BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **35**(2) (2012), 383–398

Existence of Traveling Waves of Conservation Laws with Singular Diffusion and Nonlinear Dispersion

MAI DUC THANH
Department of Mathematics, International University, Quarter 6, Linh Trung Ward, Thu Duc District,
Ho Chi Minh City, Vietnam
mdthanh@hcmiu.edu.vn

Abstract. We establish the existence of traveling waves for diffusive-dispersive conservation laws with locally Lipschitz flux function, singular diffusion and nonlinear dispersion. Because of the singular diffusion, the linearized traveling wave system at the equilibrium corresponding to the right-hand state of the shock has purely imaginary eigenvalues. We use a Lyapunov-type function and LaSalle's invariance principle to show that this equilibrium is attracting. The level sets of the Lyapunov-type function enables us to estimate its domain of attraction. The equilibrium corresponding to the left-hand state of the shock is a saddle. We show that exactly one of the two trajectories leaving the saddle enters the domain of attraction of the attractor, thus giving a traveling wave.

2010 Mathematics Subject Classification: 35L65, 74N20, 76N10, 76L05

Keywords and phrases: Shock wave, traveling wave, conservation law, diffusion, dispersion, equilibria, Lyapunov-type function, attraction domain.

1. Introduction

We consider the existence of a certain kind of smooth solution, called a *traveling wave*, of the following third-order partial differential equation

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = (R(u,\varepsilon u_x))_x + \delta(A(u)(B(u)u_x)_x)_x, \quad x \in \boldsymbol{R}, t > 0, \tag{1.1}$$

where, the term $(R(u,\varepsilon u_x))_x$ represents diffusion effect and the term $\delta(A(u)(B(u)u_x)_x)_x$ represents the dispersion effect, see [5]. The small positive numbers $\varepsilon > 0, \delta > 0$ measure the small scale of diffusion and dispersion, respectively. Throughout, we assume the following hypotheses

(H1) The flux function $f = f(u)$, the diffusion function $R = R(u,v)$ and the dispersion functions $A = A(u) > 0, B = B(u) > 0$ are continuous and locally Lipschitz.

(H2) The diffusion function $R = R(u,v)$ satisfies

$$R_u(u,0) = R_v(u,0) = 0, \quad R(u,v)v > 0, v \neq 0 \quad \text{for any} \quad u. \tag{1.2}$$

Communicated by Norhashidah Hj. Mohd. Ali.
Received: May 20, 2010; *Revised:* July 7, 2010.

As motivated by our earlier works [31, 32], we develop in this paper the attraction domain method for the model (1.1) with a singular diffusion satisfying (1.2). Observe that a regular diffusion of the type $R(u,v) \equiv v$ gives $R_v \equiv 1$ and therefore it does not fulfil (1.2). A major difference between the models with regular and singular diffusion is that with singular diffusion, the linearized traveling wave system at the equilibrium corresponding to the right-hand side state of a Lax shock does not have negative eigenvalues; instead it has purely imaginary eigenvalues. To show that the equilibrium point is asymptotically stable, we define a Lyapunov-type function and invoke LaSalle's invariance principle. However, our argument, with a slight modification, can be applied to the case of a regular diffusion and therefore improves the result in [31]. A simple example of diffusion is $R(u,v) = |v|^p v$, where $p \geq 0$, in which the case $p = 0$ corresponds to a regular diffusion as in [31], and the case $p > 0$ corresponds to a singular diffusion satisfying (1.2).

