

## Existence of Nonconstant Periodic Solutions for a Nonlinear Discrete System Involving the $p$ -Laplacian

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**Abstract.** In this paper, we study the following nonlinear discrete system involving the  $p$ -Laplacian

$$\Delta(\phi_p(\Delta u(n-1))) - a(n)|u(n)|^{p-2}u(n) + \nabla F(n, u(n)) = 0, \quad n \in \mathbb{Z}.$$

By making use of the Linking theorem, we obtain a sufficient condition under which the system has at least one nonconstant periodic solution.

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### 1. Introduction

Let  $\mathbb{N}, \mathbb{Z}, \mathbb{R}$  be the set of all natural numbers, integers and real numbers, respectively. For  $a, b \in \mathbb{Z}$ , define  $\mathbb{Z}[a] = \{a, a+1, \dots\}$ ,  $\mathbb{Z}[a, b] = \{a, a+1, \dots, b\}$  when  $a \leq b$ . Consider the following nonlinear discrete system involving the  $p$ -Laplacian

$$(1.1) \quad \Delta(\phi_p(\Delta u(n-1))) - a(n)|u(n)|^{p-2}u(n) + \nabla F(n, u(n)) = 0, \quad n \in \mathbb{Z},$$

where  $p \geq 2, q > 1, \frac{1}{p} + \frac{1}{q} = 1$ ,  $\phi_p(s) = |s|^{p-2}s$ ,  $\Delta$  is the forward difference operator defined by  $\Delta u(n) = u(n+1) - u(n)$ ,  $a : \mathbb{Z} \rightarrow \mathbb{R}$ , and for  $m \in \mathbb{N}$ ,  $F : \mathbb{Z} \times \mathbb{R}^m \rightarrow \mathbb{R}$  with  $F(n+M, x) = F(n, x)$  for any  $(n, x) \in \mathbb{Z} \times \mathbb{R}^m$  and some positive integer  $M > 1$ . Moreover,  $F(n, x)$  is continuously differentiable in  $x$  for all  $n \in \mathbb{Z}[1, M]$ .

As it is known, critical-point theory is an important tool to deal with the existence of solutions of differential equations (see [9–14, 17, 19]). For difference equations, there have also been some results (see [2–8, 15, 16, 18]). In particular, by using linking theorem, Guo and Yu have successfully proved the existence of periodic solutions for the following difference equation

$$(1.2) \quad \Delta^2 u(n-1) + f(n, u(n)) = 0, \quad n \in \mathbb{Z}(1, M),$$

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when either  $f(t, v)$  is superlinear in the second variable  $v$  [6] or  $f(t, v)$  is sublinear in the second variable [7]. In [18], Zhou, Yu and Guo generalized such results to discrete systems. In [15], by local linking theorem, and in [16], by saddle point theorem, respectively, the authors proved the existence of periodic solutions for discrete systems. For the discrete system involving  $p$ -Laplacian, recently, by using dual least principle, the authors in [8] considered system (1.1) with  $a(n) \equiv 0$  and they obtained some existence results under convex condition.

In this paper, we will use Lemma 2.4 in section 2 to study system (1.1). Obviously, system (1.1) is more general than (1.2). We obtain some solvability conditions for system (1.1). To be precise, under suitable growth conditions on  $F$ , we establish the existence of at least one nonconstant solutions for system (1.1) (Theorem 2.1). For the special case  $a \equiv 0$ , by Theorem 2.1, we establish the existence of at least one nonconstant solutions for system (1.1)(Corollary 2.1), which can be seen as the discrete form of Theorem 1 in [19]. In the proof of the Theorem, we use Lemma 2.2 which can be seen as the discrete Sobolev's inequality and Wirtinger's inequality. Such inequalities will be very useful in studying the existence of periodic solutions for many discrete dynamic systems.

