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The Linear Arboricity of the Schrijver Graph $\mathrm{SG}(2k+2,k)$

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Abstract. The linear arboricity $\mathrm{la}(G)$ of a graph G is the minimum number of linear forests which partition the edge set $E(G)$ of G. The vertex linear arboricity $\mathrm{vla}(G)$ of a graph G is the minimum number of subsets into which the vertex set $V(G)$ can be partitioned so that every subset induces a linear forest. The Schrijver graph $\mathrm{SG}(n,k)$ is the graph whose vertex set consists of all 2-stable k-subsets of the set $[n] = \{0,1,\dots,n-1\}$ and two vertices A and B are adjacent if and only if $A \cap B = \phi$. In this paper, it is proved that $\mathrm{la}(\mathrm{SG}(2k+2,k)) = \lceil (k+2)/2 \rceil$ for $k \geq 3$ and $\mathrm{vla}(\mathrm{SG}(2k+2,k)) = \mathrm{va}(\mathrm{SG}(2k+2,k)) = 2$ for $k \geq 2$.

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1. Introduction

Throughout this paper, all graphs considered are finite, undirected and simple. For a real number x, $\lceil x \rceil$ is the least integer not less than x, and $\lfloor x \rfloor$ is the most integer not more than x. For a graph G, we use $V(G), E(G), \Delta(G)$ to denote the vertex set, the edge set and the maximum degree, respectively. $N_G(v)$ denotes the set of vertices adjacent to the vertex v in G. $G[W]$ denotes the subgraph induced by $W \subseteq V(G)$ (or $W \subseteq E(G)$) in G. For disjoint subsets S and S' of $V(G)$, we denote the set of edges with one end in S and the other in S' by $[S,S']$, which is called an edge cut if $S' = \overline{S}$, where $\overline{S} = V(G) \setminus S$ is the subset obtained by removing all vertices of S from $V(G)$. Let $G \setminus H$ be the graph $G - E(H)$ that is obtained by taking away all edges of H from G. A k-path is a path with length k.

A linear forest is a graph in which each component is a path. The *linear arboricity* $\mathrm{la}(G)$ of a graph as defined by Harary [11] is the minimum number of linear forests which partition the edge set $E(G)$ of G. Akiyama *et al.* [1] conjectured that $\mathrm{la}(G) = \lceil (\Delta(G)+1)/2 \rceil$ for any regular graph G, and proved that the conjecture is true for complete graphs and graphs with $\Delta = 3,4$ [1, 2]. Enomoto and Péroche [7] proved that the conjecture is true for graphs with $\Delta = 5,6,8$. Guldan [10] proved that the conjecture is true for graphs with $\Delta = 10$. It is

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obvious that $\mathrm{la}(G) \geq \lceil(\Delta(G))/2\rceil$ for every graph G and $\mathrm{la}(G) \geq \lceil(\Delta(G)+1)/2\rceil$ for every regular graph G. So the conjecture is equivalent to the following conjecture.

Conjecture 1.1 (Linear Arboricity Conjecture(LAC))**.** [1] *For any graph G,*

$$\lceil\Delta(G)/2\rceil \leq \mathrm{la}(G) \leq \lceil(\Delta(G)+1)/2\rceil.$$

Akiyama *et al.* [1] determined the linear arboricity of complete bipartite graphs and trees. Martinov [12] determined the linear arboricity of extremal locally-tree-like graphs which have a minimal number of edges according to the number of vertices. Martinova [13] determined the linear arboricity of maximal outerplanar graphs. Wu [19] determined the linear arboricity of series-parallel graphs, moreover, Wu [20] proved the conjecture is true for a planar graph G with $\Delta(G) \neq 7$, and the case $\Delta(G) = 7$ was also settled in Wu [21]. Tan et al. [18] determined the linear arboricity of planar graphs with maximum degree at least five.

