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The Existence of Three Positive Solutions to Integral Type BVPs for Second Order ODEs with One-Dimensional p-Laplacian

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Abstract. This paper is concerned with the integral type boundary value problems of the second order differential equations with one-dimensional p-Laplacian

$$\begin{cases} [\rho(t)\Phi(x'(t))]' + f(t,x(t),x'(t)) = 0, & t \in (0,1), \\ \phi_1(x(0)) = \int_0^1 g(s)\phi_1(x(s))ds, \\ \phi_2(x'(1)) = \int_0^1 h(s)\phi_2(x'(s))ds. \end{cases}$$

Sufficient conditions to guarantee the existence of at least three positive solutions of this BVP are established. An example is presented to illustrate the main results. The emphasis is put on the one-dimensional p-Laplacian term $[\rho(t)\Phi(x'(t))]'$ involved with the function ρ, which makes the solutions un-concave.

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1. Introduction

The multi-point boundary-value problems for linear second order differential equations was initiated by Il'in and Moiseev [8]. Since then, more general nonlinear multi-point boundary-value problems (BVPs for short) were studied by several authors, see the text books [1, 3, 7] and the survey papers [9, 10] and the references cited therein. For example, in [5], the authors studied the following BVP

$$\begin{cases} [\Phi(x'(t))]' + f(t,x(t),x'(t)) = 0, & t \in (0,1), \\ x'(0) = 0, \\ \theta(x(1)) = \sum_{i=1}^n \theta(x(\xi_i))a_i(x'(\xi_i)), \end{cases}$$

where Φ, θ are two increasing homeomorphisms from R onto R with $\Phi(0) = 0$ and $\theta(0) = 0$, f is a Caratheodory function, $a_i : R \to R$ are continuous functions. By using Leray-Schauder

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fixed point theorem, the existence results to solutions of this BVP are established under the assumption $\sum_{i=1}^{n} a_i(0) = 1$ (the resonant case).

In paper [4], the following BVP was studied

$$\begin{cases} [\Phi(x'(t))]' + f(t,x(t),x'(t)) = 0, \quad t \in (a,b), \\ x(a) = 0, \\ \theta(x'(b)) = \sum_{i=1}^{n} \alpha_i \theta(x'(\xi_i)), \end{cases}$$

where Φ, θ are two increasing homeomorphisms from R onto R with $\Phi(0) = 0$ and $\theta(0) = 0$, f is a Caratheodory function, $\alpha_i \in R$ and $\xi_i \in (a,b)$. By using Leray-Schauder degree and Brouwer degree theory, the existence results to solutions of this BVP are established.

To the author's knowledge, there is no paper concerned with the existence of positive solutions of the boundary value problems to the second order differential equation

$$[\rho(t)\Phi(x'(t))]' + f(t,x(t),x'(t)) = 0, \quad t \in (0,1).$$

To fill this gap, we consider the more generalized BVP for second order differential equation with p-Laplacian coupled with the integral type BCs, i.e. the BVP

$$\begin{cases} [\rho(t)\Phi(x'(t))]' + f(t,x(t),x'(t)) = 0, \quad t \in (0,1), \\ \phi_1(x(0)) = \int_0^1 g(s)\phi_1(x(s))ds, \\ \phi_2(x'(1)) = \int_0^1 h(s)\phi_2(x'(s))ds, \end{cases} \tag{1.1}$$

where f is a nonnegative Caratheodory function and $f(t,0,0) \not\equiv 0$ on each subinterval of $[0,1]$, $g,h : [0,1] \to [0,\infty)$ satisfy $g,h \in L^1[0,1]$, $\rho \in C^0([0,1],(0,\infty))$, the integrals in mentioned equations are meant in the sense of Riemann-Stieljes, $\Phi : R \to R$ with $\Phi \in C^1(R)$ and $\Phi'(x) > 0$ for all $x > 0$, $\Phi(xy) = \Phi(x)\Phi(y)$ and $\Phi(-x) = -\Phi(x)$ for all $x,y \in R$, there exists its inverse function denoted by Φ^{-1}. It is easy to see that p-Laplacian function $\phi(x) = |x|^{p-2}x$ with $p > 1$ is such a function. ϕ_1, ϕ_2 are two increasing homeomorphisms from R onto R with $\phi_1(0) = 0$ and $\phi_2(0) = 0$.

The purpose is to establish sufficient conditions for the existence of at least three positive solutions of BVP (1.1). The result in this paper generalizes and improves some known ones since the one-dimensional p-Laplacian term $[\rho(t)\Phi(x'(t))]'$ involved with the function ρ, which makes the solutions un-concave and there exists no paper concerned with the existence of positive solutions of this kind of the BVPs. This paper fills the gap.

The remainder of this paper is organized as follows: the main result (Theorem 2.1) is presented in Section 2, and the example to show the main result is given in Section 3.

2. Main results

In this section, we first present some background definitions in Banach spaces and state an important three fixed point theorem. Then the main results are given and proved.