## 2. Main results

The Sobolev's Space  $E_M$  is defined by

$$E_M = \{u = \{u(n)\} : u(n) \in \mathbb{R}^m, u(n+M) = u(n), n \in \mathbb{Z}\}$$

and is endowed with the norm

$$\|u\| = \left( \sum_{n=1}^M |u(n)|^p \right)^{\frac{1}{p}}$$

where  $|\cdot|$  denotes the usual norm in  $\mathbb{R}^m$ . It is easy to see that  $(E_M, \|\cdot\|)$  is a finite dimensional Banach space and linear homeomorphic to  $\mathbb{R}^{mM}$ .

Let  $r > 1$ . For  $u \in E_M$ , define

$$\|u\|_r = \left( \sum_{n=1}^M |u(n)|^r \right)^{1/r}.$$

Then  $\|\cdot\|_r$  is also the norm of  $E_M$  and  $\|\cdot\|$  and  $\|\cdot\|_r$  are equivalent.

**Lemma 2.1.** *For any  $u, v \in E_M$ , the following useful equality holds:*

$$-\sum_{n=1}^M (\Delta(\phi_p(\Delta u(n-1))), v(n)) = \sum_{n=1}^M (\phi_p(\Delta u(n)), \Delta v(n)).$$

*Proof.* In fact, it follows from  $\Delta u(0) = \Delta u(M)$  and  $v(1) = v(M+1)$  that

$$\begin{aligned} & -\sum_{n=1}^M (\Delta(\phi_p(\Delta u(n-1))), v(n)) \\ &= -\sum_{n=1}^M (\phi_p(\Delta u(n)) - \phi_p(\Delta u(n-1)), v(n)) \\ &= -\sum_{n=1}^M (\phi_p(\Delta u(n)), v(n)) + \sum_{n=2}^M (\phi_p(\Delta u(n-1)), v(n)) + (\phi_p(\Delta u(0)), v(1)) \end{aligned}$$

$$\begin{aligned}
 &= - \sum_{n=1}^M (\phi_p(\Delta u(n)), v(n)) + \sum_{n=1}^{M-1} (\phi_p(\Delta u(n)), v(n+1)) + (\phi_p(\Delta u(M)), v(M+1)) \\
 &= \sum_{n=1}^M (\phi_p(\Delta u(n)), \Delta v(n)).
 \end{aligned}$$

The conclusion is achieved. ■

**Lemma 2.2.** *Let  $u \in E_M$ . If  $\sum_{n=1}^M u(n) = 0$ , then*

$$(2.1) \quad \max_{n \in \mathbb{Z}[1, M]} |u(n)| \leq \frac{(M-1)^{(q+1)/q}}{M} \left( \sum_{n=1}^M |\Delta u(n)|^p \right)^{1/p},$$

and

$$(2.2) \quad \sum_{n=1}^M |u(n)|^p \leq \frac{(M-1)^{2p-1}}{M^{p-1}} \sum_{n=1}^M |\Delta u(n)|^p.$$

*Proof.* Fix  $n \in \mathbb{Z}[1, M]$ . For every  $\tau \in \mathbb{Z}[1, n]$ , we have

$$(2.3) \quad u(n) = u(\tau) + \sum_{s=\tau}^{n-1} \Delta u(s)$$

and for every  $\tau \in \mathbb{Z}[n, M]$ ,

$$(2.4) \quad u(n) = u(\tau) - \sum_{s=n}^{\tau-1} \Delta u(s).$$

Summing (2.3) over  $\mathbb{Z}[1, n]$  and (2.4) over  $\mathbb{Z}[n, M]$ , we have

$$(2.5) \quad nu(n) = \sum_{\tau=1}^n u(\tau) + \sum_{\tau=1}^n \sum_{s=\tau}^{n-1} \Delta u(s) = \sum_{\tau=1}^n u(\tau) + \sum_{s=1}^{n-1} s \Delta u(s)$$

and

$$(2.6) \quad (M-n+1)u(n) = \sum_{\tau=n}^M u(\tau) - \sum_{\tau=n}^M \sum_{s=n}^{\tau-1} \Delta u(s) = \sum_{\tau=n}^M u(\tau) - \sum_{s=n}^{M-1} (M-s) \Delta u(s).$$