Letting $\varepsilon, \delta = 0$ in (1.1) we obtain the conservation law

$$\partial_t u + \partial_x f(u) = 0. \tag{1.3}$$

It has been known that whenever a traveling wave of (1.1) connecting the left-hand and right-hand states u_- and u_+, respectively, exists, then its point-wise limit when $\varepsilon, \delta \to 0$ gives a shock wave of (1.3) connecting the left-hand state u_- and right hand state u_+, see [18]. Conversely, given a shock wave connecting the left-hand state u_- and right hand state u_+, the question whether there is a traveling wave connecting these two states has attracted many authors. It was shown by Slemrod, see [28], that this is the case for a Lax (classical) shock of an isentropic van der Waals fluid in the domain where the pressure is a convex function of the specific volume. See also [10, 29] for the study of the viscosity-capillarity zero limit of smooth solutions. Observe that diffusive-dispersive models also provide the existence of traveling waves associated with nonclassical shocks. The reader is referred to the works of LeFloch, Bedjaoui and their collaborators, see [1–5, 11, 18] for the study of traveling waves associated with nonclassical shocks.

It is very interesting that nonlinear, degenerate diffusion and dispersion have been found useful in many applications of fluid dynamics and material sciences. Besides, in an engineering experiment, a classical shock or a nonclassical shock both could have the possibility to appear. One has seen the focus on diffusive-dispersive traveling waves associated with a nonclassical shock in the works of Bedjaoui and LeFloch [1–5, 18]. Our goal in this paper is to draw a parallel work to the one of Beadjaoui and LeFloch in [5] by focusing on diffusive-dispersive traveling waves associated with a classical shock for a general conservation law where the flux function is solely locally Lipschitz. This would provide a more comprehensive description of diffusive-dispersive traveling waves.

In the works of LeFloch-Thanh [19–22], classical and nonclassical Riemann solvers were studied. Observe that even if nonclassical shocks are privileged (for definitiveness, for example), classical shocks always appear in nonclassical Riemann solvers. Thus, a moderate nonclassical Riemann solver-something between the classical Riemann solver and the privileging-nonclassical shock Riemann solver-should contain more classical shocks. Pioneering works for classical shocks were carried out by Lax [16], Oleinik [26], and Liu [25], etc. See also [15, 23, 24, 30] for classical shocks of the extension of the p-system and gas dynamics equations and related topics. The reader is referred to the works of LeFloch and his students and collaborators, see [11, 12, 17–21], for nonclassical shock waves. The reader is referred to [7] for the Riemann problem for hyperbolic systems of conservation laws with Lipschitz flux function. We also observe that traveling waves for diffusive-dispersive scalar equations were earlier studied by Bona and Schonbek [6], Jacobs, McKinney, and Shearer [13]. Traveling waves of the hyperbolic-elliptic model of phase transition dynamics were also studied by Slemrod [10, 28] and Fan [8, 9], Shearer and Yang [27]. See also the references therein. In our recent work [33], we established the existence of traveling waves in elastodynamics with variable viscosity and capillarity.

The organization of this paper is as follows. In Section 2, we provides basic facts of shock waves and traveling waves, and the equilibrium points. Using linearization, we observe that the equilibrium point corresponding to the left-hand state of the given Lax shock is a saddle point. However, the equilibrium point corresponding to the right-hand state is a focus of the linearized system. We close the section by draw a statement that linearization does not help to bring a firm conclusion about the stability of the state on the right. In Section 3, we first introduce a Lyapunov-type function. We then apply LaSalle's invariance principle to show that the equilibrium point corresponding to the right-hand state is asymptotically stable. Then we use the level sets of the Lyapunov-type function to estimate the region of attraction of the attracting equilibrium. It turns out that there is exactly one stable trajectory of the saddle point eventually enters the attraction domain of the attracting equilibrium. This establishes a saddle-to-attractor connection and accordingly gives us a unique traveling wave, up to translation in space. Finally, in Section 4, we will show that the analysis in the current paper can be used to improve our earlier result in [31]. Moreover, we illustrate the existence of traveling waves by a numerical experiment.