Definition 2.1. *Let X be a real Banach space. The nonempty convex closed subset P of X is called a cone in X if $ax \in P$ for all $x \in P$ and $a \geq 0$ and $x \in X$ and $-x \in X$ imply $x = 0$.*

Definition 2.2. *A map $\psi : P \to [0,+\infty)$ is a nonnegative continuous concave or convex functional map provided ψ is nonnegative, continuous and satisfies $\psi(tx+(1-t)y) \geq t\psi(x)+(1-t)\psi(y)$, or $\psi(tx+(1-t)y) \leq t\psi(x)+(1-t)\psi(y)$, for all $x,y \in P$ and $t \in [0,1]$.*

Definition 2.3. *An operator $T : X \to X$ is completely continuous if it is continuous and maps bounded sets into pre-compact sets.*

Definition 2.4. *Let $a,b,c,d,h>0$ be positive constants, α,ψ be two nonnegative continuous concave functionals on the cone P, γ,β,θ be three nonnegative continuous convex functionals on the cone P. Define the convex sets as follows:*

$$P_c = \{x \in P : ||x|| < c\},$$
$$P(\gamma,\alpha;a,c) = \{x \in P : \alpha(x) \geq a,\ \gamma(x) \leq c\},$$
$$P(\gamma,\theta,\alpha;a,b,c) = \{x \in P : \alpha(x) \geq a,\ \theta(x) \leq b,\ \gamma(x) \leq c\},$$
$$Q(\gamma,\beta;,d,c) = \{x \in P : \beta(x) \leq d,\ \gamma(x) \leq c\},$$
$$Q(\gamma,\beta,\psi;h,d,c) = \{x \in P : \psi(x) \geq h,\ \beta(x) \leq d,\ \gamma(x) \leq c\}.$$

Lemma 2.1. [2] *Let X be a real Banach space, P be a nonempty cone in X, α,ψ be two nonnegative continuous concave functionals on the cone P, γ,β,θ be three nonnegative continuous convex functionals on the cone P. There exist constant $M>0$ such that*

$$\alpha(x) \leq \beta(x),\ ||x|| \leq M\gamma(x) \text{ for all } x \in P.$$

Furthermore, Suppose that $h,d,a,b,c>0$ are constants with $d<a$. Let $T:\ \overline{P_c} \to \overline{P_c}$ be a completely continuous operator. If

(C1) $\{y \in P(\gamma,\theta,\alpha;a,b,c) | \alpha(x) > a\} \neq \emptyset$ *and* $\alpha(Tx) > a$ *for every* $x \in P(\gamma,\theta,\alpha;\, a,b,c)$;
(C2) $\{y \in Q(\gamma,\theta,\psi;h,d,c) | \beta(x) < d\} \neq \emptyset$ *and* $\beta(Tx) < d$ *for every* $x \in Q(\gamma,\theta,\psi;\, h,d,c)$;
(C3) $\alpha(Ty) > a$ *for* $y \in P(\gamma,\alpha;a,c)$ *with* $\theta(Ty) > b$;
(C4) $\beta(Tx) < d$ *for each* $x \in Q(\gamma,\beta;,d,c)$ *with* $\psi(Tx) < h$,

then T has at least three fixed points y_1, y_2 and y_3 such that $\beta(y_1) < d,\ \alpha(y_2) > a,\ \beta(y_3) > d,\ \alpha(y_3) < a$.

Let us list the assumptions:

(H1) $g,h:[0,1] \to [0,\infty)$ satisfy $\int_0^1 g(s)ds < 1$ and

$$\phi_2\left(\Phi^{-1}\left(\rho(1)\right)\right)\int_0^1 h(s)\phi_2\left(\Phi^{-1}\left(\frac{1}{\rho(s)}\right)\right)ds < 1.$$

(H2) $\rho \in C^0([0,1]) \cap C^1(0,1), \rho(t) > 0$ for $t \in [0,1]$.
(H3) f is a Carathédory function, that is
  (i) $t \to f(t,x,y)$ is measurable for any $(x,y) \in [0,\infty) \times R$,
  (ii) $(x,y) \to f(t,x,y)$ is continuous for a.e. $t \in (0,1)$,
  (iii) for each $r>0$, there exists nonnegative function $\phi_r \in L^1(0,1)$ such that

$$\max\{|u|,|v|\} \leq r$$

implies

$$|f(t,u,v)| \leq \phi_r(t), a.e. t \in (0,1).$$

Choose $X = C^1[0,1]$. Define its norm by

$$||x|| = \max\left\{\max_{t\in[0,1]} |x(t)|,\ \max_{t\in[0,1]} |x'(t)|\right\}.$$

It is easy to see that X is a real Banach space. Let $\sigma \in L^1[0,1]$ be nonnegative. Consider the following BVP

$$\begin{cases} [\rho(t)\Phi(y'(t))]' + \sigma(t) = 0, \ t \in (0,1), \\ \phi_1(x(0)) = \int_0^1 g(s)\phi_1(x(s))ds, \\ \phi_2(x'(1)) = \int_0^1 h(s)\phi_2(x'(s))ds. \end{cases} \tag{2.1}$$

Lemma 2.2. *Suppose that* (H1) *and* (H2) *hold. If $y \in X$ is a solution of* BVP (2.1)*, then*

(i) *y is concave with respect to τ, where τ is defined by*

$$\tau = \tau(t) = \frac{\int_0^t \Phi^{-1}\left(\frac{1}{\rho(s)}\right)ds}{\int_0^1 \Phi^{-1}\left(\frac{1}{\rho(s)}\right)ds};$$

(ii) *$y'(t) \geq 0$ and $y(t) \geq 0$ for all $t \in [0,1]$;*

(iii) *Let $k \in (0,1)$, y satisfies that*

$$\min_{t\in[k,1]} y(t) = y(k) \geq \mu \max_{t\in[0,1]} y(t) = \mu y(1),$$

where μ is defined by

$$\mu = \frac{\int_0^k \Phi^{-1}\left(\frac{1}{\rho(s)}\right)ds}{2\int_0^1 \Phi^{-1}\left(\frac{1}{\rho(s)}\right)ds};$$

