose a numerical solution procedure and we illustrate the model with some numerical examples and a simple sensitivity analysis.

1. Introduction

In many production systems, the producer procures raw materials from external suppliers and then processes them into finished products. When raw materials are used in production, their ordering quantities are dependent on the production batch size of the finished product. Therefore, it is often desirable to consider the batch size of the finished product and the ordering quantities of the associated raw materials together by treating production and procurement in a single model. Omar and Smith [18] developed such a model that is subject to a linearly increasing time-varying demand process.

Besides fabricating the finished product from raw materials, it may be possible to reuse used products collected from the customers. Reuse of products and materials

is not a new phenomenon. Metal scrap brokers, wastepaper recycling and deposit systems for soft drink bottles are all examples that have been around for a long time. In these cases, recovery of the used products is economically more attractive than disposal. Furthermore, in the recent past, the growth of environmental concerns has given 'reuse' increasing attention [6].

In literature, extensive study has been devoted to reuse models under a constant demand process. As far as we know, the first EOQ reuse model was proposed by Schrady [23]. He assumes fixed demand and return rates, fixed lead times for external order and internal repair, infinite procurement and repair rates, and disallows shortages. His formulation treats the serviceable and recoverable inventories as interdependent parts of a total system, and jointly determines the optimal order and repair quantities using expressions derived similarly to the classical EOQ formula. He proposes policies that alternate one procurement batch with a fixed number R of repair batches (or $(1,R)$ policies for short). Nahmias and Rivera [17] extended this model by considering the case of finite repair rate. They assume that the repair rate is greater than the demand rate. Mabini *et al.* [16] also extended Schrady's model, but their formulation allows shortages to occur. Furthermore, they consider a multi-item system where items share the same repair facility. For their extended models, they propose numerical solution methods. Next, Koh *et al.* [10] proposed control policies for a joint EOQ and EPQ model where two cases were investigated: Multiple order setups (P) for a single recycling setup and vice-versa (in other words, $(P,1)$ and $(1,R)$ policies). They also assume infinite production rate and finite repair rate. However, their study is more general than that of Nahmias and Rivera [17], since they allow the repair rate to be both smaller and greater than the demand rate. For the four possible policy combinations, they derive a closed-form expression for the average total cost which is then used to determine the optimal batch sizes numerically. Konstantaras and Papachristos [12] obtained closed form expressions for both the optimal recovery and optimal ordering setup numbers for the model in Koh *et al.* [10]. Teunter [24] further generalized earlier works by deriving lot-sizing formulae for $(P,1)$ and $(1,R)$ policies under finite production and repair rates. Their results are obtained in a graphical manner, thus avoiding the tedious mathematics in previous works. Konstantaras and Papachristos [13] rectified the approximate nature of the solution algorithm in Teunter [24] by proposing an exact method.

The authors above assumed that all returned items are reuseable. However, Richter [20, 19] studied a two-shop EOQ waste disposal model in which the first shop provides products to the second shop through production of new items and repair of used items, while the second shop uses the products, stocks a portion of the used items, and disposes of the remaining portion as waste. At the end of each cycle, the stocked used items are transferred back to the first shop to be repaired. The author assumes infinite production and repair rates, and n production setups and m repair setups during each cycle. He formulates a EOQ-related cost function and a joint EOQ-related and non-EOQ-related cost function, and analyzes them separately. Later, Richter [21] extended the cost analysis of his earlier work to show an extremal property: The pure strategy of total repair or the pure strategy of total disposal (total production) dominates any mixed strategy of repair and disposal. Next, Richter and Dobos [22, 2] extended the aforementioned works by considering

part of the problem as an integer programming problem to secure the integrity of the setup numbers. They also obtain similar findings with respect to the dominance of pure strategies. After that, Dobos and Richter [3] relaxed the assumption that the production and repair rates are infinite but considered only one production and one repair setup per cycle. Subsequently, Dobos and Richter [4] generalized their former work by considering multiple production and repair setups per cycle. These works reiterated the dominance of pure strategies from an economical standpoint.