Set

$$\phi(s) = \begin{cases} s, & 1 \leq s \leq n-1, \\ M-s, & n \leq s \leq M. \end{cases}$$

Combining (2.5) with (2.6) and using the Hölder inequality, we obtain

$$\begin{aligned}
 M|u(n)| &= \left| \sum_{\tau=1}^M u(\tau) + \sum_{s=1}^{n-1} s \Delta u(s) - \sum_{s=n}^{M-1} (M-s) \Delta u(s) \right| \\
 &\leq \sum_{s=1}^{n-1} s |\Delta u(s)| + \sum_{s=n}^{M-1} (M-s) |\Delta u(s)| \\
 &= \sum_{s=1}^M \phi(s) |\Delta u(s)| \\
 &\leq \left( \sum_{s=1}^M [\phi(s)]^q \right)^{1/q} \left( \sum_{s=1}^M |\Delta u(s)|^p \right)^{1/p}
 \end{aligned}$$

$$(2.7) \quad = \left( \sum_{s=1}^{n-1} s^q + \sum_{s=n}^{M-1} (M-s)^q \right)^{1/q} \left( \sum_{s=1}^M |\Delta u(s)|^p \right)^{1/p}.$$

Since

$$(2.8) \quad \sum_{s=1}^{n-1} s^q + \sum_{s=n}^{M-1} (M-s)^q \leq \sum_{s=1}^{M-1} (M-1)^q = (M-1)^{q+1},$$

it follows from (2.7) that (2.1) holds. On the other hand, from  $\sum_{n=1}^M u(n) = 0$ , (2.7) and (2.8), we have

$$\begin{aligned} & M^p \sum_{n=1}^M |u(n)|^p \\ & \leq \left( \sum_{s=1}^M |\Delta u(s)|^p \right) \sum_{n=1}^M \left( \sum_{s=1}^{n-1} s^q + \sum_{s=n}^{M-1} (M-s)^q \right)^{p/q} \\ & \leq M(M-1)^{2p-1} \left( \sum_{s=1}^M |\Delta u(s)|^p \right). \end{aligned}$$

It follows that (2.2) holds. Thus the proof is complete. ■

**Lemma 2.3.** *For any  $u \in E_M$ , we have*

$$\sum_{n=1}^M |\Delta u(n)|^p \leq 2^p \sum_{n=1}^M |u(n)|^p.$$

*Proof.* By Hölder inequality and  $p \geq 2$ , we have

$$\begin{aligned} \sum_{n=1}^M |\Delta u(n)|^p &= \sum_{n=1}^M (|u(n+1) - u(n)|^2)^{p/2} \\ &= \sum_{n=1}^M (|u(n+1)|^2 + |u(n)|^2 - 2(u(n+1), u(n)))^{p/2} \\ &\leq \sum_{n=1}^M (2|u(n+1)|^2 + 2|u(n)|^2)^{p/2} \\ &\leq 2^{p-1} \sum_{n=1}^M (|u(n+1)|^p + |u(n)|^p) \\ &= 2^p \sum_{n=1}^M |u(n)|^p. \end{aligned}$$

The conclusion is achieved. ■

**Lemma 2.4.** [12, Theorem 2.1 and Example 3] *Let  $X = X_1 \oplus X_2$  be a Banach space, where  $X_1$  is a finite dimensional subspace of  $X$  and  $X_2 = X_1^\perp$ . Suppose that  $\varphi(\cdot) \in C^1(X, \mathbb{R})$  satisfies the Palais-Smale condition and the following conditions:*

- (i) *there are constants  $\rho > 0$  and  $\alpha$  such that  $\varphi|_{\partial B_\rho \cap X_2} \geq \alpha$ , where  $B_\rho = \{u \in X : \|u\|_X < \rho\}$ ,*
- (ii) *there is a constant  $d < \alpha$  and  $e \in X_2, \|e\|_X = 1, s_1 > 0, s_2 > \rho$  such that  $\varphi|_{\partial Q} \leq d$  where  $Q = \{u \in X | u = z + \lambda e, z \in X_1, \|z\|_X \leq s_1, \lambda \in [0, s_2]\}$ .*

Then  $\varphi$  possesses a critical value  $c \geq \alpha$ .