2. Shock waves and traveling waves

First, let us provide a brief introduction to the concept of shock waves. A *discontinuity* of the form

$$u(x,t) = \begin{cases} u_-, & x < st, \\ u_+, & x > st, \end{cases} \tag{2.1}$$

where u_-, u_+ are relatively the left-hand and right-hand states and s is the speed of discontinuity propagation, is a *weak solution* of the conservation law (1.2) in the sense of distributions iff it satisfies the Rankine-Hugoniot relation

$$-s(u_+ - u_-) + f(u_+) - f(u_-) = 0. \tag{2.2}$$

The equation (2.2) implies that the speed of discontinuity propagation s is given by

$$s = s(u_-, u_+) = \frac{f(u_+) - f(u_-)}{u_+ - u_-}.$$

It is known that weak solutions are not unique. To select a unique solution, one requires that weak solutions of the form (2.1) satisfy an entropy admissibility condition. In the case of scalar conservation laws, one often uses the *Oleinik entropy criterion*, which requires

$$\frac{f(u) - f(u_-)}{u - u_-} \geq \frac{f(u_+) - f(u_-)}{u_+ - u_-}, \quad \text{for any } u \text{ between } u_+ \text{ and } u_-. \tag{2.3}$$

The condition (2.3) is equivalent to

$$\frac{f(u) - f(u_+)}{u - u_+} \leq \frac{f(u_+) - f(u_-)}{u_+ - u_-}, \quad \text{for any } u \text{ between } u_+ \text{ and } u_-.$$

A *(classical) shock wave* of (1.3) is a weak solution of the form (2.1) and satisfies the Oleinik entropy criterion (2.3). In brief, a shock wave connecting a left-hand state u_- to a right-hand state u_+ with shock speed s is given by (2.1), where $u_\pm$ and s are such that the Rankine-Hugoniot relation (2.2) and the Oleinik criterion (2.3) hold.

In the case of a hyperbolic system of conservation, one often uses the *Lax shock inequalities* or Liu's entropy condition as an admissibility condition for shock waves. One therefore calls a *Lax shock* of (1.1) to be a discontinuity of the form (2.1) satisfying the following Lax shock inequalities

$$f'(u_-) > s(u_-,u_+) > f'(u_+), \quad u_- \neq u_+,$$

where $s(u_-,u_+)$ is the speed of discontinuity (shock speed).

Geometrically, the inequality (2.2) means that if $u_+ < u_-$, the graph of f is lying below the straight line (Δ) connecting the two points $(u_\pm, f(u_\pm))$ in the interval $[u_+,u_-]$.

A *traveling waves* of (1.1) is a smooth solution $u = u(y)$ depending on the re-scaled variable

$$y := \frac{x-st}{\varepsilon}.$$

for some constant speed s. Substituting $u = u(y), y = (x-st)/\varepsilon$, to (1.1), the traveling wave u connecting a left-hand state u_- to a right-hand state u_+ satisfies the ordinary differential equation

$$-s\frac{du}{dy} + \frac{df(u)}{dy} = \frac{dR(u,\frac{du}{dy})}{dy} + \frac{\delta}{\varepsilon^2}\frac{d}{dy}\left(A(u)\frac{d}{dy}\left(B(u)\frac{du}{dy}\right)\right), \tag{2.4}$$

and the boundary conditions

$$\begin{aligned} &\lim_{y\to\pm\infty} u(y) = u_\pm, \\ &\lim_{y\to\pm\infty} \frac{du}{dy} = \lim_{y\to\pm\infty} \frac{d^2u}{dy^2} = 0. \end{aligned} \tag{2.5}$$

Integrating (2.4) and using the boundary condition (2.5), we find u such that

$$\frac{\delta}{\varepsilon^2}A(u)\frac{d}{dy}\left(B(u)\frac{du}{dy}\right) + R\left(u,\frac{du}{dy}\right) = -s(u(y)-u_-) + f(u) - f(u_-), \quad y \in \boldsymbol{R}. \tag{2.6}$$

Using (2.5) again, we deduce from (2.6)

$$s = \frac{f(u_+) - f(u_-)}{u_+ - u_-}.$$

which means that u_-, u_+ and s satisfy the Rankine-Hugoniot relation (2.2) and they are respectively the left-hand, right-hand state and shock speed of a discontinuity of (1.3).