(iv) *there exist unique numbers A_σ, B_σ such that*

$$y(t) = B_\sigma + \int_0^t \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(\rho(1)\Phi(A_\sigma) + \int_s^1 \sigma(u)du\right)ds, \ t \in [0,1],$$

where $A_\sigma \in [0,c_0]$ satisfies

$$\phi_2(A_\sigma) = \int_0^1 h(t)\phi_2\left(\Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(\rho(1)\phi(A_\sigma) + \int_s^1 \sigma(u)du\right)\right)dt, \tag{2.2}$$

where c_0 is defined by

$$c_0 = \Phi^{-1}\left(\frac{\Phi\left(\phi_2^{-1}\left(\int_0^1 h(t)\phi_2\left(\Phi^{-1}\left(\frac{1}{\rho(t)}\right)dt\right)\right)\right)}{1-\Phi\left(\phi_2^{-1}\left(\int_0^1 h(t)\phi_2\left(\Phi^{-1}\left(\frac{1}{\rho(t)}\right)dt\right)\right)\right)\rho(1)}\int_0^1 \sigma(u)du\right),$$

and $B_\sigma \in [0,d_0]$ satisfies

$$\phi_1(B_\sigma) = \int_0^1 g(t)\Phi_1\left(B_\sigma + \int_0^t \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(\rho(1)\phi(A_\sigma) + \int_s^1 \sigma(u)du\right)ds\right)dt, \tag{2.3}$$

where d_0 is defined by

$$d_0 = \frac{\Phi^{-1}\left(\int_0^1 g(s)ds\right)}{1-\Phi^{-1}\left(\int_0^1 g(s)ds\right)}\int_0^1 \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(\rho(1)\Phi(c_0) + \int_s^1 \sigma(u)du\right)ds.$$

Proof. First we prove that y is concave with respect to τ. It is easy to see that $\tau \in C^0([0,1],[0,1])$ and

$$\frac{d\tau}{dt} = \Phi^{-1}\left(\frac{1}{\rho(t)}\right)\frac{1}{\int_0^1 \Phi^{-1}\left(\frac{1}{\rho(s)}\right)ds} > 0.$$

Thus

$$\frac{dy}{dt} = \frac{dy}{d\tau}\frac{d\tau}{dt} = \frac{dy}{d\tau}\Phi^{-1}\left(\frac{1}{\rho(t)}\right)\frac{1}{\int_0^1 \Phi^{-1}\left(\frac{1}{\rho(s)}\right)ds}. \tag{2.4}$$

It follows that

$$\rho(t)\Phi\left(\frac{dy}{dt}\right) = \Phi\left(\frac{dy}{d\tau}\right)\Phi\left(\frac{1}{\int_0^1 \phi^{-1}\left(\frac{1}{\rho(s)}\right)ds}\right).$$

Hence

$$\left[\rho(t)\phi\left(\frac{dy}{dt}\right)\right]' = \Phi'\left(\frac{dy}{d\tau}\right)\frac{d^2y}{d\tau^2}\frac{d\tau}{dt}\Phi\left(\frac{1}{\int_0^1 \Phi^{-1}\left(\frac{1}{\rho(s)}\right)ds}\right).$$

So

$$\frac{d^2y}{d\tau^2} = \frac{\left[\rho(t)\Phi\left(\frac{dy}{dt}\right)\right]'}{\Phi'\left(\frac{dy}{d\tau}\right)\frac{d\tau}{dt}}\Phi\left(\int_0^1 \Phi^{-1}\left(\frac{1}{\rho(s)}\right)ds\right).$$

Since $[\rho(t)\Phi(y'(t))]' \le 0$, $\Phi'(y) > 0 (y > 0)$ and $\frac{d\tau}{dt} > 0$, we get

$$\Phi'(-x) = -[\Phi(-x)]' = -[-\Phi(x)]' = \Phi'(x) > 0$$

for all $x > 0$. Then $\frac{d^2y}{d\tau^2} \le 0$. Hence $y(t)$ is concave with respect to τ on $[0,1]$.

Second, we prove that y is nonnegative. In fact, since $[\rho(t)\phi(y'(t))]' \le 0$, we have

$$\begin{aligned}
\phi_2(y'(1)) &= \int_0^1 h(s)\phi_2\left(\Phi^{-1}\left(\frac{1}{\rho(s)}\right)\right)\phi_2\left(\Phi^{-1}\left(\rho(s)\Phi(y'(s))\right)\right)ds \\
&\ge \int_0^1 h(s)\phi_2\left(\Phi^{-1}\left(\frac{1}{\rho(s)}\right)\right)\phi_2\left(\Phi^{-1}\left(\rho(1)\Phi(y'(1))\right)\right)ds \\
&= \phi_2(y'(1))\phi_2\left(\Phi^{-1}(\rho(1))\right)\int_0^1 h(s)\phi_2\left(\Phi^{-1}\left(\frac{1}{\rho(s)}\right)\right)ds.
\end{aligned}$$

It follows from (H1) that $\phi_2(y'(1)) \ge 0$. Hence $y'(1) \ge 0$. Then $\rho(1)\Phi(y'(1)) \ge 0$. So $\rho(t)\Phi(y'(t)) \ge 0$ for all $t \in [0,1]$. It follows that $y'(t) \ge 0$ for all $t \in [0,1]$. Now, we have

$$\phi_1(y(0)) = \int_0^1 g(s)\phi_1(y(s))ds \ge \phi_1(y(0))\int_0^1 g(s)ds.$$

From (H1), we get $y(0) \ge 0$. To gather with $y'(t) \ge 0$, we have y is nonnegative on $[0,1]$.