The vertex linear arboricity $\mathrm{vla}(G)$ of a graph is the minimum number of subsets into which the vertex set $V(G)$ can be partitioned so that every subset induces a linear forest. The vertex arboricity $\mathrm{va}(G)$ of a graph G can be defined similarly. Matsumoto [14] proved that for any finite graph G, $\mathrm{vla}(G) \leq \lceil(\Delta(G)+1)/2\rceil$, moreover, if $\Delta(G)$ is even, then $\mathrm{vla}(G) = \lceil(\Delta(G)+1)/2\rceil$ if and only if G is the complete graph of order $\Delta(G)+1$ or a cycle. Goddard [9] and Poh [15] proved that $\mathrm{vla}(G) \leq 3$ for a planar graph G. Akiyama [3] proved $\mathrm{vla}(G) \leq 2$ if G is an outerplanar graph. Alavi [4] proved that $\mathrm{vla}(G)+\mathrm{vla}(G^c) \leq 1+\lceil(n+1)/2\rceil$ for any graph G of order n, where G^c is the complement of G. Zuo [22, 23] determined the vertex linear arboricity of distance graphs and a class of integer distance graphs with special distance sets, respectively. Raspaud and Wang [16] discussed the vertex arboricity of planar graphs, and Borodin and Ivanova [5] proved that planar graphs without 4-cycles adjacent to 3-cycles are list vertex 2-arborable. The following result is obvious.

Lemma 1.1. *If* $G = G_1 \cup G_2 \cup \cdots \cup G_n$*, then* $\mathrm{la}(G) \leq \mathrm{la}(G_1)+\mathrm{la}(G_2)+\cdots+\mathrm{la}(G_n)$*. In particular,* $\mathrm{la}(G) = max\{\mathrm{la}(G_1), \mathrm{la}(G_2), \ldots, \mathrm{la}(G_n)\}$*, where* $G_i (i = 1,2,\ldots,n)$ *are connected components of G.*

The Kneser graph $KG(n,k)$ is the graph whose vertex set consists of all k-subsets of an n-set, and two vertices are adjacent if and only if they are disjoint. A subset S of $[n] = \{0,1,\ldots,n-1\}$ is said to be 2-stable if $2 \leq |x-y| \leq n-2$ for any two distinct elements x and y, i.e., S does not contain two consecutive numbers in the cyclic ordering of $[n]$.

Definition 1.1. [17] *The Schrijver graph* $\mathrm{SG}(n,k)$ *is defined as follows. Its vertices are those k-element subsets of the set* $[n] = \{0,1,\ldots,n-1\}$ *that do not contain cyclically consecutive elements* $i, i+1$ *or* $n-1, 0$*. Two such vertices are adjacent if they represent disjoint k-subsets.*

Equivalently, the Schrijver graph $\mathrm{SG}(n,k)$ is the graph whose vertex set consists of all 2-stable k-subsets of the set $[n] = \{0,1,\ldots,n-1\}$ and two vertices A and B are adjacent if and only if $A \cap B = \phi$. Clearly, the Schrijver graph $\mathrm{SG}(n,k)$ is the subgraph of $KG(n,k)$ induced by all vertices that are 2-stable subsets. The structure of Schrijver graph $\mathrm{SG}(2k+2,k)$ was studied in [6]. Now we recall some results that will be used here.

The vertex set of the Schrijver graph $\mathrm{SG}(n,k)$ has cardinality $\frac{n}{k}\binom{n-k-1}{k-1}$. In particular, $\mathrm{SG}(2k+2,k)$ has $(k+1)^2$ vertices. For $0 \leq i \leq 2k+1$, let $v(0,i) = \{i, i+2, \ldots, i+2k-2\}$, in which each element is taken modulo $2k+2$. We make the convention that all indices and elements are taken modulo $2k+2$ in the following except special instruction. We also regard $v(0,i)$ as a sequence with the elements ordered in the above manner.

