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# Weak Annihilator over Extension Rings

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**Abstract.** Let $R$ be a ring and $\mathrm{nil}(R)$ the set of all nilpotent elements of $R$. For a subset $X$ of a ring $R$, we define $N_R(X) = \{a \in R \mid xa \in \mathrm{nil}(R) \text{ for all } x \in X\}$, which is called the weak annihilator of $X$ in $R$. In this paper we mainly investigate the properties of the weak annihilator over extension rings.

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## 1. Introduction

Throughout this paper $R$ denotes an associative ring with unity, $\alpha : R \longrightarrow R$ is an endomorphism, and $\delta$ an $\alpha$-derivation of $R$, that is, $\delta$ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for $a, b \in R$. We denote by $R[x;\alpha,\delta]$ the Ore extension whose elements are the polynomials over $R$, the addition is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x + \delta(a)$ for any $a \in R$. We use $P(R)$ and $\mathrm{nil}(R)$ to represent the prime radical and the set of all nilpotent elements of $R$ respectively. Due to Birkenmeier *et al.* [3], a ring $R$ is called *2-primal* if $P(R) = \mathrm{nil}(R)$. Every reduced ring (i.e. $\mathrm{nil}(R) = 0$) is obviously a *2-primal* ring. Other examples and properties of *2-primal* rings can be founded in [4, 5, 6]. Let $\alpha$ be an endomorphism and $\delta$ an $\alpha$-derivation of a ring $R$. Following E. Hashemi and A. Moussavi [11], a ring $R$ is said to be $\alpha$-*compatible* if for each $a, b \in R, ab = 0 \Leftrightarrow a\alpha(b) = 0$. Moreover, $R$ is called to be $\delta$-*compatible* if for each $a, b \in R, ab = 0 \Rightarrow a\delta(b) = 0$. If $R$ is both $\alpha$-*compatible* and $\delta$-*compatible*, then $R$ is said to be $(\alpha,\delta)$-*compatible*.

For a subset $X$ of a ring $R$, $r_R(X) = \{a \in R \mid Xa = 0\}$ and $l_R(X) = \{a \in R \mid aX = 0\}$ will stand for the right and left annihilator of $X$ in $R$, respectively. Properties of the right (left) annihilator of a subset in a ring $R$ are studied by many authors (see [2, 8, 9, 14, 15]). As a generalization of the right (left) annihilator, in this paper we introduce the notion of a weak

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annihilator of a subset in a ring, and investigate the weak annihilator properties over the Ore extension ring $R[x;\alpha,\delta]$.

In this paper all subsets are nonempty. Let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x;\alpha,\delta]$. We say that $f(x) \in \mathrm{nil}(R)[x;\alpha,\delta]$ if and only if $a_i \in \mathrm{nil}(R)$ for all $0 \le i \le n$. Let $I$ be a subset of $R$, $I[x;\alpha,\delta]$ means $\{u_0 + u_1x + \cdots + u_nx^n \in R[x;\alpha,\delta] \mid u_i \in I\}$, that is, for any skew polynomial $f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x;\alpha,\delta]$, $f(x) \in I[x;\alpha,\delta]$ if and only if $a_i \in I$ for all $0 \le i \le n$. If $f(x) \in R[x;\alpha,\delta]$ is a nilpotent element of $R[x;\alpha,\delta]$, then we say that $f(x) \in \mathrm{nil}(R[x;\alpha,\delta])$. For $f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x;\alpha,\delta]$, we denote by $\{a_0, a_1, \cdots, a_n\}$ or $C_f$ the set comprised of the coefficients of $f(x)$, and for a subset $U \subseteq R[x;\alpha,\delta]$, $C_U = \bigcup_{f\in U} C_f$.

## 2. Weak annihilator

**Definition 2.1.** *Let $R$ be a ring. For a subset $X$ of a ring $R$, we define $N_R(X) = \{a \in R \mid xa \in \mathrm{nil}(R)$ for all $x \in X\}$, which is called the weak annihilator of $X$ in $R$. If $X$ is singleton, say $X = \{r\}$, we use $N_R(r)$ in place of $N_R(\{r\})$.*

Obviously, for any subset $X$ of a ring $R$, $N_R(X) = \{a \in R \mid xa \in \mathrm{nil}(R)$ for all $x \in X\} = \{b \in R \mid bx \in \mathrm{nil}(R)$ for all $x \in X\}$, and $r_R(X) \subseteq N_R(X)$ and $l_R(X) \subseteq N_R(X)$. If $R$ is reduced, then $r_R(X) = N_R(X) = l_R(X)$ for any subset $X$ of $R$. It is easy to see that for any subset $X \subseteq R$, $N_R(X)$ is an ideal of $R$ in case $\mathrm{nil}(R)$ is an ideal.

**Example 2.1.** Let $Z$ be the ring of integers and $T_2(Z)$ the $2\times 2$ upper triangular matrix ring over $Z$. We consider the subset $X = \left\{\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2\end{smallmatrix}\right)\right\}$. Clearly, $r_{T_2(Z)}(X) = 0$, and $N_{T_2(Z)}(X) = \left\{\left(\begin{smallmatrix} 0 & m \\ 0 & 0\end{smallmatrix}\right), \mid m \in Z\right\}$. Thus $r_{T_2(Z)}(X) \ne N_{T_2(Z)}(X)$. Hence a weak annihilator is not a trivial generalization of a annihilator.

**Proposition 2.1.** *Let $X, Y$ be subsets of $R$. Then we have the following:*

(1) *$X \subseteq Y$ implies $N_R(X) \supseteq N_R(Y)$.*
(2) *$X \subseteq N_R(N_R(X))$.*
(3) *$N_R(X) = N_R(N_R(N_R(X)))$.*

*Proof.* (1) and (2) are really easy.

(3) Applying (2) to $N_R(X)$, we obtain $N_R(X) \subseteq N_R(N_R(N_R(X)))$. Since $X \subseteq N_R(N_R(X))$, we have $N_R(X) \supseteq N_R(N_R(N_R(X)))$ by (1). Therefore we have $N_R(X) = N_R(N_R(N_R(X)))$.

Let $\delta$ be an $\alpha$-derivation of $R$. For integers $i, j$ with $0 \le i \le j$, $f_i^j \in End(R,+)$ will denote the map which is the sum of all possible words in $\alpha, \delta$ built with $i$ letters $\alpha$ and $j-i$ letters $\delta$. For instance, $f_0^0 = 1, f_j^j = \alpha^j, f_0^j = \delta^j$ and $f_{j-1}^j = \alpha^{j-1}\delta + \alpha^{j-2}\delta\alpha + \cdots + \delta\alpha^{j-1}$. The next Lemma appears in [12. Lemma 4.1]. ∎

**Lemma 2.1.** *For any positive integer $n$ and $r \in R$, we have $x^n r = \sum_{i=0}^n f_i^n(r)x^i$ in the ring $R[x;\alpha,\delta]$.*

For the proof of the next lemma, see [11].