The question of the feasibility of the implementation of pure strategies leads to Dobos and Richter [5] extending their 2004 waste disposal model by considering the quality of returned items, i.e. not all returned used items are suitable for recycling. They put forth the following question: Who should control the quality of returned used items? The authors examine two strategies: (1) The producer repurchase all used items and then reuse a maximal portion. (2) The producer repurchase only the serviceable used items and then decide how much to reuse. The authors assume that the serviceable portion of used items is known. Before minimizing the total cost, they assume that a pure recycling strategy is more economical than a pure production strategy. By minimizing the EOQ-related cost, they discover that Strategy 1 is more economical than Strategy 2. This translates to the producer's optimal policy being to conduct quality control inhouse after repurchasing all used items. In contrast, when the authors minimize the joint EOQ-related and non-EOQ-related cost, they discover that Strategy 2 is more economical than Strategy 1. So, the producer's optimal policy now is to outsource quality control and repurchase only good used items. Later, Jaber and El Saadany [9] extended Dobos and Richter's 2006 model by considering a variable return rate $R(p, q)$ as being dependent on the purchasing prince p and acceptance quality level q of the returned items. They show numerically that by computing the optimal p^* and q^*, and consequently leading to the optimal $R(p^*, q^*)$ for fixed parameter values of R, mixed strategies perform better than pure strategies. In addition to discriminating the quality of used items, the relaxation of the assumption that the quality of new and remanufactured items is indistinguishable can lead to more realistic models in general. Jaber and El Saadany [8] explored this avenue by extending the work of [19, 20] to the case where the demand for new items is different from the demand for remanufactured items, i.e. the new items and the remanufactured items are consumed by two separate markets. This leads to lost sales situations, since new items are facing shortage during the remanufacturing periods and vice versa.

Next, Konstantaras et al. [15] studied a model where not all used items qualify to be recovered: Those that qualify are recovered to an as good as new condition, while those that do not are recovered and sold at a secondary market for a lower price. They derived formulas for the optimal inventory level of used items before inspection and the optimal order quantity. Buscher and Linder [1] considered the case where production and recovery takes place on a common facility. They jointly determined the economic production and rework quantity as well as the optimal sizes of partial lots for both activities. Jaber and Rosen [7] suggested that a EOQ repair and waste disposal model can be treated as a 'disordered' physical system, and improvements to the operation of the system can be achieved by reducing the disorder through the application of the the first and second laws of thermodynamics.

Most of the works above do not account for shortages, which can be economical in some situations. Konstantaras and Papachristos [11] obtained an optimal production and recovery policy analytically for a EOQ model where excess demand are completely backordered. Later, Konstantaras and Skouri [14] obtained sufficient conditions for the optimal policy of a model with completely backlogged shortages and variable setup numbers of equal batch sizes.

In this paper, we consider the reuse of items after a simple repair process. The producer satisfies a continuous time-varying demand process for a finished product over a known and finite planning horizon, and collects used items from the customers. For satisfying the demand, he has two options: either he fabricates new items from the raw materials that he procured externally, or he repairs the used items back to an 'as new' condition. The material flow of this situation is depicted in Figure 1. In the next two sections, we present a model that treats the inventories of the raw materials, finished items, and used items as interdependent parts of a single system. Our model operates with a predetermined inventory holding policy. In Section 4, we propose a numerical solution procedure that finds the optimal solution recursively. Section 5 contains some numerical examples and sensitivity analysis as well as a conclusion.
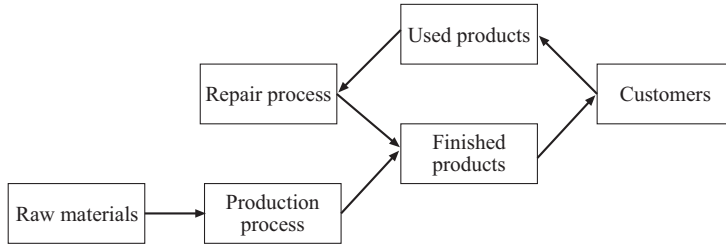
Figure 1. Material flow of the model.

2. Model description

In this model, demand for the finished product is served by either newly fabricated items or by used items that are repaired to an 'as new' condition. We assume that only one type of raw material (called raw material 1) is required to fabricate the finished product. We consider an n-cycle policy that alternates production runs with repair runs throughout a known and finite planning horizon. For each cycle, the producer orders one shipment of raw material 1 to fabricate new items in a single production run, and then repairs used items in a single repair run. At the end of the production run, all units of raw material 1 will be fully processed, and at the end of the repair run, all units of the used products will be fully repaired. Since demand varies with time, we assume that the collection rate of the used items is proportional to the demand rate. We also assume that the collection of used items incurs no setup cost. Moreover, we assume that there is no collection of used items during the period when used items are repaired and shipped. Figure 2 shows the general pattern of the inventory movement during the $(i+1)$-th cycle $(i = 0, 1, \ldots, n-1)$ when demand is

increasing over time. Before going further, we list the assumptions and nomenclature used in this paper.
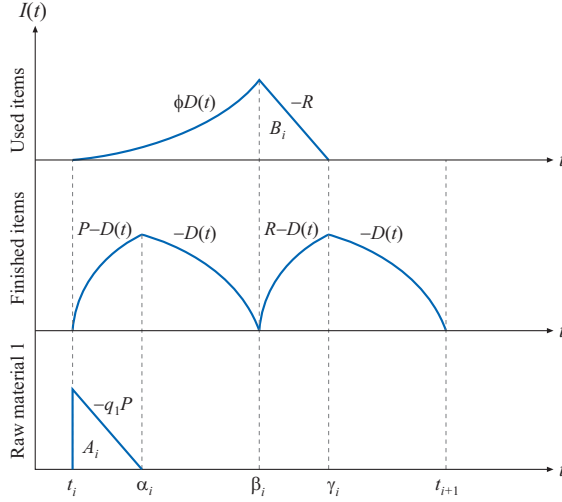

Figure 2. Inventory movement in a cycle for increasing demand.