A sequence is called a k-sequence if it has k elements. Let $m = \lfloor k/2 \rfloor$. For $1 \leq j \leq m$, let A_j be the k-sequence in which the $(k-j+1)-$th entry is equal to 2, and the other $k-1$ entries are equal to 1. Clearly, A_j can be viewed as a row vector with k components, and $v(0,i)$ and A_j can be added to $v(0,i)+A_j$. In fact, when a k-set A, regarded as a row vector, and a k-sequence B are added to get $A+B$, we just add the two sequences entry-wise to get a k-sequence if all the sums are distinct. For the sake of convenience, in addition operation, one can view $v(0,i)$ as a row vector with k components $(i, i+2, \ldots, i+2k-2)$ in R^k over real number field R, in which each element is taken modulo $2k+2$.

Now for $0 \leq i \leq 2k+1$ and $1 \leq j \leq m$, let

$$v(j,i) = v(j-1,i) + A_j$$

be the recursion formula, where $v(0,i) = (i, i+2, \ldots, i+2k-2)$ and the addition is taken modulo $2k+2$. Let $V_0 = \{v(0,i) \mid i = 0, 1, \ldots, 2k+1\}$, and $V_j = \{v(j,i) | i = 0, 1, \ldots, 2k+1\}$ for $1 \leq j \leq m$. We need the following lemmas for the proof of our main results.

Lemma 1.2. [6] *For $0 \leq j \leq m-1$, $\mid V_j \mid = 2k+2$, and*

$$\mid V_m \mid = \begin{cases} 2k+2, & \textit{if } k \textit{ is odd}, \\ k+1, & \textit{otherwise}. \end{cases}$$

Note that $\mid V_m \mid = k+1$ when k is even. Thus, in this case, the index i of $v(m,i)$ is taken modulo $k+1$ for even k henceforth.

Lemma 1.3. [6] *For each $v(0,i) \in V_0$, and $v(j,i) \in V_j$, we have*

$$N_G(v(0,i)) = \{v(0, i+p) \mid p = 1, 3, \ldots, 2k+1\} \cup \{v(1,i)\},$$

$$N_G(v(j,i)) = \{v(j,i-1), v(j,i+1), v(j-1,i), v(j+1,i)\}$$

for $1 \leq j \leq m-1$,

$$N_G(v(m,i)) = \{v(m,i-1), v(m,i+1), v(m,i+k+1), v(m-1,i)\}$$

for k is odd, and

$$N_G(v(m,i)) = \{v(m,i-1), v(m,i+1), v(m-1,i), v(m-1,i+k+1)\}$$

for k is even.

By Lemma 1.3, $\Delta(G) = k+2$ for $k \geq 3$, and the following two results are obtained immediately.

Corollary 1.1. [6] *The graph $G[V_0]$ is a complete bipartite graph with two partite subsets*

$$X = \{v(0,0), v(0,2), \ldots, v(0,2k)\} \quad \textit{and} \quad Y = \{v(0,1), v(0,3), \ldots, v(0,2k+1)\}.$$

Corollary 1.2. [6] *The graph $G[V_j]$ is a cycle with length $2k+2$ for $1 \leq j \leq m-1$, $G[V_m]$ is a cycle with length $k+1$ for even k, and $G[V_m]$ is a 3-regular graph for odd k.*

2. The linear arboricity of $\mathrm{SG}(2k+2,k)$

Let $G(X,Y)$ be a balanced bipartite graph with partite sets $X=\{x_i \mid i \in Z_n\}$ and $Y=\{y_i \mid i \in Z_n\}$. In [8], it was defined that the *bipartite difference* α of an edge $x_p y_q$ in $G(X,Y)$ by the value $(q-p)(mod\ n)$, i.e., $\alpha=(q-p)(mod\ n)$. It is obvious that an edge subsets in $G(X,Y)$ containing the edges with the same bipartite difference must be a matching. In particular, this edge subset is also a perfect matching if $G(X,Y)$ is $K_{n,n}$.

Let M_α be the edge set consisting of edges with bipartite difference α. The following lemmas give a decomposition of $K_{n,n}$.