**Lemma 2.2.** *Let $R$ be an $(\alpha,\delta)$-compatible ring. Then we have the following:*

(1) *If $ab = 0$, then $a\alpha^n(b) = \alpha^n(a)b = 0$ for all positive integers $n$.*
(2) *If $\alpha^k(a)b = 0$ for some positive integer $k$, then $ab = 0$.*
(3) *If $ab = 0$, then $\alpha^n(a)\delta^m(b) = 0 = \delta^m(a)\alpha^n(b)$ for all positive integers $m, n$.*

**Lemma 2.3.** *Let $\delta$ be an $\alpha$-derivation of $R$. If $R$ is $(\alpha,\delta)$-compatible, then $abc=0$ implies $abf_i^j(c)=0$ and $af_i^j(b)c=0$ for all $0\leq i\leq j$ and $a$, $b$, $c\in R$.*

*Proof.* Let $abc=0$ for $a$, $b$, $c\in R$. Then $ab\alpha(c)=ab\delta(c)=0$ since $R$ is $(\alpha,\delta)$-compatible. Thus $abf_i^j(c)=0$ is clear. To see $af_i^j(b)c=0$, it suffices to show that if $abc=0$, then $a\alpha(b)c=0$ and $a\delta(b)c=0$. Take $a$, $b$, $c\in R$ such that $abc=0$. Then because $R$ is $(\alpha,\delta)$-compatible,

$$abc=0\Rightarrow a\alpha(bc)=a\alpha(b)\alpha(c)=0\Rightarrow a\alpha(b)c=0,$$

and

$$a\alpha(b)c=0\Rightarrow a\alpha(b)\delta(c)=0.$$

Moreover,

$$abc=0\Rightarrow a\delta(bc)=a\alpha(b)\delta(c)+a\delta(b)c=0\Rightarrow a\delta(b)c=0.$$

Therefore we obtain $af_i^j(b)c=0$. ∎

**Corollary 2.1.** *Let $R$ be an $(\alpha,\delta)$-compatible ring. Then $a_1a_2\cdots a_n=0$ implies*

$$f_{s_1}^{t_1}(a_1)f_{s_2}^{t_2}(a_2)\cdots f_{s_n}^{t_n}(a_n)=0$$

*for all $t_i\geq s_i\geq 0$ and $a_i\in R$, $i=1,2,\cdots,n$.*

*Proof.* It follows from Lemma 2.3. ∎

**Lemma 2.4.** *Let $\delta$ be an $\alpha$-derivation of $R$. If $R$ is $(\alpha,\delta)$-compatible, then $ab\in \mathrm{nil}(R)$ implies $af_i^j(b)\in \mathrm{nil}(R)$ for all $j\geq i\geq 0$ and $a,b\in R$.*

*Proof.* Since $ab\in\mathrm{nil}(R)$, there exists some positive integer $k$ such that $(ab)^k=abab\cdots ab=0$. Then by Corollary 2.1, it is easy to see that $af_i^j(b)\in\mathrm{nil}(R)$. ∎

**Lemma 2.5.** *Let $R$ be an $(\alpha,\delta)$-compatible ring. If $a\alpha^m(b)\in\mathrm{nil}(R)$ for $a,b\in R$, and $m$ is a positive integer, then $ab\in\mathrm{nil}(R)$.*

*Proof.* Since $a\alpha^m(b)\in\mathrm{nil}(R)$, there exists some positive integer $n$ such that $(a\alpha^m(b))^n=0$. In the following computations, we use freely the condition that $R$ is $(\alpha,\delta)$-*compatible*:

$$\begin{aligned}(a\alpha^m(b))^n=&\underbrace{a\alpha^m(b)a\alpha^m(b)\cdots a\alpha^m(b)}_{n}=0\\ \Rightarrow& a\alpha^m(b)a\alpha^m(b)\cdots a\alpha^m(b)ab=0\\ \Rightarrow& a\alpha^m(b)a\alpha^m(b)\cdots a\alpha^m(b)\alpha^m(ab)=0\\ \Rightarrow& a\alpha^m(b)a\alpha^m(b)\cdots a\alpha^m(b)a\alpha^m(bab)=0\\ \Rightarrow& a\alpha^m(b)a\alpha^m(b)\cdots a\alpha^m(b)abab=0\\ \Rightarrow&\cdots\Rightarrow ab\in\mathrm{nil}(R).\end{aligned}$$

∎

**Lemma 2.6.** *Let $R$ be an $(\alpha,\delta)$-compatible 2-primal ring and $f(x)=a_0+a_1x+\cdots+a_nx^n\in R[x;\alpha,\delta]$. Then $f(x)\in\mathrm{nil}(R[x;\alpha,\delta])$ if and only if $a_i\in\mathrm{nil}(R)$ for all $0\leq i\leq n$.*

*Proof.* $(\Longrightarrow)$ Suppose $f(x)\in\mathrm{nil}(R[x;\alpha,\delta])$. There exists some positive integer $k$ such that $f(x)^k=(a_0+a_1x+\cdots+a_nx^n)^k=0$. Then

$$0=f(x)^k=\text{“}lower\ terms\text{”}+a_n\alpha^n(a_n)\alpha^{2n}(a_n)\cdots\alpha^{(k-1)n}(a_n)x^{nk}.$$

Hence $a_n\alpha^n(a_n)\alpha^{2n}(a_n)\cdots\alpha^{(k-1)n}(a_n)=0$, and $\alpha$-compatibility of $R$ gives $a_n\in \text{nil}(R)$. So by Lemma 2.4, $a_n = 1\cdot a_n\in \text{nil}(R)$ implies $1\cdot f_i^j(a_n)=f_i^j(a_n)\in \text{nil}(R)$ for all $0\le i\le j$. Let $Q=a_0+a_1x+\cdots+a_{n-1}x^{n-1}$. Then we have

$$\begin{aligned}0 &= (Q+a_nx^n)^k\\ &= (Q+a_nx^n)(Q+a_nx^n)\cdots(Q+a_nx^n)\\ &= (Q^2+Q\cdot a_nx^n+a_nx^n\cdot Q+a_nx^n\cdot a_nx^n)(Q+a_nx^n)\cdots(Q+a_nx^n)\\ &= \cdots = Q^k+\Delta,\end{aligned}$$

where $\Delta\in R[x;\alpha,\delta]$. Note that the coefficients of $\Delta$ can be written as sums of monomials in $a_i$ and $f_u^v(a_j)$ where $a_i,a_j\in\{a_0,a_1,\cdots,a_n\}$ and $v\ge u\ge 0$ are positive integers, and each monomial has $a_n$ or $f_s^t(a_n)$. Since nil$(R)$ of a *2-primal* ring $R$ is an ideal, we obtain that each monomial is in nil$(R)$, and so $\Delta\in \text{nil}(R)[x;\alpha,\delta]$. Thus we obtain

$$\begin{aligned}&(a_0+a_1x+\cdots+a_{n-1}x^{n-1})^k\\ =\ &\text{“}\textit{lower terms}\text{”}+a_{n-1}\alpha^{n-1}(a_{n-1})\cdots\alpha^{(n-1)(k-1)}(a_{n-1})x^{(n-1)k}\in \text{nil}(R)[x;\alpha,\delta]\end{aligned}$$

since nil$(R)$ is an ideal of $R$. Hence

$$a_{n-1}\alpha^{n-1}(a_{n-1})\cdots\alpha^{(k-1)(n-1)}(a_{n-1})\in \text{nil}(R)$$

and so $a_{n-1}\in \text{nil}(R)$ by Lemma 2.5. Using induction on $n$ we obtain $a_i\in \text{nil}(R)$ for all $0\le i\le n$.