2.1. Assumptions

- A single product inventory system is considered over a known and finite planning horizon which is H units of time long.
- The demand rate at time t is given by the deterministic and continuous function $D(t)$.
- The production rate is a known constant P and $P > D(t)$ for all t.
- The repair rate is a known constant R and $R > D(t)$ for all t.
- The collection rate of the used items, $C(t)$, is proportional to the demand rate, i.e. $C(t) = \phi D(t)$, $0 \leq \phi \leq 1$.
- All used items are repaired to an 'as new' condition. There is no collection of used items during the repair period.
- Only one type of raw material (called raw material 1) is required to fabricate the finished product. After an order is placed, raw material 1 is immediately replenished.
- There is a single production run, a single repair run, and a single replenishment of raw material 1 per cycle.
- Newly fabricated or repaired items are immediately shipped out.
- Shortages are not allowed during the planning horizon.
- The following cost parameters are considered:
    - c_P, the setup cost of the production run (cost/setup).
    - c_R, the setup cost of the repair run (cost/setup).
    - c_1, the ordering cost of raw material 1 (cost/order).
    - h_P, the inventory holding cost of finished items (cost/unit/time).

- h_R, the inventory holding cost of used items (cost/unit/time).
- h_1, the inventory holding cost of raw material 1 (cost/unit/time).
- s_P, the unit production cost finished items (cost/unit).
- s_R, the unit repair cost of used items (cost/unit).

2.2. Nomenclature

- q_1, the quantity of raw material 1 required to produce one unit of the finished product.
- n, the number of cycles during the planning horizon $(n = 1, 2, \ldots)$.
- t_i, the total elapsed time up to the start of the $(i + 1)$-th cycle's production run $(i = 0, 1, \ldots, n - 1)$, where $t_0 = 0$ and $t_n = H$.
- α_i, the total time that is elapsed up to the end of the $(i + 1)$-th cycle's production run.
- β_i, the total elapsed time up to the start of the $(i + 1)$-th cycle's repair run.
- γ_i, the total time that is elapsed up to the end of the $(i + 1)$-th cycle's repair run.

3. Mathematical formulation

The total relevant cost of the system when there are n cycles is given by

$$(3.1) \qquad TC(n, \mathbf{t}) = n(c_R + c_P + c_1) + H_R + H_P + H_1,$$

where $\mathbf{t}$ are the starting times of the cycles, and H_R, H_P and H_1 are the total inventory holding cost throughout the planning horizon, respectively for the used items, the finished items and raw material 1.

First, we consider the used items inventory during the $(i + 1)$-th cycle. Since the amount of units collected is equal to the amount of units repaired, we have

$$(3.2) \qquad \gamma_i = \beta_i + \frac{\phi}{R} \int_{t_i}^{\beta_i} D(t)\mathrm{d}t.$$

Now, the inventory level of the used items at time t that spans the production period $(t_i \le t \le \beta_i)$, $I_R(t)$, is given by the amount of used items collected from time t_i to t, that is

$$(3.3) \qquad I_R(t) = \int_{t_i}^{t} \phi D(u)\mathrm{d}u, \quad t_i \le t \le \beta_i.$$

Then, from the top graph in Figure 2, we observe that the time-weighted inventory holding of the used items during the $(i + 1)$-th cycle is given by

$$(3.4) \qquad \int_{t_i}^{\beta_i} I_R(t)\mathrm{d}t + B_i,$$

where B_i is the area of the corresponding right triangle. Next, by assuming that $g(u)$ is the antiderivative of $D(u)$, then the total inventory holding cost of the used items throughout the planning horizon, H_R, is given by

$$H_R = h_R \sum_{i=0}^{n-1} \left(\int_{t_i}^{\beta_i} I_R(t)\mathrm{d}t + B_i \right)$$

$$(3.5) \qquad = h_R \phi \sum_{i=0}^{n-1} \left(\int_{t_i}^{\beta_i} g(t)\mathrm{d}t - g(t_i)(\beta_i - t_i) + \frac{\phi}{2R} \left[\int_{t_i}^{\beta_i} D(t)\mathrm{d}t \right]^2 \right).$$

Secondly, we consider the finished items inventory during the $(i+1)$-th cycle. Since the production during the production uptime period must satisfy the demand during the production period, we have

$$(3.6) \qquad \alpha_i = t_i + \frac{1}{P} \int_{t_i}^{\beta_i} D(t)\mathrm{d}t.$$