Third, we prove that

$$\min_{t\in[k,1]} y(t) \ge \mu \sup_{t\in[0,1]} y(t). \tag{2.5}$$

It follows from the second step that y is nondecreasing on $[0,1]$. So

$$\min_{t\in[k,1]} y(t) = y(k), \quad \max_{t\in[0,1]} y(t) = y(1).$$

From step 1, we have $y(t)$ is concave with respect to τ. Let $t=\tau(t)$ be the inverse function of $\tau=\tau(t)$. Then, one has

$$\begin{aligned} y(k) &= y(t(\tau(k))) \\ &= y\left(t\left(\frac{1-\tau(k)+\tau(1)}{1+\tau(1)} \frac{\tau(k)}{1-\tau(k)+\tau(1)} + \frac{\tau(k)}{1+\tau(1)} \tau(1) \right)\right). \end{aligned}$$

Noting that $1>\tau(k)$ and $y(t)$ is concave with respect to τ, then

$$\begin{aligned} y(k) &\geq \frac{1-\tau(k)+\tau(1)}{1+\tau(1)} y\left(t\left(\frac{\tau(k)}{1-\tau(k)+\tau(1)} \right)\right) \\ &\quad + \frac{\tau(k)}{1+\tau(1)} y\left(t\left(\tau(1)\right)\right) \\ &\geq \frac{\int_0^k \Phi^{-1}\left(\frac{1}{\rho(s)}\right) ds}{2\int_0^1 \Phi^{-1}\left(\frac{1}{\rho(s)}\right) ds} y(1) \\ &= \mu \sup_{t\in[0,1]} y(t). \end{aligned}$$

Hence (2.3) holds.

Finally, we prove (iv). In fact, from (2.1), we have

$$y(t) = y(0) + \int_0^t \Phi^{-1}\left(\frac{1}{\rho(s)}\right) \Phi^{-1}\left(\rho(1)\Phi(y'(1)) + \int_s^1 \sigma(u)du\right) ds, \ \ t\in[0,1].$$

The BCs in (2.1) imply that

$$\begin{aligned} \phi_1(y(0)) = \int_0^1 g(t)\phi_1\Bigg(y(0) + \int_0^t \Phi^{-1}\left(\frac{1}{\rho(s)}\right) \Phi^{-1}\big(\rho(1)\Phi(y'(1)) \\ + \int_s^1 \sigma(u)du\big) ds\Bigg) dt. \end{aligned} \tag{2.6}$$

and

$$\phi_2(y'(1)) = \int_0^1 h(t)\phi_2\left(\Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(\rho(1)\Phi(y'(1)) + \int_s^1 \sigma(u)du\right)\right) dt. \tag{2.7}$$

Let

$$G(c) = \phi_2(c) - \int_0^1 h(t)\phi_2\left(\Phi^{-1}\left(\frac{1}{\rho(t)}\right)\Phi^{-1}\left(\rho(1)\Phi(c) + \int_t^1 \sigma(u)du\right)\right) dt.$$

If $c\neq 0$, we know

$$\frac{G(c)}{\phi_2(c)} = 1 - \int_0^1 h(t)\phi_2\left(\Phi^{-1}\left(\frac{1}{\rho(t)}\right)\Phi^{-1}\left(\rho(1) + \frac{1}{\Phi(c)}\int_t^1 \sigma(u)du\right)\right) dt.$$

Since $\frac{G(c)}{\phi_2(c)}$ is increasing on $(-\infty,0)$ and $(0,+\infty)$ respectively, and

$$\lim_{t\to 0^+} \frac{G(c)}{\phi_2(c)} = -\infty, \ \ \lim_{t\to 0^-} \frac{G(c)}{\phi_2(c)} = +\infty,$$

$$\lim_{t\to -\infty} \frac{G(c)}{\phi_2(c)} = 1 - \phi_2\left(\Phi^{-1}(\rho(1))\right) \int_0^1 h(t)\phi_2\left(\Phi^{-1}\left(\frac{1}{\rho(s)}\right)\right) dt > 0,$$

and

$$\frac{G(c_0)}{\phi_2(c_0)} = 0,$$

we know that there exists an unique number $A_\sigma \in [0, c_0]$ such that $G(A_\sigma) = 0$. Then $y'(1) = A_\sigma \in [0, c_0]$.