Lemma 2.1. *Let $K_{n,n}$ be a balanced complete bipartite graph with partite sets $X=\{x_i \mid i=0,1,\dots,n-1\}$ and $Y=\{y_i \mid i=0,1,\dots,n-1\}$, then $K_{n,n}$ can be decomposed into the union of $n/2$ Hamiltonian paths and a matching for even n, and decomposed into the union of $(n-1)/2$ Hamilton paths and a linear forest for odd n.*

Proof. If n is even, then $K_{n,n}$ can be decomposed into the union of $n/2$ Hamiltonian cycles $M_\alpha \cup M_{\alpha+1} (\alpha=0,2,\dots,n-2)$. Next, we take away one edge $x_{\alpha/2}y_{n-\alpha/2-1}$ from each $M_\alpha \cup M_{\alpha+1} (\alpha=0,2,\dots,n-2)$. Then

$$H_{\alpha/2}=M_\alpha \cup M_{\alpha+1} \setminus \{x_{\alpha/2}y_{n-\alpha/2-1}\} \quad (\alpha=0,2,\dots,n-2)$$

are $n/2$ Hamiltonian paths of $K_{n,n}$, and $M=\{x_{\alpha/2}y_{n-\alpha/2-1} \mid \alpha=0,2,\dots,n-2\}$ is a matching.

Similarly, for odd n, each $M_\alpha \cup M_{\alpha+1} (\alpha=0,2,\dots,n-3)$ generates a Hamiltonian cycle. Therefore $M_\alpha \cup M_{\alpha+1} \setminus \{x_{\alpha/2}y_{n-\alpha/2-1}\}$ is a Hamiltonian path. Let

$$H_{\alpha/2}=M_\alpha \cup M_{\alpha+1} \setminus \{x_{\alpha/2}y_{n-\alpha/2-1}\} \quad (\alpha=0,2,\dots,n-3).$$

Moreover, it is clear that $M=M_{n-1} \cup \{x_{\alpha/2}y_{n-\alpha/2-1} \mid \alpha=0,2,\dots,n-3\}$ forms a linear forest. ∎

Therefore, $\mathrm{la}(K_{n,n}) \le \lceil (n+1)/2 \rceil$ and $\mathrm{la}(K_{n,n} \setminus M) \le \lceil (n-1)/2 \rceil$.

Lemma 2.2. [1] *The linear arboricity of every* 3*-regular graph is 2.*

Lemma 2.3. [2] *The linear arboricity of every* 4*-regular graph is* 3.

Now we give the main result of this paper.

Theorem 2.1. *Let $G=\mathrm{SG}(2k+2,k)$ $(k \ge 2)$ be a Schrijver graph, then*

$$\mathrm{la}(G)=\begin{cases} 3, & \text{for } k=2,\\ \lceil (k+2)/2 \rceil, & \text{for } k \ge 3. \end{cases}$$

Proof. For $k=2$, $\mathrm{SG}(2k+2,k)=\mathrm{SG}(6,2)$ is a 4-regular graph, and the result holds by Lemma 2.3. So, in this section, suppose that $k \ge 3$ hereafter. It is obvious that $\mathrm{la}(G) \ge \lceil (k+2)/2 \rceil$ since $\Delta(G)=k+2$. So it suffices to show that $\mathrm{la}(G) \le \lceil (k+2)/2 \rceil$. By Lemma 1.3,

$$[V_j,V_{j+1}]=\{v(j,i)v(j+1,i) \mid i=0,1,\dots,2k+1\}$$

for $0 \le j \le m-2$,

$$[V_{m-1},V_m]=\{v(m-1,i)v(m,i) \mid i=0,1,\dots,2k+1\}$$

which is a matching if k is odd, and

$$G[[V_{m-1},V_m]]=\{v(m-1,i)v(m,i)v(m-1,i+k+1) \mid i=0,1,\dots,k\}$$

if k is even, in which each component is a 2-path.