The main result of this paper are the following theorems:

**Theorem 2.1.** Assume that the following conditions hold:

$$(2.9) \quad \min_{n \in \mathbb{Z}[1, M]} a(n) > \frac{-M^{p-1}}{(M-1)^{2p-1}}, \quad \sum_{n=1}^M a(n) \leq 0,$$

$$(2.10) \quad \sum_{n=1}^M F(n, x) \geq 0 \quad \text{for all } x \in \mathbb{R}^m,$$

$$(2.11) \quad \lim_{|x| \rightarrow 0} \frac{F(n, x)}{|x|^p} < \frac{M^{p-1}}{p(M-1)^{2p-1}} + \frac{\min_{n \in \mathbb{Z}[1, M]} a(n)}{p} \quad \text{for all } n \in \mathbb{Z}[1, M],$$

$$(2.12) \quad \lim_{|x| \rightarrow \infty} \frac{F(n, x)}{|x|^p} > \frac{M^{p/2}}{p \sum_{n=1}^M |\frac{1+M}{2} - n|^p} + \frac{\max_{n \in \mathbb{Z}[1, M]} a(n)}{p} \quad \text{for all } n \in \mathbb{Z}[1, M].$$

If there exist constant  $\mu > 0$  such that

$$(2.13) \quad \liminf_{|x| \rightarrow \infty} \frac{(\nabla F(n, x), x) - pF(n, x)}{|x|^\mu} > 0 \quad \text{for all } n \in \mathbb{Z}[1, M],$$

then system (1.1) has at least one non-constant  $M$ -periodic solution.

For the special case  $a \equiv 0$ , in Theorem 2.1, it is easy to obtain the following corollary.

**Corollary 2.1.** Assume that  $F$  satisfies (2.10), (2.13) and the following conditions:

$$\lim_{|x| \rightarrow 0} \frac{F(n, x)}{|x|^p} < \frac{M^{p-1}}{p(M-1)^{2p-1}} \quad \text{for all } n \in \mathbb{Z}[1, M],$$

$$\lim_{|x| \rightarrow \infty} \frac{F(n, x)}{|x|^p} > \frac{M^{p/2}}{p \sum_{n=1}^M |\frac{1+M}{2} - n|^p} \quad \text{for all } n \in \mathbb{Z}[1, M].$$

Then system (1.1) has at least one non-constant  $M$ -periodic solution.

**Remark 2.1.** In [19], the authors consider the differential form of system (1.1) with  $a \equiv 0$  and it is required that there exist constants  $r > p$  such that  $\mu > r - p$  and

$$(2.14) \quad \limsup_{|x| \rightarrow \infty} \frac{F(t, x)}{|x|^r} < \infty \quad \text{uniformly for a.e. } t \in [0, T].$$

Corollary 2.1 shows that in dimensional space, such condition (2.14) above is unnecessary and  $\mu > r - p$  can be extended to  $\mu > 0$ .

### 3. Proof of Theorem 2.1

Consider the functional  $\varphi$  defined on  $E_M$  by

$$\varphi(u) = \sum_{n=1}^M \left[ \frac{1}{p} |\Delta u(n)|^p + \frac{a(n)}{p} |u(n)|^p - F(n, u(n)) \right].$$

It is well known that the functional  $\varphi$  on  $E_M$  is continuously differentiable. Moreover, for any  $u, v \in E_M$ , we have

$$\langle \varphi'(u), v \rangle = \sum_{n=1}^M [(\phi_p(\Delta u(n)), \Delta v(n)) + (a(n)|u(n)|^{p-2}u(n), v(n))]$$

$$-(\nabla F(n, u(n)), v(n))]$$

for any  $u, v \in E_M$ (see [11]). Then,  $u \in E_M$  is a critical point of  $\varphi$  if and only if

$$\sum_{n=1}^M (\phi_p(\Delta u(n)), \Delta v(n)) = \sum_{n=1}^M (-a(n)|u(n)|^{p-2}u(n) + \nabla F(n, u(n)), v(n)).$$