($\Longleftarrow$) Consider the finite subset $S=\{a_0,a_1,\cdots,a_n\}\subseteq \text{nil}(R)$. Since $R$ is a *2-primal* ring, there exists an integer $k$ such that any product of $k$ elements $a_{i1}a_{i2}\cdots a_{ik}$ from $\{a_0,a_1,\cdots,a_n\}$ is zero. Then by Corollary 2.1, we obtain

$$a_{i1}f_{s_{i2}}^{t_{i2}}(a_{i2})f_{s_{i3}}^{t_{i3}}(a_{i3})\cdots f_{s_{ik}}^{t_{ik}}(a_{ik})=0.$$

Now we claim that

$$f(x)^k=(a_0+a_1x+\cdots+a_nx^n)^k=0.$$

From

$$(\sum_{i=0}^n a_ix^i)^2=\sum_{k=0}^{2n}\left(\sum_{s+t=k}(\sum_{i=s}^n a_if_s^i(a_t))\right)x^k,$$

it is easy to check that the coefficients of $(\sum_{i=0}^n a_ix^i)^k$ can be written as sums of monomials of length $k$ in $a_i$ and $f_u^v(a_j)$, where $a_i,a_j\in\{a_0,a_1,\cdots,a_n\}$ and $v\ge u\ge 0$ are positive integers. Since each monomial $a_{i1}f_{s_{i2}}^{t_{i2}}(a_{i2})\cdots f_{s_{ik}}^{t_{ik}}(a_{ik})=0$, where $a_{i1}$, $a_{i2}$, $\cdots$, $a_{ik}\in\{a_0,a_1,\cdots,a_n\}$ and $s_{ip}$, $t_{ip}$ are nonnegative integers for all $2\le p\le k$. We obtain $f(x)^k=0$. Hence $f(x)$ is a nilpotent element of $R[x;\alpha,\delta]$. ∎

**Corollary 2.2.** *Let R be an $(\alpha,\delta)$-compatible 2-primal. Then we have the following:*

(1) nil$(R[x;\alpha,\delta])$ *is an ideal.*

(2) nil$(R[x;\alpha,\delta])=\text{nil}(R)[x;\alpha,\delta]$.

*In particular, if R is an $\alpha$-compatible ring, then* nil$(R[x;\alpha])$ *is an ideal and* nil$(R[x;\alpha])=$ nil$(R)[x;\alpha]$.

**Theorem 2.1.** *Let R be an $(\alpha,\delta)$-compatible 2-primal ring. If for each subset $X\not\subseteq$ nil$(R)$, $N_R(X)$ is generated as an ideal by a nilpotent element, then for each subset $U\not\subseteq$ nil$(R[x;\alpha,\delta])$, $N_{R[x;\alpha,\delta]}(U)$ is generated as an ideal by a nilpotent element.*

*Proof.* Let $U$ be a subset of $R[x;\alpha,\delta]$ with $U \not\subseteq \mathrm{nil}(R[x;\alpha,\delta])$. Then by Corollary 2.2, we have $C_U \not\subseteq \mathrm{nil}(R)$. So there exists $c \in \mathrm{nil}(R)$ such that $N_R(C_U) = c\cdot R$. Now we show that $N_{R[x;\alpha,\delta]}(U) = c\cdot R[x;\alpha,\delta]$. For any $d(x) = d_0 + d_1x + \cdots + d_ux^u \in U$ and $h(x) = h_0 + h_1x + \cdots + h_vx^v \in R[x;\alpha,\delta]$, we have

$$d(x)\cdot ch(x) = \sum_{k=0}^{u+v}\left(\sum_{s+t=k}(\sum_{i=s}^{u} d_i f_s^i(ch_t))\right)x^k.$$

Since $c \in \mathrm{nil}(R)$ and $\mathrm{nil}(R)$ of a *2-primal* ring is an ideal, we obtain $d_ich_t \in \mathrm{nil}(R)$, and so $d_if_s^i(ch_t) \in \mathrm{nil}(R)$ by Lemma 2.4. Hence $\sum_{s+t=k}(\sum_{i=s}^{u} d_if_s^i(ch_t)) \in \mathrm{nil}(R)$, and so $d(x)\cdot ch(x) \in \mathrm{nil}(R[x;\alpha,\delta])$ by Lemma 2.6, and so $N_{R[x;\alpha,\delta]}(U) \supseteq c\cdot R[x;\alpha,\delta]$. Let $g(x) = b_0 + b_1x + \cdots + b_nx^n \in N_{R[x;\alpha,\delta]}(U)$, then $f(x)g(x) \in \mathrm{nil}(R[x;\alpha,\delta])$ for any $f(x) = a_0 + a_1x + \cdots + a_mx^m \in U$. Then

$$f(x)g(x) = \sum_{k=0}^{m+n}\left(\sum_{s+t=k}(\sum_{i=s}^{m} a_if_s^i(b_t))\right)x^k = \sum_{k=0}^{m+n}\Delta_kx^k \in \mathrm{nil}(R[x;\alpha,\delta]).$$

Then we have the following equations by Lemma 2.6:

$$\Delta_{m+n} = a_m\alpha^m(b_n), \tag{2.1}$$

$$\Delta_{m+n-1} = a_m\alpha^m(b_{n-1}) + a_{m-1}\alpha^{m-1}(b_n) + a_mf_{m-1}^m(b_n), \tag{2.2}$$

$$\Delta_{m+n-2} = a_m\alpha^m(b_{n-2}) + \sum_{i=m-1}^{m} a_if_{m-1}^i(b_{n-1}) + \sum_{i=m-2}^{m} a_if_{m-2}^i(b_n), \tag{2.3}$$