By Corollary 1.1, $G[V_0]$ is a balanced complete bipartite graph with two partite subsets

$$X = \{v(0,0), v(0,2), \ldots, v(0,2k)\} \quad \text{and} \quad Y = \{v(0,1), v(0,3), \ldots, v(0,2k+1)\}.$$

Let $v(0,2i) = x_i$ and $v(0,2i+1) = y_i$. Then $G[V_0] = K_{k+1,k+1}$ is a balanced complete bipartite graph with two partite subsets

$$X = \{x_i \mid i = 0,1,\ldots,k\} \quad \text{and} \quad Y = \{y_i \mid i = 0,1,\ldots,k\}.$$

Case 1. $k \geq 3$ is odd.

It is not difficult to see that

$$G[\cup_{j=0}^{m-1}[V_j, V_{j+1}]] = \{v(0,i)v(1,i)\cdots v(m,i) \mid i = 0,1,\ldots,2k+1\}$$

is a linear forest in which each component is an m-path. Let $B = \cup_{j=0}^{m-1}[V_j, V_{j+1}]$ and $S_j = V_0 \cup V_1 \cup \cdots \cup V_j$. By Lemma 1.3, it is not difficult to see that every $[V_j, V_{j+1}] = [S_j, \overline{S_j}\,](0 \leq j \leq m-1)$ is an edge cut of G. Hence $G \setminus B$ is a graph whose components are $G[V_0], G[V_1], \ldots, G[V_m]$. Next, we will take away a matching from $G[V_0]$. By Lemma 2.1, $G[V_0] = K_{k+1,k+1}$ can be decomposed into the union of $(k+1)/2$ Hamiltonian path and a matching $M = \{x_{\alpha/2}y_{k-\alpha/2} \mid \alpha = 0,2,\ldots,k-1\}$. Then $M \cup B$ forms a linear forest. Moreover, we have $G = (G[V_0] \setminus M) \cup G[V_1] \cup \cdots \cup G[V_m] \cup (M \cup B)$. Thus by Lemma 1.1, Corollary 1.2 and Lemma 2.2, $\mathrm{la}(G) \leq \mathrm{la}((G[V_0] \setminus M) \cup G[V_1] \cup \cdots \cup G[V_m]) + 1 = \mathrm{la}(G[V_0] \setminus M) + 1 \leq (k+1)/2 + 1 = \lceil (k+2)/2 \rceil$.

Case 2. $k \geq 4$ is even.

Let

$$B' = G[\cup_{j=0}^{m-2}[V_j, V_{j+1}]] = \{v(0,i)v(1,i)\cdots v(m-1,i) \mid i = 0,1,3,\ldots,2k+1\},$$

then

$$G = G[V_0] \cup G[V_1] \cup \cdots \cup G[V_m] \cup B' \cup G[[V_{m-1}, V_m]].$$

In the following, we first decompose $G[V_j](j = 0,1,\ldots,m)$ and B'. Let

$$P_i = v(0,i)v(1,i)\cdots v(m-1,i) \quad \text{for} \quad 0 \leq i \leq 2k+1.$$

By Lemma 2.1, $G[V_0] = K_{k+1,k+1}$ can be decomposed into the union of $k/2$ Hamiltonian paths

$$H_{\alpha/2} = M_\alpha \cup M_{\alpha+1} \setminus \{x_{\alpha/2}y_{k-\alpha/2}\}(\alpha = 0,2,4,\ldots,k-2)$$

and a linear forest

$$M = M_k \cup \{x_{\alpha/2}y_{k-\alpha/2} \mid \alpha = 0,2,4,\ldots,k-2\}.$$

Hence $G[V_0] = H_0 \cup H_1 \cup \cdots \cup H_{k/2-1} \cup M$. For $1 \leq j \leq k/2-1$, let $G[V_j] = P_{j,1} \cup P_{j,2}$, where

$$P_{j,1} = v(j,0)v(j,1)\cdots v(j,2k) \quad \text{and} \quad P_{j,2} = v(j,0)v(j,2k+1)v(j,2k).$$

For $j = m$, let $G[V_m] = P_{m,1} \cup P_{m,2}$, where

$$P_{m,1} = v(m,0)v(m,1)\cdots v(m,k-1) \quad \text{and} \quad P_{m,2} = v(m,0)v(m,k)v(m,k-1).$$

Subcase 2.1. $k \equiv 0 \pmod 4$.