Now, four expressions for the inventory level of the finished items with respect to time, $I_j(t)$ $(j = 1, 2, 3, 4)$ can be defined as follows; they are for the production uptime and downtime periods as well as the repair uptime and downtime periods respectively:

$$(3.7) \qquad I_1(t) = P(t - t_i) - \int_{t_i}^{t} D(u)\mathrm{d}u, \quad t_i \le t \le \alpha_i,$$

$$(3.8) \qquad I_2(t) = \int_{t}^{\beta_i} D(u)\mathrm{d}u, \quad \alpha_i \le t \le \beta_i.$$

$$(3.9) \qquad I_3(t) = R(t - \beta_i) - \int_{\beta_i}^{t} D(u)\mathrm{d}u, \quad \beta_i \le t \le \gamma_i,$$

$$(3.10) \qquad I_4(t) = \int_{t}^{t_{i+1}} D(u)\mathrm{d}u, \quad \gamma_i \le t \le t_{i+1}.$$

Then, from the middle graph in Figure 2, we observe that the time-weighted inventory holding of the finished items during the $(i+1)$-th batch is given by

$$(3.11) \qquad \int_{t_i}^{\alpha_i} I_1(t)\mathrm{d}t + \int_{\alpha_i}^{\beta_i} I_2(t)\mathrm{d}t + \int_{\beta_i}^{\gamma_i} I_3(t)\mathrm{d}t + \int_{\gamma_i}^{t_{i+1}} I_4(t)\mathrm{d}t.$$

Next, the total inventory holding cost of the finished items throughout the planning horizon, H_P, is given by

$$H_P = h_P \sum_{i=0}^{n-1} \left(\int_{t_i}^{\alpha_i} I_1(t)\mathrm{d}t + \int_{\alpha_i}^{\beta_i} I_2(t)\mathrm{d}t + \int_{\beta_i}^{\gamma_i} I_3(t)\mathrm{d}t + \int_{\gamma_i}^{t_{i+1}} I_4(t)\mathrm{d}t \right).$$

$$= h_P \sum_{i=0}^{n-1} \left(\left[\frac{\phi^2}{2R} - \frac{1}{2P} \right] \left[\int_{t_i}^{\beta_i} D(t)\mathrm{d}t \right]^2 + g(t_{i+1})(t_{i+1} - \beta_i) + g(\beta_i)(\beta_i - t_i) \right.$$

$$(3.12)$$

$$\left. - \frac{\phi}{R} \int_{t_i}^{\beta_i} D(t)\mathrm{d}t \int_{\beta_i}^{t_{i+1}} D(t)\mathrm{d}t - \int_{t_i}^{t_{i+1}} g(t)\mathrm{d}t \right).$$

Thirdly, we consider the raw material 1 inventory during the $(i+1)$-th cycle. From the bottom graph in Figure 2, we observe that the inventory holding of raw material 1 during the $(i+1)$-th batch is given by A_i, the area of the corresponding right triangle. Hence, it follows that the total inventory holding cost of raw material

1 throughout the planning horizon, H_1, is given by

$$(3.13) \qquad H_1 \; = \; h_1 \sum_{i=0}^{n-1} A_i \; = \; \frac{h_1 q_1}{2P} \sum_{i=0}^{n-1} \left(\int_{t_i}^{\beta_i} D(t) \mathrm{d}t \right)^2 .$$

Finally, using equations (3.5), (3.12) and (3.13), equation (3.1) can be rewritten as

$$TC(n, \mathbf{t}) = n(c_R + c_P + c_1) + \sum_{i=0}^{n-1} \left(\left[\frac{\phi^2 (h_R - h_P)}{2R} + \frac{h_1 q_1 - h_P}{2P} \right] \left[\int_{t_i}^{\beta_i} D(t) \mathrm{d}t \right]^2 \right.$$

$$+ \; h_R \phi \left[\int_{t_i}^{\beta_i} g(t) \mathrm{d}t - g(t_i)(\beta_i - t_i) \right] + h_P \left[g(\beta_i)(\beta_i - t_i) \right.$$

$$(3.14) \qquad \left. + \; g(t_{i+1})(t_{i+1} - \beta_i) - \int_{t_i}^{t_{i+1}} g(t) \mathrm{d}t \right] \right) .$$

The problem is to minimize $TC(n, \mathbf{t})$ by seeking the optimal integer value of n and the optimal real values of $\mathbf{t}$ for that n, subject to the following constraints:

$$(3.15) \qquad t_i \le \beta_i \le t_{i+1}, \quad t_0 = 0, \quad t_n = H.$$

It can easily be shown that β_i is a function of (t_i, t_{i+1}). Hence, we present the following lemma.