$$\vdots$$

$$\Delta_k = \sum_{s+t=k}(\sum_{i=s}^{m} a_if_s^i(b_t)), \tag{2.4}$$

with $\Delta_i \in \mathrm{nil}(R)$ for all $0 \le i \le m+n$. From Lemma 2.5 and Equation (2.1), we obtain $a_mb_n \in \mathrm{nil}(R)$, and so $b_na_m \in \mathrm{nil}(R)$. Now we show that $a_ib_n \in \mathrm{nil}(R)$ for all $0 \le i \le m$. If we multiply Equation (2.2) on the left side by $b_n$, then $b_na_{m-1}\alpha^{m-1}(b_n) = b_n\Delta_{m+n-1} - (b_na_m\alpha^m(b_{n-1}) + b_na_mf_{m-1}^m(b_n)) \in \mathrm{nil}(R)$ since the $\mathrm{nil}(R)$ of a *2-primal* ring is an ideal. Thus by Lemma 2.5, we obtain $b_na_{m-1}b_n \in \mathrm{nil}(R)$, and so $b_na_{m-1} \in \mathrm{nil}(R), a_{m-1}b_n \in \mathrm{nil}(R)$. If we multiply Equation (2.3) on the left side by $b_n$, then we obtain $b_na_{m-2}f_{m-2}^{m-2}(b_n) = b_na_{m-2}\alpha^{m-2}(b_n) = b_n\Delta_{m+n-2} - b_na_m\alpha^m(b_{n-2}) - b_na_{m-1}f_{m-1}^{m-1}(b_{n-1}) - b_na_mf_{m-1}^m(b_{n-1}) - b_na_{m-1}f_{m-2}^{m-1}(b_n) - b_na_mf_{m-2}^m(b_n) = b_n\Delta_{m+n-2} - (b_na_m)\alpha^m(b_{n-2}) - (b_na_{m-1})f_{m-1}^{m-1}(b_{n-1}) - (b_na_m)f_{m-1}^m(b_{n-1}) - (b_na_{m-1})f_{m-2}^{m-1}(b_n) - (b_na_m)\ f_{m-2}^m(b_n) \in \mathrm{nil}(R)$ since $\mathrm{nil}(R)$ is an ideal of $R$. Thus we obtain $a_{m-2}b_n \in \mathrm{nil}(R)$ and $b_na_{m-2} \in \mathrm{nil}(R)$. Continuing this procedure yields that $a_ib_n \in \mathrm{nil}(R)$ for all $0 \le i \le m$, and so $a_if_s^t(b_n) \in \mathrm{nil}(R)$ for any $t \ge s \ge 0$ and $0 \le i \le m$ by Lemma 2.4. Thus it is easy to verify that $(\sum_{i=0}^{m} a_ix^i)(\sum_{j=0}^{n-1} b_jx^j) \in \mathrm{nil}(R)[x;\alpha,\delta]$. Applying the preceding method repeatedly, we obtain $a_ib_j \in \mathrm{nil}(R)$ for all $0 \le i \le m$ and $0 \le j \le n$. Thus $b_j \in N_R(C_U) = c\cdot R$ for all $0 \le j \le n$. Thus there exists $r_j \in R$ such that $b_j = cr_j$. Hence $g(x) = b_0 + b_1x + \cdots + b_nx^n = c(r_0 + r_1x + \cdots + r_nx^n) \in c\cdot R[x;\alpha,\delta]$. Therefore $N_{R[x;\alpha,\delta}(U) = c\cdot R[x;\alpha,\delta]$ where $c \in \mathrm{nil}(R[x;\alpha,\delta])$. ∎

**Corollary 2.3.** *Let $R$ be an $(\alpha,\delta)$-compatible 2-primal ring, and $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j \in R[x;\alpha,\delta]$. Then $f(x)g(x) \in \mathrm{nil}(R[x;\alpha,\delta])$ if and only if $a_ib_j \in \mathrm{nil}(R)$ for all $i$, $j$.*

*Proof.* ($\Leftarrow$) Suppose $a_ib_j \in \text{nil}(R)$ for all $i,j$. Then $a_if_s^i(b_j) \in \text{nil}(R)$ for all $i,j$ and all positive integer $i \geq s \geq 0$ by Lemma 2.4. Thus

$$\sum_{s+t=k}\left(\sum_{i=s}^{m} a_if_s^i(b_t)\right) \in \text{nil}(R), k = 0,1,2,\cdots m+n.$$

Hence $f(x)g(x) = \sum_{k=0}^{m+n}(\sum_{s+t=k}(\sum_{i=s}^{m} a_if_s^i(b_t)))x^k \in \text{nil}(R[x;\alpha,\delta])$ by Lemma 2.6.

($\Rightarrow$) By analogy with the proof of Theorem 2.1, we complete the proof. ∎

**Theorem 2.2.** *Let $R$ be an $\alpha$-compatible 2-primal ring. Then the following statements are equivalent:*

(1) *For each subset $X \nsubseteq \text{nil}(R)$, $N_R(X)$ is generated as an ideal by a nilpotent element.*
(2) *For each subset $U \nsubseteq \text{nil}(R[x;\alpha])$, $N_{R[x;\alpha]}(U)$ is generated as an ideal by a nilpotent element.*

*Proof.* By Theorem 2.1, it suffices to show (2) $\Rightarrow$ (1). Let $X$ be a subset of $R$ with $X \nsubseteq \text{nil}(R)$. Then $X \nsubseteq \text{nil}(R[x;\alpha])$. So there exists $f(x) = a_0 + a_1x + \cdots + a_mx^m \in \text{nil}(R[x;\alpha])$ such that $N_{R[x;\alpha]}(X) = f(x)\cdot R[x;\alpha]$. Note that $f(x) = a_0 + a_1x + \cdots + a_mx^m \in \text{nil}(R[x;\alpha])$, we have $a_i \in \text{nil}(R)$ for all $0 \leq i \leq m$ by Corollary 2.2. Clearly, we may assume that $a_0 \neq 0$. Now we show that $N_R(X) = a_0R$. Since $a_0 \in \text{nil}(R)$ and $\text{nil}(R)$ is an ideal of $R$, we obtain $p\cdot a_0R \subseteq \text{nil}(R)$ for each $p \in X$. So $N_R(X) \supseteq a_0R$. If $m \in N_R(X)$, then $m \in N_{R[x;\alpha]}(X)$. Thus there exists $h(x) = h_0 + h_1x + \cdots + h_qx^q \in R[x;\alpha]$ such that

$$m = f(x)h(x) = \sum_{s=0}^{m+q}\left(\sum_{i+j=s} a_i\alpha^i(h_j)\right)x^s.$$

Thus we have $m = a_0h_0 \in a_0R$, and so $N_R(X) \subseteq a_0R$. Hence $N_R(X) = a_0R$ where $a_0 \in \text{nil}(R)$.

For any $p \in R$, we denote by $p\cdot R$ the principal right ideal of $R$ generated by $p$. Then we have the following results. ∎

**Theorem 2.3.** *Let $R$ be an $(\alpha,\delta)$-compatible 2-primal ring. If for each principal right ideal $p\cdot R \nsubseteq \text{nil}(R)$, $N_R(p\cdot R)$ is generated as an ideal by a nilpotent element, then for each principal right ideal $f(x)\cdot R[x;\alpha,\delta] \nsubseteq \text{nil}(R[x;\alpha,\delta])$, $N_{R[x;\alpha,\delta]}(f(x)\cdot R[x;\alpha,\delta])$ is generated as an ideal by a nilpotent element.*

*Proof.* Let $f(x) = a_0 + a_1x + \cdots + a_mx^m \in R[x;\alpha,\delta])$ with $f(x)\cdot R[x;\alpha,\delta] \nsubseteq \text{nil}(R[x;\alpha,\delta])$. We show that $N_{R[x;\alpha,\delta]}(f(x)\cdot R[x;\alpha,\delta])$ is generated as an ideal by a nilpotent element. If $a_iR \subseteq \text{nil}(R)$ for all $0 \leq i \leq m$, then by Corollary 2.2, it is easy to see that $f(x)\cdot R[x;\alpha,\delta] \subseteq \text{nil}(R[x;\alpha,\delta])$, a contradiction. So there exists $0 \leq i \leq m$ such that $a_iR \nsubseteq \text{nil}(R)$. Thus there exists $c \in \text{nil}(R)$ such that $N_R(a_iR) = c\cdot R$. Now we show that $N_{R[x;\alpha,\delta]}(f(x)\cdot R[x;\alpha,\delta]) = c\cdot R[x;\alpha,\delta]$. For any $u(x) = u_0 + u_1x + \cdots + u_tx^t \in R[x;\alpha,\delta]$ and $v(x) = v_0 + v_1x + \cdots + v_qx^q \in R[x;\alpha,\delta]$, we have $a_iu_jcv_k \in \text{nil}(R)$ for each $i,j,k$, since $c \in \text{nil}(R)$ and $\text{nil}(R)$ is an ideal of $R$. Thus $a_if_s^i(u_j)cv_k \in \text{nil}(R)$ for all $i,j,k$ and $s \leq i$ by Lemma 2.4, and so it is easy to see that $(f(x)u(x))\cdot cv(x) \in \text{nil}(R[x;\alpha,\delta])$ for all $u(x) \in R[x;\alpha,\delta]$ and $v(x) \in R[x;\alpha,\delta]$ by Corollary 2.3. Hence $cv(x) \in N_{R[x;\alpha,\delta]}(f(x)\cdot R[x;\alpha,\delta])$ and so $N_{R[x;\alpha,\delta]}(f(x)\cdot R[x;\alpha,\delta]) \supseteq c\cdot R[x;\alpha,\delta]$. On the other hand, assume that $p(x) = p_0 + p_1x + \cdots + p_sx^s \in N_{R[x;\alpha,\delta]}(f(x)\cdot R[x;\alpha,\delta])$. Then $f(x)\cdot R[x;\alpha,\delta]\cdot p(x) \subseteq \text{nil}(R[x;\alpha,\delta])$ and so $f(x)\cdot R\cdot$