Let $P_0 = M_0 \cup M_0'$, where

$$M_0 = \{v(2t,0)v(2t+1,0) \mid t = 0,1,\ldots,k/4-1\},$$

and

$$M'_0 = \{v(2t+1,0)v(2t+2,0) \mid t = 0,1,\ldots,k/4-2\}.$$

And let $P_{2k} = M_{2k} \cup M'_{2k}$, where

$$M_{2k} = \{v(2t,2k)v(2t+1,2k) \mid t = 0,1,\ldots,k/4-1\},$$

and

$$M'_{2k} = \{v(2t+1,2k)v(2t+2,2k) \mid t = 0,1,\ldots,k/4-2\}.$$

Then

$$H_0 \cup M_0 \cup M'_{2k} \cup P_{2k+1} \cup (\cup_{j=1}^{m} P_{j,1}) \cup \{v(m-1,2k)v(m,k-1), v(m-1,2k+1)v(m,k)\}$$

forms a Hamiltonian path of G. Let

$$T = [V_{m-1}, V_m] \setminus \{v(m-1,2k+1)v(m,k), v(m-1,2k)v(m,k-1)\}.$$

Then

$$H_1 \cup P_2 \cup P_{2k-1} \cup (\cup_{j=1}^{m-1} P_{j,2}) \cup T$$

forms a linear forest. For $2 \le j \le m-1$, each $H_j \cup P_{2j} \cup P_{2k-2j+1}$ forms a linear forest. Finally,

$$M \cup M'_0 \cup M_{2k} \cup P_{m,2} \cup (\cup_{j=0}^{m} P_{2j+1}) \cup (\cup_{j=0}^{m-1} P_{2j+k})$$

forms a linear forest. Thus, the edge set $E(G)$ is partitioned into $(k+2)/2$ linear forest. Hence $\mathrm{la}(G) \le (k+2)/2$.

Subcase 2.2. $k \equiv 2 \pmod 4$.

Similar to Subcase 2.1, let $P_0 = N_0 \cup N'_0$, where

$$N_0 = \{v(2t,0)v(2t+1,0) \mid t = 0,1,\ldots,(k-6)/4\},$$

and

$$N'_0 = \{v(2t+1,0)v(2t+2,0) \mid t = 0,1,\ldots,(k-6)/4\}.$$

Let $P_{2k} = N_{2k} \cup N'_{2k}$, where

$$N_{2k} = \{v(2t,2k)v(2t+1,2k) \mid t = 0,1,\ldots,(k-6)/4\},$$

and

$$N'_{2k} = \{v(2t+1,2k)v(2t+2,2k) \mid t = 0,1,\ldots,(k-6)/4\}.$$

Then it is not difficult to see that

$$H_0 \cup N_0 \cup N'_{2k} \cup P_{2k+1} \cup (\cup_{j=1}^{m} P_{j,1}) \cup \{v(m-1,0)v(m,0), v(m-1,2k+1)v(m,k)\},$$

forms a Hamiltonian path of G. Let

$$T' = [V_{m-1}, V_m] \setminus \{v(m-1,0)v(m,0), v(m-1,2k+1)v(m,k)\}.$$

Clearly,

$$H_1 \cup P_2 \cup P_{2k-1} \cup (\cup_{j=1}^{m-1} P_{j,2}) \cup T'$$

forms a linear forest, and for $2 \le j \le m-1$, each $H_j \cup P_{2j} \cup P_{2k-2j+1}$ forms a linear forest. Finally, it is not difficult to verify that

$$M \cup N'_0 \cup N_{2k} \cup P_{m,2} \cup (\cup_{j=0}^{m} P_{2j+1}) \cup (\cup_{j=0}^{m-1} P_{2j+k})$$

forms a linear forest. Thus, the edge set $E(G)$ is partitioned into $(k+2)/2$ linear forests. Hence we have $\mathrm{la}(G) \le (k+2)/2$, too.