By the arbitrariness of  $v$  and Lemma 2.1, we conclude that

$$\Delta(\phi_p(\Delta u(n-1))) - a(n)|u(n)|^{p-2}u(n) + \nabla F(n, u(n)) = 0, \quad \forall n \in \mathbb{Z}[1, M].$$

Hence  $u \in E_M$  is a critical point of  $\varphi$  if and only if  $u$  satisfies system (1.1). Thus the problem of finding the solutions for system (1.1) is reduced to one of seeking the critical points of functional  $\varphi$  on  $E_M$ .

**Lemma 3.1.** *Assume that condition (2.13) holds. Then the functional  $\varphi$  satisfies condition (C), that is  $\{u_k\}$  has a convergent subsequence in  $E_M$ , whenever  $\varphi(u_k)$  is bounded and  $\|\varphi'(u_k)\| \times (1 + \|u_k\|) \rightarrow 0$  as  $k \rightarrow \infty$ .*

*Proof.* Let  $\{u_k\}$  be a sequence in  $E_M$  such that  $\varphi(u_k)$  is bounded and  $\|\varphi'(u_k)\| \times (1 + \|u_k\|) \rightarrow 0$  as  $k \rightarrow \infty$ . Then there exists a constant  $M$  such that

$$(3.1) \quad |\varphi(u_k)| \leq M, \quad \|\varphi'(u_k)\|(1 + \|u_k\|) \leq M$$

for every  $k \in \mathbb{N}$ . By (2.13), there are constants  $C_1 > 0$  and  $\delta_1 > 0$  such that

$$(\nabla F(n, x), x) - pF(n, x) \geq C_1|x|^\mu > 0,$$

for all  $|x| > \delta_1$  and all  $n \in \mathbb{Z}[1, M]$ . Hence,

$$(\nabla F(n, x), x) - pF(n, x) \geq C_1|x|^\mu - C_1\delta_1^\mu - C_2$$

for all  $x \in \mathbb{R}^m$  and all  $n \in \mathbb{Z}[1, M]$ , where  $C_2 = \delta_1 \max\{|\nabla F(n, x)| | n \in \mathbb{Z}[1, M], |x| \leq \delta_1\} + p \max\{|F(n, x)| | n \in \mathbb{Z}[1, M], |x| \leq \delta_1\}$ . Then we have for all large  $k$ ,

$$\begin{aligned} & (p+1)M \\ & \geq p\varphi(u_k) - (\varphi'(u_k), u_k) \\ & = \sum_{n=1}^M |\Delta u_k(n)|^p + \sum_{n=1}^M a(n)|u_k(n)|^p - p \sum_{n=1}^M F(n, u_k(n)) \\ & \quad - \sum_{n=1}^M (|\Delta u_k(n)|^{p-2}\Delta u_k(n), \Delta u_k(n)) \\ & \quad - \sum_{n=1}^M (a(n)|u_k(n)|^{p-2}u_k(n), u_k(n)) + \sum_{n=1}^M (\nabla F(n, u_k(n)), u_k(n)) \\ & = \sum_{n=1}^M [(\nabla F(n, u_k(n)), u_k(n)) - pF(n, u_k(n))] \\ & \geq C_1 \sum_{n=1}^M |u_k(n)|^\mu - MC_1\delta_1^\mu - C_2M. \end{aligned}$$

So  $\sum_{n=1}^M |u_k(n)|^\mu$  is bounded. If  $\mu \geq p$ , by Hölder's inequality, we have

$$\sum_{n=1}^M |u_k(n)|^p \leq M^{\frac{\mu-p}{\mu}} \left( \sum_{n=1}^M |u_k(n)|^\mu \right)^{\frac{p}{\mu}}.$$

Then  $\|u_k\|$  is bounded. If  $\mu < p$ , since

$$\left( \sum_{n=1}^M |u_k(n)|^p \right)^{\mu/p} \leq \sum_{n=1}^M |u_k(n)|^\mu.$$