$p(x) \subseteq \text{nil}(R[x;\alpha,\delta])$. Thus we obtain $a_iR \cdot p_j \subseteq \text{nil}(R)$ for all $0 \leq j \leq s$. So $p_j \in N_R(a_iR) = cR$. Thus there exists $r_j \in R$ such that $p_j = cr_j$ for all $0 \leq j \leq s$. Hence $p(x) = p_0 + p_1x + \cdots + p_sx^s = c(r_0 + r_1x + \cdots + r_sx^s) \in c \cdot R[x;\alpha,\delta]$. Hence $N_{R[x;\alpha,\delta]}(f(x) \cdot R[x;\alpha,\delta]) \subseteq c \cdot R[x;\alpha,\delta]$. Therefore $N_{R[x;\alpha,\delta]}(f(x) \cdot R[x;\alpha,\delta]) = c \cdot R[x;\alpha,\delta]$. ∎

**Theorem 2.4.** *Let $R$ be an $\alpha$-compatible 2-primal ring. Then the following statements are equivalent:*

(1) *For each principal right ideal $p \cdot R \not\subseteq \text{nil}(R)$, $N_R(p \cdot R)$ is generated as an ideal by a nilpotent element.*
(2) *For each principal right ideal $f(x) \cdot R[x;\alpha] \not\subseteq \text{nil}(R[x;\alpha])$, $N_{R[x;\alpha]}(f(x) \cdot R[x;\alpha])$ is generated as an ideal by a nilpotent element.*

*Proof.* It follows by the same method of proof as in Theorem 2.2. ∎

Using the same way as above, we also obtain the next two theorems:

**Theorem 2.5.** *Let $R$ be an $(\alpha,\delta)$-compatible 2-primal ring. If for each $p \notin \text{nil}(R)$, $N_R(p)$ is generated as an ideal by a nilpotent element, then for each $f(x) \notin \text{nil}(R[x;\alpha,\delta])$, $N_{R[x;\alpha,\delta]}(f(x))$ is generated as an ideal by a nilpotent element.*

**Theorem 2.6.** *Let $R$ be an $\alpha$-compatible 2-primal ring. Then the following statements are equivalent:*

(1) *For each $p \notin \text{nil}(R)$, $N_R(p)$ is generated as an ideal by a nilpotent element.*
(2) *for each skew polynomial $f(x) \notin \text{nil}(R[x;\alpha])$, $N_{R[x;\alpha]}(f(x))$ is generated as an ideal by a nilpotent element.*

**Example 2.2.** Let $R$ be a domain and let

$$R_3 = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_1 & a_2 \\ 0 & 0 & a_1 \end{pmatrix} \mid a_i \in R \right\}$$

be the subring of $3 \times 3$ upper triangular matrix ring. Let $X$ be any subset of $R_3$ with $X \not\subseteq \text{nil}(R_3)$. We show that $N_{R_3}(X)$ is generated as a ideal by a nilpotent element. Let

$$U = \left\{ x \in R \mid \begin{pmatrix} x & y & z \\ 0 & x & y \\ 0 & 0 & x \end{pmatrix} \in X \right\}.$$

If $U = \{0\}$, then $X \subseteq \text{nil}(R_3)$. This is contrary to the fact that $X \not\subseteq \text{nil}(R_3)$. Thus we have $U \neq \{0\}$. In this case, we have

$$N_{R_3}(X) = \left\{ \begin{pmatrix} 0 & u & v \\ 0 & 0 & u \\ 0 & 0 & 0 \end{pmatrix} \mid u,v \in R \right\} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot R_3,$$

where $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right) \in \text{nil}(R_3)$ by a routine computations. Therefore $N_{R_3}(X)$ is generated as an ideal by a nilpotent element.

**Example 2.3.** Let $Z$ be the ring of integers, and $T(Z,Z) = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a,b \in Z \right\}$ the trivial extension of $Z$ by $Z$. Let $p = \left(\begin{smallmatrix} a & m \\ 0 & a \end{smallmatrix}\right) \in T(Z,Z)$. If $a = 0$, then we have $p \cdot T(Z,Z) \subseteq \text{nil}(T(Z,Z))$. So we assume that $a \neq 0$. By a routine computations, we obtain

$$N_{T(Z,Z)}(p \cdot T(Z,Z)) = \left\{ \left(\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right) \mid m \in Z \right\} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \cdot T(Z,Z),$$

where $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is a nilpotent element.

## 3. Nilpotent associated primes

Given a right $R$-module $N_R$, the right annihilator of $N_R$ is denoted by $r_R(N_R) = \{a \in R \mid Na = 0\}$. We say that $N_R$ is prime if $N_R \neq 0$, and $r_R(N_R) = r_R(N'_R)$ for every nonzero submodule $N'_R \subseteq N_R$ (see [1]). Let $M_R$ be a right $R$-module, an ideal $\wp$ of $R$ is called an associated prime of $M_R$ if there exists a prime submodule $N_R \subseteq M_R$ such that $\wp = r_R(N_R)$. The set of associated primes of $M_R$ is denoted by $\mathrm{Ass}(M_R)$ (see [1]). Associated primes are well-known in commutative algebra for their important role in the primary decomposition, and has attracted a lot of attention in recent years. In [7], Brewer and Heinzer used localization theory to prove that, over a commutative ring $R$, the associated primes of the polynomial ring $R[x]$ (viewed as a module over itself) are all extended: that is, every $\wp \in \mathrm{Ass}(R[x])$ may be expressed as $\wp = \wp_0[x]$, where $\wp_0 = \wp \cap R \in \mathrm{Ass}(R)$. Using results of Shock in [13] on good polynomials, C. Faith has provided a new proof in [10] of the same result which does not rely on localization or other tools from commutative algebra. In [1], Scott Annin showed that Brewer and Heinzer's result still holds in the more general setting of a polynomial module $M[x]$ over a skew polynomial ring $R[x;\alpha]$, with possibly noncommutative base $R$. So the properties of associated primes over a commutative ring can be profitably generalized to noncommutative setting as well.

Motivated by the results in [1], [7], [10], in this section, we continue the study of nilpotent associated primes over Ore extension rings. We first introduce the notion of nilpotent associated primes, which are a generalization of associated primes. We next describe all nilpotent associated primes of the Ore extension ring $R[x;\alpha,\delta]$ in terms of the nilpotent associated primes of the ring $R$.