Up to now, we have shown that $\mathrm{la}(G) \le \lceil (k+2)/2 \rceil$, and then the theorem holds. ∎

Therefore, the linear arboricity conjecture holds for Schrijver graph $\mathrm{SG}(2k+2,k)$ for $k \geq 2$.

3. The vertex linear arboricity and vertex arboricity of Schrijver graph $\mathrm{SG}(2k+2,k)$

In this section, we discuss the vertex linear arboricity and the vertex arboricity of the Schrijver graph.

Theorem 3.1. *The vertex linear arboricity for the Schrijver graph $G = \mathrm{SG}(2k+2,k)$, $(k \geq 2)$, is two.*

Proof. The proof will be split into three cases. The main idea is to partition the vertex set $V(G)$ into two subsets such that every subset induces a linear forest.

Case 1. $k \geq 3$ is odd.

Let

$$Q = \{v(j,i) \mid 0 \leq j \leq m,\ i = 0,2,\ldots,2k\},$$

and

$$R = \{v(j,i) \mid 0 \leq j \leq m,\ i = 1,3,\ldots,2k+1\}.$$

By Lemma 1.3,

$$G[Q] = \{v(0,i)v(1,i)\ldots v(m,i)v(m,i+k+1)v(m-1,i+k+1)\cdots v(0,i+k+1)| \\ i = 0,2,\ldots,k-1\}$$

and

$$G[R] = \{v(0,i)v(1,i)\cdots v(m,i)v(m,i+k+1)v(m-1,i+k+1)\cdots v(0,i+k+1)| \\ i = 1,3,\ldots,k\}$$

are two linear forests in which every component is a k-path.

Case 2. $k \geq 4$ is even.

Let

$$Q' = \{v(j,i) \mid 0 \leq j \leq m-1,\ i = 0,2,\ldots,2k\} \cup \{v(m,i) \mid i = 0,2,\ldots,k\},$$

and

$$R' = \{v(j,i) \mid 0 \leq j \leq m-1,\ i = 1,3,\ldots,2k+1\} \cup \{v(m,i) \mid i = 1,3,\ldots,k-1\}.$$

By Lemma 1.3,

$$\begin{aligned} G[Q'] = &\{v(0,0)v(1,0)\cdots v(m,0)v(m,k)v(m-1,k)\cdots v(0,k)\} \\ &\cup \{v(0,i)v(1,i)\cdots v(m,i) \mid i = 2,4,\ldots,k-2\} \\ &\cup \{v(0,i)v(1,i)\cdots v(m-1,i) \mid i = k+2,k+4,\ldots,2k\} \end{aligned}$$

and

$$\begin{aligned} G[R'] = &\{v(0,i)v(1,i)\cdots v(m,i) \mid i = 1,3,\ldots,k-1\} \\ &\cup \{v(0,i)v(1,i)\cdots v(m-1,i) \mid i = k+1,k+3,\ldots,2k+1\} \end{aligned}$$

are two linear forests.

Case 3. $k = 2$.

One can partition the vertex set $V(G)$ into two subsets

$$\{v(0,0),v(0,2),v(0,4),v(1,0),v(1,2)\} \quad \text{and} \quad \{v(0,1),v(0,3),v(0,5),v(1,1)\}.$$

It is easy to verify that every subset induces a linear forest. ∎

The following result follows from the fact that the Schrijver graph $G = \mathrm{SG}(2k+2,k)$ contains a cycle for $k \geq 2$.

Corollary 3.1. *The vertex arboricity for the Schrijver graph* $\mathrm{SG}(2k+2,k)$*,* $(k \geq 2)$*, is 2.*

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