Fix $c > 0$. Let

$$H(d) = \phi_1(d) - \int_0^1 g(t)\phi_1\left(d + \int_0^t \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(\rho(1)\Phi(c) + \int_s^1 \sigma(u)du\right)ds\right)dt.$$

If $d \neq 0$, we see

$$\frac{H(d)}{d} = 1 - \int_0^1 g(t)\phi_1\left(1 + \frac{1}{d}\int_0^t \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(\rho(1)\Phi(c) + \int_s^1 \sigma(u)du\right)ds\right)dt.$$

It is easy to see $\frac{H(d)}{d}$ is increasing on $(0, +\infty)$ and $(-\infty, 0)$ respectively. Denote

$$\delta = \frac{\Phi\left(\phi_2^{-1}\left(\int_0^1 h(t)\phi_2\left(\Phi^{-1}\left(\frac{1}{\rho(t)}\right)dt\right)\right)\right)}{1 - \Phi\left(\phi_2^{-1}\left(\int_0^1 h(t)\phi_2\left(\Phi^{-1}\left(\frac{1}{\rho(t)}\right)dt\right)\right)\right)\rho(1)}.$$

Then $c_0 = \Phi^{-1}\left(\delta \int_0^1 \sigma(u)du\right)$. Since

$$\lim_{t\to 0^+} \frac{H(d)}{d} = -\infty, \quad \lim_{t\to 0^-} \frac{H(d)}{d} = +\infty,$$

$$\lim_{t\to -\infty} \frac{H(d)}{d} = 1 - \int_0^1 g(t)dt > 0,$$

and

$$\begin{aligned}
&\frac{H(d_0)}{d_0} \\
&= 1 - \int_0^1 g(t)\phi_1\left(1 + \frac{1}{d_0}\int_0^t \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(\rho(1)\Phi(c) + \int_s^1 \sigma(u)du\right)ds\right)dt \\
&\geq 1 - \int_0^1 g(t)\phi_1\left(1 + \frac{1}{d_0}\int_0^t \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(\rho(1)\Phi(c_0) + \int_s^1 \sigma(u)du\right)ds\right)dt \\
&\geq 1 - \int_0^1 g(t)\phi_1\left(1 + \frac{1 - \Phi^{-1}\left(\int_0^1 g(s)ds\right)}{\Phi^{-1}\left(\int_0^1 g(s)ds\right)}\right)dt \\
&= 0,
\end{aligned}$$

we know that there exists an unique number $B_\sigma \in [0, d_0]$ such that $H(B_\sigma) = 0$. Then $y(0) = B_\sigma \in [0, d_0]$. The proof is complete. ∎

Choose $k \in (0, 1)$. Define the cone $P \subseteq X$ by

$$P = \left\{ x \in X : \begin{array}{c} x(t) \geq 0, t \in [0,1], x'(t) \geq 0, t \in [0,1], \\ x(0) \leq \phi_1^{-1}\left(\int_0^1 g(s)ds\right)x(1), \\ \min_{t\in[k,1]} x(t) \geq \mu \max_{t\in[0,1]} x(t) \end{array} \right\}.$$

Define the functionals on $P : P \to R$ by

$$\gamma(y) = \max_{t\in[0,1]} |y'(t)|,\ y \in P,$$
$$\beta(y) = \max_{t\in[0,1]} |y(t)|,\ y \in P,$$
$$\theta(y) = \max_{t\in[0,1]} |y(t)|,\ y \in P,$$
$$\alpha(y) = \min_{t\in[k,1]} |y(t)|,\ y \in P,$$
$$\psi(y) = \min_{t\in[k,1]} |y(t)|,\ y \in P.$$

It is easy to see that α, ψ are two nonnegative continuous concave functionals on the cone P, γ, β, θ are three nonnegative continuous convex functionals on the cone P and $\alpha(y) \le \beta(y)$ for all $y \in P$.

Define the operator $T : P \to X$ by

$$(Tx)(t) = B_x + \int_0^t \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(\rho(1)\phi(A_x) + \int_s^1 \sigma(u)du\right)ds,\ \ t \in [0,1],$$

where $A_x \in [0, c_0]$ satisfies

$$\phi_2(A_x) = \int_0^1 h(t)\phi_2\left(\Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(\rho(1)\phi(A_x) + \int_s^1 f(u,x(u),x'(u))du\right)\right)dt, \tag{2.8}$$

where c_0 is defined by

$$c_0 = \Phi^{-1}\left(\frac{\Phi\left(\phi_2^{-1}\left(\int_0^1 h(t)\phi_2\left(\Phi^{-1}\left(\frac{1}{\rho(t)}\right)dt\right)\right)\right)}{1-\Phi\left(\phi_2^{-1}\left(\int_0^1 h(t)\phi_2\left(\Phi^{-1}\left(\frac{1}{\rho(t)}\right)dt\right)\right)\right)\rho(1)}\int_0^1 f(u,x(u),x'(u))du\right),$$

and $B_x \in [0, d_0]$ satisfies

$$\begin{aligned}\phi_1(B_x) = \int_0^1 g(t)\phi_1\Bigg(B_x + \int_0^t \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\Big(\rho(1)\Phi(A_x)\\ + \int_s^1 f(u,x(u),x'(u))du\Big)ds\Bigg)dt,\end{aligned} \tag{2.9}$$

where d_0 is defined by

$$\begin{aligned}d_0 = {}& \frac{\Phi^{-1}\left(\int_0^1 g(s)ds\right)}{1-\Phi^{-1}\left(\int_0^1 g(s)ds\right)}\\ &\times \int_0^1 \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(\rho(1)\Phi(c_0) + \int_s^1 f(u,x(u),x'(u))du\right)ds.\end{aligned}$$

It follows from Lemma 2.2(ii) and (iii) that $Tx \in P$ for all $x \in P$. Then $T : P \to P$ is well defined. Suppose (H1)-(H3) hold. Similarly to Lemma 3.7 in [11], we can prove that T is completely continuous.