**Definition 3.1.** *Let $I$ be a right ideal of a nonzero ring $R$. We say that $I$ is a right quasi-prime ideal if $I \not\subseteq \mathrm{nil}(R)$ and $N_R(I) = N_R(I')$ for every right ideal $I' \subseteq I$ and $I' \not\subseteq \mathrm{nil}(R)$.*

Let $R$ be a domain and Let

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

be the subring of $n \times n$ upper triangular matrix ring. Then $\mathrm{nil}(R_n)$ is an ideal of $R_n$ and

$$\mathrm{nil}(R_n) = \left\{ \begin{pmatrix} 0 & x_{12} & \cdots & x_{1n} \\ 0 & 0 & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \mid x_{ij} \in R \right\}.$$

By a routine computations, we know that each right ideal $I \not\subseteq \mathrm{nil}(R_n)$ is a right quasi-prime ideal.

**Definition 3.2.** *Let $\mathrm{nil}(R)$ be an ideal of a ring $R$. An ideal $\wp$ of $R$ is called a nilpotent associated prime of $R$ if there exists a right quasi-prime ideal $I$ such that $\wp = N_R(I)$. The set of nilpotent associated primes of $R$ is denoted by $\mathrm{NAss}(R)$.*

Recall that an ideal $\wp$ in a ring $R$ is said to be a prime ideal if $\wp \neq R$, and for $a, b \in R$, $aRb \subseteq \wp$ implies that $a \in \wp$ or $b \in \wp$. Suppose $\text{nil}(R)$ is an ideal. Then it is easy to see that if $I$ is a right quasi-prime ideal, then $\wp = N_R(I)$ is a prime ideal of $R$.

**Example 3.1.** We now provide the following examples:

**(a)** Let

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

be the subring of $n \times n$ upper triangular matrix ring. Then it is easy to see that $\text{NAss}(R_n) = \{\text{nil}(R_n)\}$.

**(b)** Let $k$ be any field, and consider the ring $R = \left(\begin{smallmatrix} k & 0 \\ k & k \end{smallmatrix}\right)$ of $2 \times 2$ lower triangular matrices over $k$. One easily checks that $\left(\begin{smallmatrix} k & 0 \\ k & k \end{smallmatrix}\right) \supsetneq \left(\begin{smallmatrix} k & 0 \\ k & 0 \end{smallmatrix}\right) \supsetneq \left(\begin{smallmatrix} k & 0 \\ 0 & 0 \end{smallmatrix}\right) \supsetneq \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is a composition series for $R_R$. In particular, $R_R$ has finite length.

Next we shall determine the set $\text{Ass}(R)$. By an easy ad hoc calculation, we can write down all of the proper nonzero ideals of $R$ :

$$\left\{ m_1 = \begin{pmatrix} 0 & 0 \\ k & k \end{pmatrix}, m_2 = \begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix}, \alpha = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix} \right\}.$$

Now since $\alpha^2 = 0$, 0 is not a prime ideal. Moreover, since $m_1 R m_2 \subseteq \alpha$, $\alpha$ is not a prime ideal. So the only candidates for the associated primes of $R$ are the maximal ideals $m_1$ and $m_2$.

We claim that $m_2 \notin \text{Ass}(R)$. Otherwise, there would exists a right ideal $I \supsetneq 0$ of $R$ with $m_2 = r_R(I)$. So $I \cdot m_2 = 0$. Now, given $\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) \in I$, we have $0 = \left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right)$, so $a = b = 0$. Also, $0 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & c \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ c & 0 \end{smallmatrix}\right)$ implies that $c = 0$. Thus $I = 0$, a contradiction. Hence $m_2 \notin \text{Ass}(R)$.

By virtue of $R_R$ being noetherian, we know that $\text{Ass}(R) \neq 0$. Hence $\text{Ass}(R) = \{m_1\}$.

Finally, we should determine the set of $\text{NAss}(R)$. Clearly, $\text{nil}(R) = \alpha$. Thus $\text{nil}(R)$ is an ideal. Now we show that $m_1 = N_R(m_2)$ and $m_2$ is a right quasi-prime ideal. Clearly, $m_1 \subseteq N_R(m_2)$ since $m_2 m_1 = 0$. Given $\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) \in N_R(m_2)$, we have $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \text{nil}(R)$. Then $a = 0$ and so $\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) \in m_1$. Hence $m_1 = N_R(m_2)$. Next we see that $m_2$ is a right quasi-prime ideal. Let $n \not\subseteq \text{nil}(R)$ and $n \subseteq m_2$. Since $N_R(n) \supseteq N_R(m_2)$ is clear, we now assume that $\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) \in N_R(n)$, and find $\left(\begin{smallmatrix} h & 0 \\ k & 0 \end{smallmatrix}\right) \in n$ with $h \neq 0$. Then we have $\left(\begin{smallmatrix} h & 0 \\ k & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) = \left(\begin{smallmatrix} ha & 0 \\ ka & 0 \end{smallmatrix}\right) \in \text{nil}(R)$. Thus $a = 0$ and so $\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) \in N_R(m_2)$. Hence we obtain $N_R(n) = N_R(m_2)$ and so $m_2$ is a right quasi-prime ideal. Thus we obtain $m_1 \in \text{NAss}(R)$. Similarly, we have $m_2 \in \text{NAss}(R)$. Therefore $\text{NAss}(R) = \{m_1, m_2\} \neq \text{Ass}(R)$.

If $R$ is reduced, then $\wp$ is a nilpotent associated prime of $R$ if and only if $\wp$ is an associated prime of $R$. So $\text{NAss}(R) = \text{Ass}(R)$ in case $R$ is reduced.

Given a polynomial $f(x) \in R[x]$. If the polynomial $f(x)$ has the property that each nonzero coefficient has the same right annihilator in $R$, then we say that such a polynomial is a good polynomial. Shock showed in [13] that, given any nonzero polynomial $f(x) \in R[x]$, one can find $r \in R$ such that $f(x)r$ is good. In order to prove the main result of this section, we will need a generalized version of Shock's result which applies in our skew polynomial setting.

Let $m(x) = m_0 + m_1x + \cdots + m_kx^k + \cdots + m_nx^n \notin \text{nil}(R)[x;\alpha,\delta]$. If $m_k \notin \text{nil}(R)$, and $m_i \in \text{nil}(R)$ for all $i > k$, then we say that the nilpotent degree of $m(x)$ is $k$. To simplify notations, we write $\text{Ndeg}(m(x))$ for the nilpotent degree of $m(x)$. If $m(x) \in \text{nil}(R)[x;\alpha,\delta]$, then we define $\text{Ndeg}(m(x)) = -1$.