Thus we know that  $\|u_k\|$  is bounded in  $E_M$ . Since  $E_M$  is finite dimensional Banach space, it is easy to know that there exists a convergent subsequence of  $\{u_k\}$ , which implies that  $\varphi$  satisfies the (C) condition. The proof is complete.  $\blacksquare$

*Theorem 2.1.* As shown in [1], a deformation lemma can be proved with the weaker condition (C) replacing the usual Palais-Smale condition, and it turns out that Lemma 2.4 holds under the condition (C). Let  $\tilde{E}_M = \{u \in E_M \mid \sum_{n=1}^M u(n) = 0\}$ . Obviously,  $E_M = \mathbb{R}^m \oplus \tilde{E}_M$ . Let  $X = E_M$ ,  $X_1 = \mathbb{R}^m$  and  $X_2 = \tilde{E}_M$ . Then, by Lemma 3.1, we only need to prove (i) and (ii) in Lemma 2.4 hold.

By (2.11), there is

$$0 < \varepsilon_0 < \frac{M^{p-1}}{2p(M-1)^{2p-1}} + \frac{\min_{n \in \mathbb{Z}[1,M]} a(n)}{2p}$$

such that

$$\lim_{|x| \rightarrow 0} \frac{F(n,x)}{|x|^p} \leq \frac{M^{p-1}}{p(M-1)^{2p-1}} + \frac{\min_{n \in \mathbb{Z}[1,M]} a(n)}{p} - 2\varepsilon_0.$$

Thus, there is a constant  $\delta_2 > 0$  such that

$$F(n,x) \leq \left( \frac{M^{p-1}}{p(M-1)^{2p-1}} + \frac{\min_{n \in \mathbb{Z}[1,M]} a(n)}{p} - \varepsilon_0 \right) |x|^p$$

for all  $|x| \leq \delta_2$  and all  $n \in \mathbb{Z}[1, M]$ . For every  $u \in \tilde{E}_M$  and  $\rho > 0$  with

$$\|u\| = \rho \leq \frac{M\delta_2}{2(M-1)^{(q+1)/q}},$$

by (2.1) and Lemma 2.3, it is easy to know that  $\max_{n \in \mathbb{Z}[1,M]} |u(n)| \leq \delta_2$ . Note that (2.9) implies  $\frac{-M^{p-1}}{(M-1)^{2p-1}} < \min_{n \in \mathbb{Z}[1,M]} a(n) \leq 0$ . Thus, by (2.2), we have

$$\begin{aligned} \varphi(u) &= \frac{1}{p} \sum_{n=1}^M |\Delta u(n)|^p + \frac{1}{p} \sum_{n=1}^M a(n) |u(n)|^p - \sum_{n=1}^M F(n, u(n)) \\ &\geq \frac{1}{p} \sum_{n=1}^M |\Delta u(n)|^p + \frac{1}{p} \sum_{n=1}^M a(n) |u(n)|^p \\ &\quad - \left( \frac{M^{p-1}}{p(M-1)^{2p-1}} + \frac{\min_{n \in \mathbb{Z}[1,M]} a(n)}{p} - \varepsilon_0 \right) \sum_{n=1}^M |u(n)|^p \\ &\geq \frac{1}{p} \sum_{n=1}^M |\Delta u(n)|^p + \frac{\min_{n \in \mathbb{Z}[1,M]} a(n)}{p} \cdot \frac{(M-1)^{2p-1}}{M^{p-1}} \sum_{n=1}^M |\Delta u(n)|^p \\ &\quad - \left( \frac{M^{p-1}}{p(M-1)^{2p-1}} + \frac{\min_{n \in \mathbb{Z}[1,M]} a(n)}{p} - \varepsilon_0 \right) \frac{(M-1)^{2p-1}}{M^{p-1}} \sum_{n=1}^M |\Delta u(n)|^p \\ &\geq \varepsilon_0 \frac{(M-1)^{2p-1}}{M^{p-1}} \sum_{n=1}^M |\Delta u(n)|^p \geq \varepsilon_0 \|u\|^p. \end{aligned}$$

Hence, there exists a constant  $\alpha > 0$  such that

$$\varphi(u) \geq \alpha, \text{ for every } u \in \tilde{E}_M \text{ and } \|u\| = \rho.$$

which shows that (i) holds.