Lemma 3.1. *For any demand function $D(t)$ that is integrable in the interval $[t_i, t_{i+1}]$, β_i is a function of (t_i, t_{i+1}).*

Proof. Since the repair run during the repair uptime period must satisfy the demand during the repair period, we have

$$R(\gamma_i - \beta_i) = \int_{\beta_i}^{t_{i+1}} D(t) \mathrm{d}t.$$

From equation (3.2), it follows that

$$(3.16) \qquad \int_{t_i}^{\beta_i} \phi D(t) \mathrm{d}t = \int_{\beta_i}^{t_{i+1}} D(t) \mathrm{d}t.$$

Now, since $0 \le \phi \le 1$, then for equation (3.16) to hold, $t_i \le \beta_i \le t_{i+1}$ must be true. Finally, it is easily observable that β_i, which is a solution of equation (3.16), is a function of (t_i, t_{i+1}). ∎

To illustrate, we consider two common time-varying demand functions, i.e. the linearly-varying demand function and the exponentially-varying demand function. The linearly-varying demand function has the form $D(t) = a + bt$, and $D(t) > 0$, $b \neq 0$ for $0 \le t \le H$. Then, using Lemma 3.1, we have

$$(3.17) \qquad \beta_i = \frac{-a(1 + \phi) + \sqrt{a^2 (1 + \phi)^2 + 2b(1 + \phi)[g(t_{i+1}) + \phi g(t_i)]}}{b(1 + \phi)}.$$

The exponentially-varying demand function has the form $D(t) = a e^{bt}$, $a > 0$, $b \neq 0$ for $0 \le t \le H$. Then, using Lemma 3.1, we have

$$(3.18) \qquad \beta_i = \frac{1}{b} \ln \left| \frac{1}{1 + \phi} (e^{bt_{i+1}} + \phi e^{bt_i}) \right|.$$

For fixed n, the set of equations tial $TC/$tial $t_{i+1} = 0$ $(i = 0, 1, \ldots, n-1)$ gives the following set of optimality conditions:

$$2A\left[\int_{t_i}^{\beta_i} D(t)\mathrm{d}t - \int_{t_{i+1}}^{\beta_{i+1}} D(t)\mathrm{d}t\right] + h_P\left[(t_{i+1} - t_i) + \phi(\beta_{i+1} - \beta_i)\right.$$

$$- \frac{\phi+1}{D(t_{i+1})}\int_{t_{i+1}}^{\beta_{i+1}} D(t)\mathrm{d}t\right] + \phi(h_R - h_P)\left[\frac{1}{D(\beta_i)}\int_{t_i}^{\beta_i} D(t)\mathrm{d}t\right.$$

$$(3.19) \qquad + \frac{1}{D(\beta_{i+1})}\int_{\beta_{i+1}}^{t_{i+2}} D(t)\mathrm{d}t\right] - h_R\phi(\phi+1)(\beta_{i+1} - t_{i+1}) = 0,$$

where

$$A = \frac{\phi^2(h_R - h_P)}{2R} + \frac{h_1 q_1 - h_P}{2P}.$$

Now, from the set of equations (3.19) and using Lemma 3.1, we observe that t_{i+2} is a function of (t_i, t_{i+1}). Moreover, since $t_0 = 0$, it is easy to see that once t_1 is fixed, the set of equations (3.19) will determine all other t_i recursively as functions of t_1. Hence, a procedure for finding the optimal $\mathbf{t}$ for a given number of cycles n is to fix t_1, get the other t_i $(i = 2, 3, \ldots, n)$ by solving the set of equations (3.19), and finally check if $t_n = H$.

A proof of the existence and uniqueness of the optimal $\mathbf{t}$ is difficult to derive. However, under certain conditions, some arguments can be made, as presented in the following lemma.

Lemma 3.2. *For an increasing demand function $D(t)$, if $B \geq 0$ and $h_R - h_P \geq 0$, then the solution of equation (3.19) is unique.*

Proof. Let $F(t_i, t_{i+1}, t_{i+2})$ be the function representing the left hand side of equation (3.19). If $t_{i+2} = t_{i+1}$, then using equation (3.16), $\beta_{i+1} = t_{i+1}$ and $\int_{t_{i+1}}^{\beta_{i+1}} D(t)\mathrm{d}t = 0$. Then, for fixed t_i and t_{i+1}, $F(t_i, t_{i+1}, t_{i+2})$ is reduced to

$$F(t_{i+2}) = 2B\int_{t_i}^{\beta_i} D(t)\mathrm{d}t + h_P[(t_{i+1} - t_i) + \phi(\beta_{i+1} - \beta_i)]$$

$$(3.20) \qquad + \phi(h_R - h_P)\frac{1}{D(\beta_i)}\int_{t_i}^{\beta_i} D(t)\mathrm{d}t,$$

which is positive if $B \geq 0$ and $h_R - h_P \geq 0$. Next, $dF(t_{i+2})/dt_{i+2}$ is given by

$$\frac{dF(t_{i+2})}{dt_{i+2}} = \frac{D(t_{i+1})D(t_{i+2})}{(\phi+1)^2}\left(-2B - \frac{h_P(\phi+1)}{D(t_{i+1})} + \frac{\phi}{D(\beta_{i+1})}[h_P(1-\phi) - h_R]\right.$$

$$(3.21) \qquad - \phi(h_R - h_P)\frac{D'(\beta_{i+1})}{D^3(\beta_{i+1})}\right),$$

and it is negative if $B \geq 0$ and $h_R - h_P \geq 0$. If follows that $F(t_{i+2})$ is a decreasing function of t_{i+2}, and hence, there exists an unique value of t_{i+2} such that $t_{i+2} > t_{i+1}$ and $F(t_{i+2}) = 0$. ∎