**Definition 3.3.** *Let $m(x) = m_0 + m_1x + \cdots + m_kx^k + \cdots + m_nx^n \notin \text{nil}(R)[x;\alpha,\delta]$ and the nilpotent degree of $m(x)$ be $k$. If $N_R(m_k) \subseteq N_R(m_i)$ for all $i \leq k$, then we say that $m(x)$ is a nilpotent good polynomial.*

**Lemma 3.1.** *Let $R$ be an $(\alpha,\delta)$-compatible 2-primal ring. For any $m(x) = m_0 + m_1x + \cdots + m_kx^k + \cdots + m_nx^n \notin \text{nil}(R)[x;\alpha,\delta]$, there exists $r \in R$ such that $m(x)r$ is a nilpotent good polynomial.*

*Proof.* Assume the result is false, and let $m(x) = m_0 + m_1x + \cdots + m_kx^k + \cdots + m_nx^n \notin \text{nil}(R)[x;\alpha,\delta]$ be a counterexample of minimal nilpotent degree $\text{Ndeg}(m(x)) = k \geq 1$. In particular, $m(x)$ is not a nilpotent good polynomial. Hence there exists $i < k$ such that $N_R(m_k) \nsubseteq N_R(m_i)$. So we can find $b \in R$ with $m_ib \notin \text{nil}(R)$, and $m_kb \in \text{nil}(R)$. Note that the degree $k$ coefficient of $m(x)b$ is $m_k\alpha^k(b) + \sum_{i=k+1}^{n} m_if_k^i(b)$ and $m_k\alpha^k(b) \in \text{nil}(R)$ due to the $(\alpha,\delta)$-compatibility of $R$. On the other hand, we have $\text{Ndeg}(m(x)) = k$, so $m_i \in \text{nil}(R)$ for all $i > k$. Since $\text{nil}(R)$ of a *2-primal* ring is an ideal, $m_if_k^i(b) \in \text{nil}(R)$ for all $i > k$. Hence it is easy to see that $m(x)b$ has nilpotent degree at most $k-1$. Since $m_ib \notin \text{nil}(R)$, by Corollary 2.3, we have $m(x)b \notin \text{nil}(R)[x;\alpha,\delta]$. By the minimality of $k$, we know that there exists $c \in R$ with $m(x)bc$ nilpotent good. But this contradicts the fact that $m(x)$ is a counterexample to the statement. ∎

**Theorem 3.1.** *Let $R$ be an $(\alpha,\delta)$-compatible 2-primal ring. Then*

$$\text{NAss}(R[x;\alpha,\delta]) = \{\wp[x;\alpha,\delta] \mid \wp \in \text{NAss}(R)\}.$$

*Proof.* We first prove $\supseteq$. Let $\wp \in \text{NAss}(R)$. By definition, there exists a right ideal $I \nsubseteq \text{nil}(R)$ with $I$ a right quasi-prime ideal of $R$ and $\wp = N_R(I)$. It suffices to prove

$$\wp[x;\alpha,\delta] = N_{R[x;\alpha,\delta]}(I[x;\alpha,\delta]) \tag{3.1}$$

and

$$I[x;\alpha,\delta] \quad \text{is quasi-prime.} \tag{3.2}$$

For Equation (3.1), let $f(x) = a_0 + a_1x + \cdots + a_lx^l \in \wp[x;\alpha,\delta]$, and let $i(x) = i_0 + i_1x + \cdots + i_mx^m \in I[x;\alpha,\delta]$. Since $i_ka_j \in \text{nil}(R)$ for each $k$, $j$, applying Corollary 2.3 yields that $i(x)f(x) \in \text{nil}(R[x;\alpha,\delta])$. Hence $\wp[x;\alpha,\delta] \subseteq N_{R[x;\alpha,\delta]}(I[x;\alpha,\delta])$.

Conversely, if $f(x) = a_0 + a_1x + \cdots + a_lx^l \in N_{R[x;\alpha,\delta]}(I[x;\alpha,\delta])$, then $i(x)f(x) \in \text{nil}(R[x;\alpha,\delta])$ for all $i(x) = i_0 + i_1x + \cdots + i_mx^m \in I[x;\alpha,\delta]$. Using Corollary 2.3 again, we obtain that $i_ka_j \in \text{nil}(R)$ for each $k$, $j$. Thus for all $0 \leq j \leq l$, $a_j \in N_R(I) = \wp$, and so $f(x) \in \wp[x;\alpha,\delta]$. Hence $N_{R[x;\alpha,\delta]}(I[x;\alpha,\delta]) \subseteq \wp[x;\alpha,\delta]$. Therefore $\wp[x;\alpha,\delta] = N_{R[x;\alpha,\delta]}(I[x;\alpha,\delta])$.

Note that the right ideal $I$ is a right quasi-prime ideal. Then we have $I \nsubseteq \text{nil}(R)$. Thus

$$I[x;\alpha,\delta] \nsubseteq \text{nil}(R)[x;\alpha,\delta] = \text{nil}(R[x;\alpha,\delta]).$$

To see (3.2), we must show that if a right ideal $\mho \nsubseteq \text{nil}(R[x;\alpha,\delta])$ and $\mho \subseteq I[x;\alpha,\delta]$, then

$$N_{R[x;\alpha,\delta]}(\mho) = N_{R[x;\alpha,\delta]}(I[x;\alpha,\delta]).$$

To this end, let $D$ be a subset of $R$ consisting of all coefficients of elements of $\mho$. Then let $\wp_0$ denote the right ideal of $R$ generated by $D$. Since $\mho \not\subseteq \mathrm{nil}(R[x;\alpha,\delta]) = \mathrm{nil}(R)[x;\alpha,\delta]$, $D \not\subseteq \mathrm{nil}(R)$, and hence $\wp_0 \subseteq I$, $\wp_0 \not\subseteq \mathrm{nil}(R)$. So we have $N_R(\wp_0) = N_R(I) = \wp$ because $I$ is a right quasi-prime ideal. Since $N_{R[x;\alpha,\delta]}(\mho) \supseteq N_{R[x;\alpha,\delta]}(I[x;\alpha,\delta])$ is clear, we now assume that

$$h(x) = h_0 + h_1x + \cdots + h_ux^u \in N_{R[x;\alpha,\delta]}(\mho),$$

and

$$s(x) = s_0 + s_1x + \cdots + s_vx^v \in \mho.$$

Then we have $s(x)h(x) \in \mathrm{nil}(R[x;\alpha,\delta])$. By Corollary 2.3, we obtain

$$s_ih_j \in \mathrm{nil}(R) \text{ for all } 0 \le i \le v,\ 0 \le j \le u.$$

Since $\mathrm{nil}(R)$ of a *2-primal* ring is an ideal, $s_ih_j \in \mathrm{nil}(R)$ implies $h_js_i \in \mathrm{nil}(R)$ and so $s_iRh_js_iRh_j = (s_iRh_j)^2 \in \mathrm{nil}(R)$. Hence $s_iRh_j \in \mathrm{nil}(R)$. Thus we obtain

$$h_j \in N_R(\wp_0) = N_R(I) = \wp \text{ for all } 0 \le j \le u.$$

Let $i(x) = i_0 + i_1x + \cdots + i_px^p \in I[x;\alpha,\delta]$, we have $i_mh_j \in \mathrm{nil}(R)$ for all $0 \le m \le p$, $0 \le j \le u$. Then $i(x)h(x) \in \mathrm{nil}(R[x;\alpha,\delta])$ by Corollary 2.3. Hence $N_{R[x;\alpha,\delta]}(\mho) \subseteq N_{R[x;\alpha,\delta]}(I[x;\alpha,\delta])$ is proved, and so is $\supseteq$ in Theorem 3.1.