It is easy to show it holds that

$$\begin{cases} [\rho(t)\Phi((Tx)'(t))]' + f(t,x(t),x'(t)) = 0, \quad t \in (0,1), \\ \phi_1((Tx)(0)) = \int_0^1 g(s)\phi_1((Tx)(s))ds, \\ \phi_2((Tx)'(1)) = \int_0^1 h(s)\phi_2((Tx)'(s))ds, \end{cases}$$

Similarly to Lemma 2.2, we can prove that $Tx \in P$ if $x \in P$.

Theorem 2.1. *Suppose that* (H1)-(H3) *hold and there exist positive constants* e_1, e_2, c, *and* Q, W *and* E *are given by*

$$M = \frac{1}{1-\phi_1^{-1}\left(\int_0^1 g(s)ds\right)},$$

$$Q = \Phi\left(\frac{c}{M}\right) \min_{t\in[0,1]} \rho(t)\left[1-\Phi\left(\phi_2^{-1}\left(\int_0^1 h(t)\phi_1\left(\frac{1}{\rho(t)}\right)dt\right)\right)\rho(1)\right],$$

$$W = \frac{1}{1-k}\Phi\left(\frac{e_2}{\int_0^1 \Phi^{-1}\left(\frac{1}{\rho(s)}\right)ds}\right),$$

$$E = \Phi\left(\frac{e_1}{M}\right) \min_{t\in[0,1]} \rho(t)\left[1-\Phi\left(\phi_2^{-1}\left(\int_0^1 h(t)\phi_1\left(\frac{1}{\rho(t)}\right)dt\right)\right)\rho(1)\right].$$

such that

$$c \geq \frac{e_2}{\mu} > e_2 > e_1 > 0, \;\; Q \geq W.$$

If

(A1) $f(t,u,v) \leq Q$ *for all* $t \in (0,1), u \in [0,c], v \in [-c,c]$;
(A2) $f(t,u,v) \geq W$ *for all* $t \in [k,1], u \in [e_2, e_2/\mu], v \in [-c,c]$;
(A3) $f(t,u,v) \leq E$ *for all* $t \in (0,1), u \in [0,e_1], v \in [-c,c]$;

then BVP (1.1) *has at least three increasing positive solutions* x_1, x_2, x_3 *such that*

$$x_1(1) < e_1, \; x_2(k) > e_2, \;\; x_3(1) > e_1, \; x_3(k) < e_2. \tag{2.10}$$

Proof. To apply Lemma 2.1, we prove that all conditions in Lemma 2.1 are satisfied. By the definitions, it is easy to see that α, ψ are two nonnegative continuous concave functionals on the cone P, γ, β, θ are three nonnegative continuous convex functionals on the cone P and $\alpha(y) \leq \beta(y)$ for all $y \in P$.

For $y \in P$, since $y(t) \geq 0, y'(t) \geq 0$ for all $t \in [0,1]$ and $y(0) \leq \phi_1^{-1}\left(\int_0^1 g(s)ds\right)y(1)$, we have

$$\begin{aligned} |y(t)| &= \left|\int_0^t y'(s)ds + y(0)\right| \\ &= \left|\int_0^t y'(s)ds\right| + \frac{y(0) - y(0)\phi_1^{-1}\left(\int_0^1 g(s)ds\right)}{1-\phi_1^{-1}\left(\int_0^1 g(s)ds\right)} \\ &\leq \int_0^t |y'(s)|ds + \frac{\phi_1^{-1}\left(\int_0^1 g(s)ds\right)y(1) - \phi_1^{-1}\left(\int_0^1 g(s)ds\right)y(0)}{1-\phi_1^{-1}\left(\int_0^1 g(s)ds\right)} \end{aligned}$$

$$\leq \frac{1}{1-\phi_1^{-1}\left(\int_0^1 g(s)ds\right)} \sup_{t\in[0,1]} |y'(t)|.$$

It follows that

$$||y|| \leq M\beta_1(y) \text{ with } M = \frac{1}{1-\phi_1^{-1}\left(\int_0^1 g(s)ds\right)}.$$

for all $y \in P$. One sees $x = x(t)$ is a positive solution of BVP (1.1) if and only if x is a solution of the operator equation $x = Tx$.

Corresponding to Lemma 2.1,

$$c = c,\ \ h = \mu e_1,\ \ d = e_1,\ \ a = e_2,\ \ b = \frac{e_2}{\mu}.$$

Now, we prove that (C1)–(C4) of Lemma 2.1 hold. One sees that $0 < d < a$. The remainder is divided into four steps.

Step 1. Prove that $T : \overline{P_c} \to \overline{P_c}$;

For $y \in \overline{P_c}$, we have $||y|| \leq c$. Then $0 \leq y(t) \leq c$ for $t \in [0,1]$ and $-c \leq y'(t) \leq c$ for all $t \in [0,1]$. So (A1) implies that

$$f(t,y(t),y'(t)) \leq Q,\ t \in (0,1).$$

We have

$$\begin{aligned}
|(Ty)'(t)| &= \left|\Phi^{-1}\left(\frac{1}{\rho(t)}\right)\Phi^{-1}\left(\rho(1)\Phi(A_y)+\int_t^1 f(r,y(r),y'(r))dr\right)\right| \\
&\leq \left|\Phi^{-1}\left(\frac{1}{\rho(t)}\right)\Phi^{-1}\left(\rho(1)\Phi(c_0)+\int_t^1 f(r,y(r),y'(r))dr\right)\right| \\
&\leq \Phi^{-1}\left(\frac{1}{\min_{t\in[0,1]}\rho(t)}\right)\Phi^{-1}\left(\frac{\int_0^1 f(r,y(r),y'(r))dr}{1-\Phi\left(\phi_2^{-1}\left(\int_0^1 h(t)\phi_1\left(\frac{1}{\rho(t)}\right)dt\right)\right)\rho(1)}\right) \\
&\leq \Phi^{-1}\left(\frac{1}{\min_{t\in[0,1]}\rho(t)}\right)\Phi^{-1}\left(\frac{Q}{1-\Phi\left(\phi_2^{-1}\left(\int_0^1 h(t)\phi_1\left(\frac{1}{\rho(t)}\right)dt\right)\right)\rho(1)}\right) \\
&\leq c.
\end{aligned}$$