4. Solution procedure

There is an integer decision variable (n) in this problem in addition to the real decision variables $\mathbf{t} = \{t_1, t_2, \ldots, t_{n-1}\}$. Therefore, to deal with this mixture of integer and real variables, we first fix n and find t_1 such that the set of equations (3.19) will generate $t_n = H$. The resultant $\mathbf{t}^*$ minimizes the total relevant cost for the fixed n, and we will write this minimized total relevant cost as $TC^*(n)$. Next, we change n to improve the total relevant cost, until the first $n = N$ that satisfies the conditions $TC^*(N) < TC^*(N-1)$ and $TC^*(N) < TC^*(N+1)$ is found.

We should observe an improvement on $TC^*(n)$ as n increases until $n = N$. Beyond this, increasing n should increase $TC^*(n)$. When n is fixed, $TC^*(n)$ is given by

$$(4.1) \qquad TC^*(n) = n(c_R + c_P + c_1) + H_R(\mathbf{t}^*) + H_P(\mathbf{t}^*) + H_1(\mathbf{t}^*).$$

Since H_R, H_P and H_1 are all positive, then $TC^*(N + k)$, where k is a positive integer, is guaranteed to be greater than $TC^*(N)$ when

$$(4.2) \qquad (N + k)(c_R + c_P + c_1) > TC^*(N).$$

If $k = k_0$ is the first solution of the inequality in (4.2), then $n = N + k_0$ is an upper bound until which the evaluation of $TC^*(n)$ should be conducted to test the validity of $TC^*(N) < TC^*(n)$ for any $n > N$. If this inequality holds until the upper bound, then $TC^*(N)$ is the absolute minimum of all $TC^*(n)$.

Finally, the computer algorithm of the solution procedure is outlined below:

1. Let $n = 1$.
2. Compute the total relevant cost, $TC(1)$. We note that $TC(1)$ is already minimized.
3a. Increase n by 1.
3b. Find t_1 such that the set of equations (3.19) will generate $t_n = H$. Calculate the corresponding $TC^*(n)$.
3c. If $TC^*(n) > TC^*(n-1)$, let $N = n - 1$. Go to Step 4.
3d. If $TC^*(n) < TC^*(n-1)$, return to Step 3a.
4. Compute the integer k_0 such that the inequality in (4.2) is solved.
5. Validate that $TC^*(N + k)$ from $k = 1$ to $k = k_0$ are all $\geq TC^*(N)$.

5. Numerical example and sensitivity analysis

We present two numerical examples to illustrate the solution procedure described in the preceding section. For Example 1, the demand function is a linearly-increasing function of the form $D(t) = 6 + 15t$. The associated parameter values are $c_R = 100$, $c_P = 300$, $c_1 = 50$, $h_R = 30$, $h_P = 30$, $h_1 = 5$, $\phi = 0.7$, $P = 100$, $R = 100$, $H = 5$, and $q_1 = 1$. $N + k_0$ is computed to be 10. For varying values of n, the corresponding optimal total relevant costs, $TC^*(n)$, are tabulated in Table 1. For Example 2, the demand function is an exponential-increasing function of the form $D(t) = 60e^{0.5t}$. The associated parameter values are similar to those used in Example 1 except that $H = 2$. $N + k_0$ is computed to be 6. For varying values of n, the corresponding $TC^*(n)$ are tabulated in Table 2.

Table 1. Optimal total relevant costs for varying n for Example 1.

n	1	2	3	4	5*
$TC^*(n)$	12,983.01	6,342.21	4,800.88	4,321.87	$4,235.60^*$

n	6	7	8	9	10
$TC^*(n)$	4,336.88	4,542.11	4,810.90	5,121.36	5,460.62

Table 2. Optimal total relevant costs for varying n for Example 2.

n	1	2	3*
$TC^*(n)$	3,826.44	2,401.68	$2,314.16^*$

n	4	5	6
$TC^*(n)$	2,509.88	2,811.71	3,164.70

5.1. Sensitivity analysis

One may ask the pertinent question: "How do the optimal total relevant costs respond to parameter changes?" To attempt answering this question, we perform a sensitivity analysis. Since the objective functions are quite complicated and the optimal values of the decision variables are a mixture of integer and real values that are computed through a search procedure, we perform a numerical sensitivity analysis by solving many sample problems.