Next it will be shown that (ii) also holds. For the sake of convenience, denote

$$A = \left( \sum_{n=1}^M \left| \frac{1+M}{2} - n \right|^p \right)^{1/p}.$$

Let

$$\varepsilon_1 = \min_{n \in \mathbb{Z}[1, M]} \liminf_{|x| \rightarrow \infty} \frac{F(n, x)}{|x|^p} - \frac{M^{p/2}}{pA^p} - \frac{\max_{n \in \mathbb{Z}[1, M]} a(n)}{p} > 0,$$

By (2.12), there exists sufficiently large  $\delta_3 > 0$  such that

$$F(n, x) \geq \left( \frac{M^{p/2}}{pA^p} + \frac{\max_{n \in \mathbb{Z}[1, M]} a(n)}{p} + \frac{\varepsilon_1}{2} \right) |x|^p$$

for all  $|x| \geq \delta_3$  and all  $n \in \mathbb{Z}[1, M]$  and

$$(2C_3)^{1/p} M^{1/2} / \left( \varepsilon_1^{1/p} \sqrt{\sum_{n=1}^M \left| \frac{1+M-2n}{2A} \right|^2} \right) := B > \rho,$$

where

$$C_3 := \left( \frac{M^{p/2}}{pA^p} + \frac{\max_{n \in \mathbb{Z}[1, M]} a(n)}{p} + \frac{\varepsilon_1}{2} \right) \delta_3^p + \max\{|F(n, x)| \mid n \in \mathbb{Z}[1, M], |x| \leq \delta_3\}.$$

It is easy to verify that  $M^{p-1}/(M-1)^{2p-1} < M^{p/2}/A^p$ . So by (2.9), we know that  $\max_{n \in \mathbb{Z}[1, M]} a(n) > -M^{p/2}/A^p$ . Thus, for all  $x \in \mathbb{R}^m$  and all  $n \in \mathbb{Z}[1, M]$ , we have

$$F(n, x) \geq \left( \frac{M^{p/2}}{pA^p} + \frac{\max_{n \in \mathbb{Z}[1, M]} a(n)}{p} + \frac{\varepsilon_1}{2} \right) |x|^p - C_3.$$

Let  $e = \{e(n)\}$ ,  $e(n+M) = e(n)$ , where

$$(3.2) \quad e(n) = \frac{\left(\frac{1+M}{2} - n\right) e_1}{\left(\sum_{n=1}^M \left|\frac{1+M}{2} - n\right|^p\right)^{1/p}}, \quad e_1 = (1, 0, \dots, 0)^\tau.$$

Obviously,  $e \in \tilde{E}_M$ . By calculation, it is easy to know that  $|e(n)| = |(1+M-2n)/2A|$ ,  $\|e\| = 1$ . Let  $Q = \{u \in E_M \mid u = x + se, x \in \mathbb{R}^m, \|x\| \leq M^{1/p}(2C_3)^{1/p}/\varepsilon_1^{1/p}, s \in [0, B]\}$ . It follows from  $p \geq 2$  and Hölder's inequality that

$$\sum_{n=1}^M |x + se(n)|^2 \leq \left[ \sum_{n=1}^M (|x + se(n)|^2)^{\frac{p}{2}} \right]^{\frac{2}{p}} \cdot M^{1-\frac{2}{p}}.$$

for all  $x \in \mathbb{R}^m$  and  $s \in [0, \infty)$ . Thus we have

$$\frac{M^{p/2}}{A^p} \sum_{n=1}^M |x + se(n)|^p \geq \frac{M^{p/2}}{A^p} M^{1-\frac{p}{2}} \left( \sum_{n=1}^M |x + se(n)|^2 \right)^{\frac{p}{2}}$$