Now we turn our attention to proving $\subseteq$ in Theorem 3.1. Let $I \in \mathrm{NAss}(R[x;\alpha,\delta])$. By definition, we have a right quasi-prime ideal $\pounds$ of $R[x;\alpha,\delta]$ with $I = N_{R[x;\alpha,\delta]}(\pounds)$. Pick any

$$m(x) = m_0 + m_1x + \cdots + m_kx^k + \cdots + m_nx^n \notin \mathrm{nil}(R)[x;\alpha,\delta]$$

in $\pounds$. By $\pounds \not\subseteq \mathrm{nil}(R[x;\alpha,\delta])$ and Lemma 3.1, we may assume that $m(x)$ is nilpotent good, and $\mathrm{Ndeg}(m(x)) = k$. Set $\pounds_0 = m(x)\cdot R[x;\alpha,\delta]$. Note that $m(x) \notin \mathrm{nil}(R)[x;\alpha,\delta])$, so we get

$$\pounds_0 = m(x)R[x;\alpha,\delta] \not\subseteq \mathrm{nil}(R)[x;\alpha,\delta] = \mathrm{nil}(R[x;\alpha,\delta]).$$

Then we have

$$N_{R[x;\alpha,\delta]}(\pounds) = N_{R[x;\alpha,\delta]}(\pounds_0) = N_{R[x;\alpha,\delta]}(m(x)\cdot R[x;\alpha,\delta]) = I$$

because $\pounds$ is quasi-prime. Consider the right ideal $m_kR$, and assume that $U = N_R(m_kR)$. We wish to claim that $I = U[x;\alpha,\delta]$. Let

$$g(x) = b_0 + b_1x + \cdots + b_lx^l \in U[x;\alpha,\delta].$$

Then

$$m_kRb_j \in \mathrm{nil}(R) \text{ for all } 0 \le j \le l.$$

Since $m(x)$ is nilpotent good, and $\mathrm{Ndeg}(m(x)) = k$, $m_iRb_j \in \mathrm{nil}(R)$ for all $0 \le i \le k$, and $0 \le j \le l$. On the other hand, for all $i > k$, $m_i \in \mathrm{nil}(R)$. Thus we have $m_iRb_j \in \mathrm{nil}(R)$ for all $0 \le i \le n$, $0 \le j \le l$. Choose any

$$h(x) = h_0 + h_1x + \cdots + h_px^p \in R[x;\alpha,\delta].$$

From $m_ih_db_j \in \mathrm{nil}(R)$ for all $0 \le i \le n$, $0 \le d \le p$ and $0 \le j \le l$ and $(\alpha,\delta)$-compatibility of $R$, we obtain $m(x)h(x)g(x) \in \mathrm{nil}(R[x;\alpha,\delta])$ by a routine computations. Hence $g(x) \in N_{R[x;\alpha,\delta]}(m(x)R[x;\alpha,\delta]) = I$, and so $U[x;\alpha,\delta] \subseteq I$. Conversely, let $g(x) = b_0 + b_1x + \cdots + b_lx^l \in I$. Then

$$m(x)Rg(x) \in \mathrm{nil}(R[x;\alpha,\delta]).$$

By Corollary 2.3, we get $m_iRb_j \in \mathrm{nil}(R)$ for all $0 \le i \le n$, and $0 \le j \le l$. Thus $b_j \in N_R(m_kR)$ for all $0 \le j \le l$, and so $g(x) \in U[x;\alpha,\delta]$. Hence $I \subseteq U[x;\alpha,\delta]$. Therefore $I = U[x;\alpha,\delta]$.

We are now to check that $m_kR$ is quasi-prime. Since $m_k \notin \mathrm{nil}(R)$, $m_kR \not\subseteq \mathrm{nil}(R)$. Assume that a right ideal $Q \subseteq m_kR$, and $Q \not\subseteq \mathrm{nil}(R)$. Then $N_R(Q) \supseteq N_R(m_kR)$ is clear. Now we show that

$$N_R(Q) \subseteq N_R(m_kR).$$

Set $W = \{m(x)r \mid r \in Q\}$, and let $WR[x;\alpha,\delta]$ be the right ideal of $R[x;\alpha,\delta]$ generated by $W$. It is obvious that $WR[x;\alpha,\delta] \subseteq m(x)R[x;\alpha,\delta]$. Since $Q \not\subseteq \mathrm{nil}(R)$, there exists $a \in R$ such that $m_ka \in Q$ and $m_ka \notin \mathrm{nil}(R)$. If $m_k \cdot m_ka \in \mathrm{nil}(R)$, then we have $m_ka \in \mathrm{nil}(R)$. This contradicts to the fact that $m_ka \notin \mathrm{nil}(R)$. Thus $m_k \cdot m_ka \notin \mathrm{nil}(R)$ and so $m(x) \cdot m_ka \not\subseteq \mathrm{nil}(R[x;\alpha,\delta])$ by Corollary 2.3, and this implies that $WR[x;\alpha,\delta] \not\subseteq \mathrm{nil}(R[x;\alpha,\delta])$. Since £ is quasi-prime, we obtain

$$N_{R[x;\alpha,\delta]}(WR[x;\alpha,\delta]) = N_{R[x;\alpha,\delta]}(m(x)R[x;\alpha,\delta]) = I.$$

Suppose $q \in N_R(Q)$. Then $rq \in \mathrm{nil}(R)$ for each $r \in Q$. For any $m(x)rf(x) \in WR[x;\alpha,\delta]$ where $f(x) = a_0 + a_1x + \cdots + a_lx^l \in R[x;\alpha,\delta]$. The typical term of $m(x)rf(x)$ is $m_ix^ira_jx^j$. From $rq \in \mathrm{nil}(R)$ and $\mathrm{nil}(R)$ of a *2-primal* ring is an ideal, we have

$$rq \in \mathrm{nil}(R) \Rightarrow qr \in \mathrm{nil}(R) \Rightarrow ra_jqra_jq \in \mathrm{nil}(R) \Rightarrow ra_jq \in \mathrm{nil}(R) \Rightarrow m_ira_jq \in \mathrm{nil}(R).$$

Thus $m_ix^ira_jx^jq \in \mathrm{nil}(R)[x;\alpha,\delta]$ due to the $(\alpha,\delta)-$compatibility of $R$, and so

$$m(x)rf(x)q \in \mathrm{nil}(R)[x;\alpha,\delta] = \mathrm{nil}(R[x;\alpha,\delta]).$$

Thus for any

$$\sum m(x)r_if_i(x) \in WR[x;\alpha,\delta],$$

it is easy to see that

$$\left(\sum m(x)r_if_i(x)\right)q \in \mathrm{nil}(R[x;\alpha,\delta]).$$

Hence $q \in N_{R[x;\alpha,\delta]}(WR[x;\alpha,\delta]) = I = U[x;\alpha,\delta]$, and so $q \in U = N_R(m_kR)$. So $N_R(Q) \subseteq N_R(m_kR)$, and this implies that $N_R(Q) = N_R(m_kR)$. Thus $m_kR$ is quasi-prime.

Assembling the above results, we finish the proof of Theorem 3.1. ∎

**Corollary 3.1.** *Let R be a 2-primal ring. Then* $\mathrm{NAss}(R[x]) = \{\wp[x] \mid \wp \in \mathrm{NAss}(R)\}$.

*Proof.* Take $\alpha = id$ and $\delta = 0$ in Theorem 3.1. ∎

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