We use the following parameter values as the standard values of the parameters: $D(t) = 6 + 15t$, $c_R = 100$, $c_P = 300$, $c_1 = 50$, $h_R = 15$, $h_P = 30$, $h_1 = 5$, $\phi = 0.7$, $P = 1000$, $R = 1000$, $H = 5$, and $q_1 = 1$. For the cost parameters of c_R, c_P, c_1, h_R, h_P and h_1, we choose seven different levels by multiplying their standard values by 1/5, 1/3, 1/2, 1, 2, 3 and 5, respectively. The results are depicted in Figures 3 to 8.

Figures 3 to 5 show that the optimal total relevant costs increase with c_P, c_R and c_1. We observe that when c_R, c_P or c_1 is small, the system favors larger n, i.e. more frequent production and repair runs in the planning horizon, and vice-versa.

Figures 6 to 8 show that the optimal total relevant costs increase with h_P, h_R and h_1 as well. The noticeably smaller effect h_1 has compared to h_R or h_P is because of its smaller base value and because the time-weighted inventory holding of raw material 1 constitutes a small portion of the time-weighted inventory holding of the whole system. Here, we observe that when h_R or h_P is small, the system favors smaller n this time, that is, less frequent production and repair runs in the planning horizon, and vice-versa. However, this trend is not apparent for h_1 due to aforementioned reasons.

For the non-cost parameters, i.e. the demand function parameter b, the production rate P and the repair rate R, we multiply their standard values by 1/10, 1/4, 1/2, 1, 1.5, 2.5 and 5, respectively. The results are depicted in Figures 9 to 11.

In Figure 9, we find that the demand rate has a relatively large effect on the optimal total relevant costs. Besides increasing the optimal total relevant costs, a demand that grows more rapidly compels the system to perform more frequent production and repair runs in the planning horizon for more savings, i.e. the larger

the b, the larger the n. The former behavior is self-explanatory, while the latter
is because the larger the demand is at any point of time, the larger the on-hand
inventories are at that time, and thus more production and repair runs are required
to reduce the peak inventory levels. On the other hand, Figures 10 and 11 show
that larger production or repair rates lead to higher optimal total relevant costs
but this effect stagnates as the rates increase because the production and repair
uptime periods during which the finished items are kept in stock, and the repair
uptime period during which the used items are kept in stock, become increasingly
brief, thus reducing their contribution to the time-weighted inventory holding of the
whole system.

For the used products collection rate proportionality constant, ϕ, we choose the
levels of 0.1, 0.25, 0.5, 0.7, 0.8, 0.9 and 1.0 respectively. The result is depicted in
Figures 12, which shows that the optimal total relevant costs improve as ϕ increases.
This behavior is predictable and is true for cost structures that favors used products
holding as well as the repair process. In addition, it is easily observable that a pure
repair policy is better than a mixed policy.
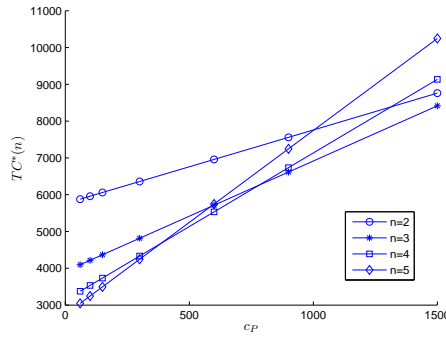
Figure 3. Effect of c_P on $TRC^*(n)$ for varying n.
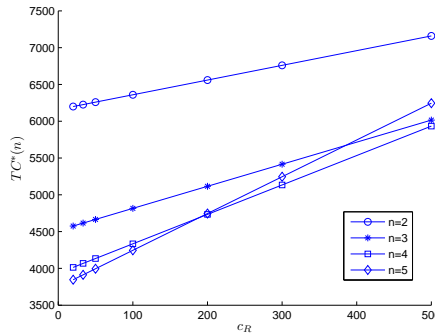
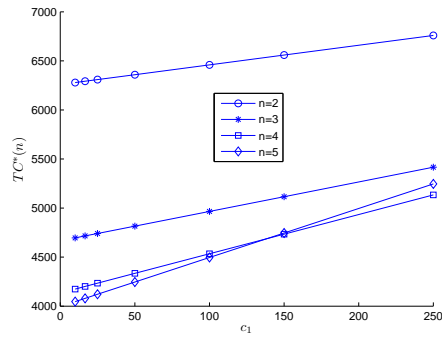Figure 4. Effect of c_R on $TRC^*(n)$ for varying n.