It follows from $Ty \in P$ that

$$\begin{aligned}
0 &\leq (Ty)(t) \leq M \max_{t\in[0,1]} |(Ty)'(t)| \\
&\leq M\Phi^{-1}\left(\frac{1}{\min_{t\in[0,1]}\rho(t)}\right)\Phi^{-1}\left(\frac{Q}{1-\Phi\left(\phi_2^{-1}\left(\int_0^1 h(t)\phi_1\left(\frac{1}{\rho(t)}\right)dt\right)\right)\rho(1)}\right) \\
&\leq c.
\end{aligned}$$

It follows that

$$||Ty|| = \max\left\{\max_{t\in[0,1]} |(Ty)(t)|, \max_{t\in[0,1]} |(Ty)'(t)|\right\} \leq c.$$

Then $T : \overline{P_c} \to \overline{P_c}$.

Step 2. Prove that

$$\{y \in P(\gamma,\theta,\alpha;a,b,c)|\alpha(y) > a\} = \left\{y \in P\left(\gamma,\theta,\alpha;e_2,\frac{e_2}{\mu},c\right)|\alpha(y) > e_2\right\} \neq \emptyset$$

and $\alpha(Ty) > e_2$ for every $y \in P\left(\gamma,\theta,\alpha;e_2,\frac{e_2}{\mu},c\right)$;

Choose $y(t) = \frac{e_2}{2\mu}$ for all $t \in [0,1]$. Then $y \in P$ and

$$\alpha(y) = \frac{e_2}{2\mu} > e_2,\ \theta(y) = \frac{e_2}{2\mu} \leq \frac{e_2}{\mu},\ \gamma(y) = 0 < c.$$

It follows that $\{y \in P(\gamma,\theta,\alpha;a,b,c)|\alpha(y) > a\} \neq \emptyset$.

For $y \in P(\gamma,\theta,\alpha;a,b,c)$, one has that

$$\alpha(y) = \min_{t\in[k,1]} y(t) \geq e_2,\ \theta(y) = \max_{t\in[0,1]} y(t) \leq \frac{e_2}{\mu},\ \gamma(y) = \max_{t\in[0,1]} |y'(t)| \leq c.$$

Then

$$e_2 \leq y(t) \leq \frac{e_2}{\mu},\ t \in [k,1],\ \ |y'(t)| \leq c.$$

Thus (A2) implies that

$$f(t,y(t),y'(t)) \geq W,\ \ t \in [k,1].$$

We get

$$\begin{aligned}
\alpha(Ty) &= (Ty)(k) \\
&= B_y + \int_0^t \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(\rho(1)\phi(A_y) + \int_s^1 \sigma(u)du\right)ds \\
&> \int_0^k \Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(\int_k^1 f(r,y(r),y'(r))dr\right)ds \\
&\geq \int_0^k \Phi^{-1}\left(\frac{1}{\rho(s)}\right)ds\Phi^{-1}\left((1-k)W\right) \\
&\geq e_2.
\end{aligned}$$

This completes Step 2.

Step 3. Prove that

$$\{y \in Q(\gamma,\theta,\psi;h,d,c)|\beta(y) < d\} = \{y \in Q(\gamma,\theta,\psi;\mu e_1,e_1,c)|\ \beta(y) < e_1\} \neq \emptyset$$

and

$$\beta(Ty) < e_1 \text{ for every } y \in Q(\gamma,\theta,\psi;h,d,c) = Q(\gamma,\theta,\psi;\mu e_1,e_1,c);$$

Choose $y(t) = \mu e_1$. Then $y \in P$, and

$$\psi(y) = \mu e_1 \geq h,\ \beta(y) = \theta(y) = \mu e_1 < e_1 = d,\ \gamma(y) = 0 \leq c.$$

It follows that $\{y \in Q(\gamma,\theta,\psi;h,d,c)|\beta(y) < d\} \neq \emptyset$.

For $y \in Q(\gamma,\theta,\psi;h,d,c)$, one has that

$$\psi(y) = \min_{t\in[k,1]} y(t) \geq h = \mu e_1,\ \theta(y) = \max_{t\in[0,1]} y(t) \leq d = e_1,\ \gamma(y) = \max_{t\in[0,1]} |y'(t)| \leq c.$$

Hence we get that

$$0 \leq y(t) \leq e_1,\ t \in [0,1];\ \ -c \leq y'(t) \leq c,\ t \in [0,1].$$

Then (A3) implies that

$$f(t,y(t),y'(t)) \le E,\ \ t \in (0,1).$$

So

$$\begin{aligned}\beta(Ty) &\le M \max_{t\in[0,1]} |(Ty)'(t)|\\ &\le M\Phi^{-1}\left(\frac{1}{\min_{t\in[0,1]}\rho(t)}\right)\Phi^{-1}\left(\frac{E}{1-\Phi\left(\phi_2^{-1}\left(\int_0^1 h(t)\phi_1\left(\frac{1}{\rho(t)}\right)dt\right)\right)\rho(1)}\right)\\ &= e_1 = d.\end{aligned}$$

This completes Step 3.