$$\begin{aligned}
 &= \frac{M}{A^p} \left( \sum_{n=1}^M |x|^2 + \sum_{n=1}^M |se(n)|^2 \right)^{\frac{p}{2}} \\
 (3.3) \quad &\geq \frac{M}{A^p} \left( \sum_{n=1}^M |se(n)|^2 \right)^{\frac{p}{2}} \geq \frac{Ms^p}{A^p} \left( \sum_{n=1}^M |e(n)|^p \right) = \frac{Ms^p}{A^p} = \sum_{n=1}^M |s\Delta e(n)|^p
 \end{aligned}$$

for all  $x \in \mathbb{R}^m$  and  $s \in [0, \infty)$ . Then for every  $x + se \in Q$ , by (3.2) and (3.3), we have

$$\begin{aligned}
 \varphi(x + se) &= \frac{1}{p} \sum_{n=1}^M |s\Delta e(n)|^p + \frac{1}{p} \sum_{n=1}^M a(n)|x + se(n)|^p - \sum_{n=1}^M F(n, x + se(n)) \\
 &\leq \left( \frac{M^{p/2}}{pA^p} + \frac{\max_{n \in \mathbb{Z}[1, M]} a(n)}{p} \right) \sum_{n=1}^M |x + se(n)|^p \\
 &\quad - \left( \frac{M^{p/2}}{pA^p} + \frac{\max_{n \in \mathbb{Z}[1, M]} a(n)}{p} + \frac{\varepsilon_1}{2} \right) \sum_{n=1}^M |x + se(n)|^p + MC_3 \\
 &= -\frac{\varepsilon_1}{2} \sum_{n=1}^M |x + se(n)|^p + MC_3 \\
 &\leq -\frac{\varepsilon_1}{2} M^{1-\frac{p}{2}} \left( \sum_{n=1}^M |x|^2 + s^2 \sum_{n=1}^M \left| \frac{1+M-2n}{2A} \right|^2 \right)^{\frac{p}{2}} + MC_3.
 \end{aligned}$$

For every  $x + se \in Q$ , where  $|x| = (2C_3)^{1/p} / \varepsilon_1^{1/p}$ , we have

$$(3.4) \quad \varphi(x + se) \leq -\frac{\varepsilon_1}{2} M^{1-\frac{p}{2}} \left( \sum_{n=1}^M |x|^2 \right)^{\frac{p}{2}} + MC_3 = 0.$$

For every  $x + se \in Q$ , where

$$s = B = (2C_3)^{1/p} M^{1/2} / \left( \varepsilon_1^{1/p} \sqrt{\sum_{n=1}^M \left| \frac{1+M-2n}{2A} \right|^2} \right),$$

we have

$$(3.5) \quad \varphi(x + se) \leq -\frac{\varepsilon_1}{2} M^{1-\frac{p}{2}} \left( s^2 \sum_{n=1}^M \left| \frac{1+M-2n}{2A} \right|^2 \right)^{\frac{p}{2}} + MC_3 = 0.$$

If  $s = 0$ , for all  $x \in \mathbb{R}^m$ , by (2.9) and (2.10), we have

$$(3.6) \quad \varphi(x) = \frac{1}{p} \sum_{n=1}^M a(n)|x|^p - \sum_{n=1}^M F(n, x) \leq 0.$$

Therefore, by (3.4)–(3.6), we have  $\varphi|_{\partial Q} \leq 0$ . Let  $d = 0, s_1 = M^{1/p}(2C_3)^{1/p} / \varepsilon_1^{1/p}$  and  $s_2 = B$ . Thus (ii) in Lemma 2.4 is proved. Hence, by Lemma 2.4,  $\varphi$  has one critical value  $c \geq \alpha > 0$ . Then system (1.1) has at least one nonconstant  $M$ -periodic solution. In fact, assume that  $x \in \mathbb{R}^m$  is the solution of system (1.1). Then by (2.9) and (2.10), one has (3.6) which contradicts  $c > 0$ . ■

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