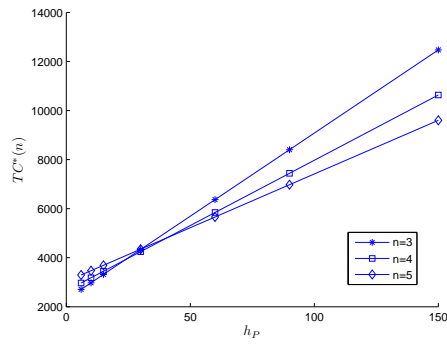Figure 5. Effect of c_1 on $TRC^*(n)$ for varying n.

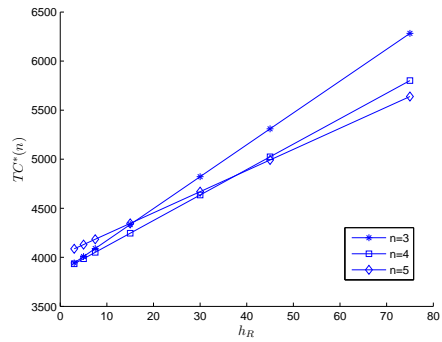Figure 6. Effect of h_P on $TRC^*(n)$ for varying n.

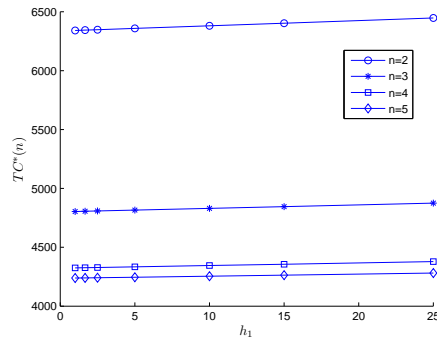Figure 7. Effect of h_R on $TRC^*(n)$ for varying n.

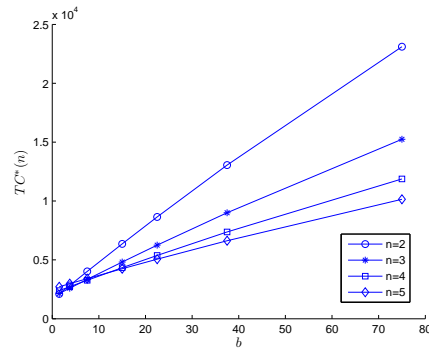Figure 8. Effect of h_1 on $TRC^*(n)$ for varying n.

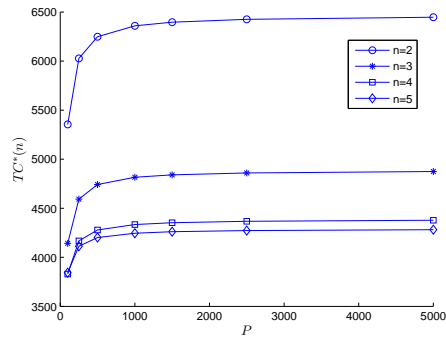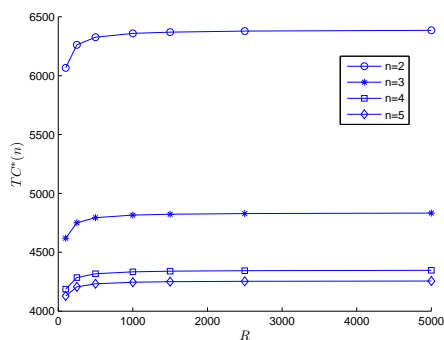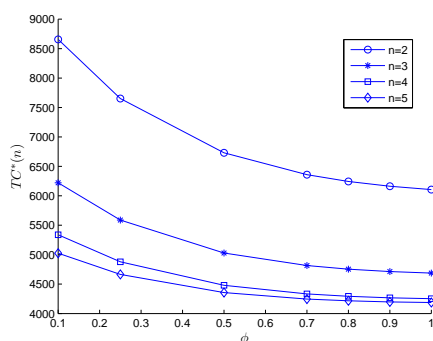Figure 9. Effect of b on $TRC^*(n)$ for varying n.

Figure 10. Effect of P on $TRC^*(n)$ for varying n.

Figure 11. Effect of R on $TRC^*(n)$ for varying n.

Figure 12. Effect of ϕ on $TRC^*(n)$ for varying n.

5.2. Conclusion

In this paper, we have proposed a model of a production system where a continuous time-varying demand for a finished product can be satisfied by newly fabricated items or by repaired items. We assume that only one type of raw material is required to fabricate the finished product, and there is no collection of used items during the repair period. Moreover, we assume that the system procures raw materials, fabricate finished items and repair used items in single lots during each cycle throughout the planning horizon. For the optimal joint policy, the continuous variables are computed numerically while an exhaustive search procedure is used for the integer variable n. We also presented two numerical examples to illustrate the solution procedure. Furthermore, using a sensitivity analysis, we observed characteristics of our model under various situations. Finally, we conclude the paper by noting that our model may be extended to the case where used items are collected during the repair period and shortages are allowed.

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