Step 4. Prove that $\alpha(Ty) > a$ for $y \in P(\gamma,\alpha;a,c)$ with $\theta(Ty) > b$;

For $y \in P(\gamma,\alpha;a,c) = P(\gamma,\alpha;e_2,c)$ with $\theta(Ty) > b = \frac{e_2}{\mu}$, we have that $\alpha(y) = \min_{t\in[k,1]} y(t) \ge e_2$ and $\gamma(y) = \max_{t\in[0,1]} |y'(t)| \le c$ and $\max_{t\in[0,1]}(Ty)(t) > \frac{e_2}{\mu}$. Then

$$\alpha(Ty) = \min_{t\in[k,1]}(Ty)(t) \ge \mu\beta(Ty) > \mu\frac{e_2}{\mu} = e_2 = a.$$

This completes Step 4.

Step 5. Prove that $\beta(Ty) < d$ for each $y \in Q(\gamma,\beta;d,c)$ with $\psi(Ty) < h$.

For $y \in Q(\gamma,\beta;d,c)$ with $\psi(Ty) < d$, we have $\gamma(y) = \max_{t\in[0,1]} |y'(t)| \le c$ and $\beta(y) = \max_{t\in[0,1]} y(t) \le d = e_1$ and $\psi(Ty) = \min_{t\in[k,1]}(Ty)(t) < h = e_1\mu$. Then $Ty \in P$ implies

$$\beta(Ty) = \max_{t\in[0,1]}(Ty)(t) \le \frac{1}{\mu}\min_{t\in[k,1]}(Ty)(t) < \frac{1}{\mu}e_1\mu = e_1 = d.$$

This completes the Step 5.

Then Lemma 2.1 implies that T has at least three fixed points $x_1 \in P$, $x_2 \in P$ and $x_3 \in P$ such that

$$\beta(x_1) < e_1,\ \alpha(x_2) > e_2,\ \beta(x_3) > e_1,\ \alpha(x_3) < e_2.$$

Hence BVP (1.1) has three increasing positive solutions x_1, x_2 and x_3 such that (2.10) holds. The proof is complete. ∎

3. Examples

Now, we present an example, whose three positive solutions can not be obtained by theorems in known papers, to illustrate the main results.

Example 3.1. Consider the following BVP

$$\begin{cases} [e^t(x'(t))^3]' + f(t,x(t),x'(t)) = 0,\ \ t \in (0,1),\\ x(0) = 0,\\ x'(1) = 0.\end{cases} \tag{3.1}$$

Corresponding to BVP (1.1), one sees that $\Phi(x) = x^3, \Phi^{-1}(x) = x^{\frac{1}{3}}$, $g(t) = h(t) \equiv 0$, $\rho(t) = e^t$, $\phi_1(x) = \phi_2(x) = x$, $f : (0,1)\times[0,\infty)\times R \to [0,\infty)$ is continuous.

Choose $k = 1/4$. Choose $e_1 = 50, e_2 = 250, c = 400000$ and Q, W and E are given by

$$\mu = \frac{\int_0^{1/4} \Phi^{-1}\left(\frac{1}{\rho(s)}\right) ds}{2\int_0^1 \Phi^{-1}\left(\frac{1}{\rho(s)}\right) ds} = \frac{1-e^{-\frac{1}{12}}}{3-3e^{-\frac{1}{3}}} > \frac{1}{3},$$

$$M = \frac{1}{1-\phi_1\left(\int_0^1 g(s)ds\right)} = 1,$$

$$Q = \Phi\left(\frac{c}{M}\right) \min_{t\in[0,1]} \rho(t) = 4\times 10^{15},$$

$$W = \frac{1}{1-k}\Phi\left(\frac{e_2}{\int_0^1 \Phi^{-1}\left(\frac{1}{\rho(s)}\right) ds}\right) = \frac{4}{3}\times 250^3;$$

$$E = \Phi\left(\frac{e_1}{M}\right) \min_{t\in[0,1]} \rho(t) = 50^3.$$

such that

$$c \geq \frac{e_2}{\mu} > e_2 > e_1 > 0, \;\; Q \geq W.$$

If

(A_1) $f(t,u,v) \leq 4\times 10^{15}$ for all $t \in (0,1), u \in [0,400000], v \in [-400000,400000]$;
(A_2) $f(t,u,v) \geq \frac{4}{3}\times 250^3$ for all $t \in [1/4,1], u \in [250,1000], v \in [-400000,400000]$;
(A_3) $f(t,u,v) \leq 50^3$ for all $t \in (0,1), u \in [0,50], v \in [-400000,400000]$;

then Theorem 2.1 implies that BVP (3.1) has at least three increasing positive solutions x_1, x_2, x_3 such that

$$x_1(1) < 50, \; x_2(1/4) > 250,$$

and

$$x_3(1) > 50, \; x_3(1/4) < 250.$$

Remark 3.1. Example 3.1 implies that there is a large number of functions that satisfy the conditions of Theorem 2.1. In addition, the conditions of Theorem 2.1 are also easy to check.

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