Setting

$$v = B(u)\frac{du}{dy}$$

we can re-write the second-order differential equation (2.6) to the following second-order system

$$\begin{aligned} \frac{du(y)}{dy} &= b(u(y))v(y), \\ \frac{dv(y)}{dy} &= \gamma a(u(y))(-Q(u(y),v(y)) + h(u(y))), \quad -\infty < y < +\infty, \end{aligned} \tag{2.7}$$

where

$$a(u) = \frac{1}{A(u)}, \quad b(u) = \frac{1}{B(u)}, \quad \gamma = \frac{\varepsilon^2}{\delta}, \tag{2.8}$$
$$Q(u,v) = R(u,b(u)v), \quad h(u) = -s(u-u_-) + f(u) - f(u_-).$$

Setting

$$U = (u,v), \quad F(U) = (b(u)v, \gamma a(u)(-Q(u,v) + h(u))),$$

we can re-write the system (2.7) in the form

$$\frac{dU}{dy} = F(U), \quad -\infty < y < +\infty. \tag{2.9}$$

It is easy to check that a point U in the (u,v)-phase plane is an equilibrium point of the autonomous differential equations (2.7) if and only if U has the form $U = (u_+,0)$ for some constant u_+ so that the states $u_\pm$ and the shock speed s satisfy the Rankine-Hugoniot relation (2.2). Consequently, $u = u(x,t)$ defined by (2.1) is a weak solution of the conservation law (1.3). Conversely, a jump of (1.3) of the form (2.1) gives equilibria $(u_-,0),(u_+,0)$ of the differential equation (2.7).

Suppose now that f is differentiable at some point u. Then, the Jacobian matrix $DF(U)$ of the system (2.9) at $U = (u,v)$ is given by

$$DF(U) = \begin{pmatrix} b'(u)v & b(u) \\ \gamma a'(u)(-Q(u,v)+h) + \gamma a(u)(-Q_u(u,v) + f'(u) - s) & -\gamma a(u) Q_v(u,v) \end{pmatrix}.$$

Recall that $Q(u,v) = R(u,b(u)v)$. Therefore, it holds that

$$Q_u(u,v) = R_u(u,b(u)v) + R_v(u,b(u)v)b'(u)v,$$
$$Q_v(u,v) = R_v(u,b(u)v)b(u).$$

This yields, under the hypotheses (H1) and (H2):

$$Q_u(u,0) = Q_v(u,0) = Q(u,0) = 0.$$

Thus, if f is differentiable at $u_\pm$, using $h(u_\pm) = 0$ and (1.2) we obtain

$$DF(u_\pm,0) = \begin{pmatrix} 0 & b(u_\pm) \\ \gamma a(u_\pm)(f'(u_\pm) - s) & 0 \end{pmatrix}.$$

The characteristic equation of $DF(u_\pm,0)$ is then given by

$$\lambda^2 - \gamma a(u_\pm)b(u_\pm)(f'(u_\pm) - s) = 0.$$

Thus, we arrive at the following conclusion.

Proposition 2.1. *Assume the hypotheses (H1) and (H2). In addition, assume that $f'(u_\pm)$ exist and the states $u_\pm$ correspond to a Lax shock. Then, for the linearized system at $(u_\pm,0)$:*

(i) *The Jacobian matrix at $(u_-,0)$ admits two eigenvalues having opposite signs*

$$\lambda_1(u_-,0) = -\sqrt{\gamma a(u_-)b(u_-)(f'(u_-) - s)} < 0, \tag{2.10}$$
$$\lambda_2(u_-,0) = \sqrt{\gamma a(u_-)b(u_-)(f'(u_-) - s)} > 0.$$

The point $(u_-,0)$ is a saddle point.

(ii) *The Jacobian matrix at* $(u_+,0)$ *admits two purely imaginary eigenvalues*

$$\lambda_{1,2}(u_+,0) = \pm i\sqrt{\gamma a(u_+)b(u_+)(s - f'(u_+))}, \quad \textit{where} \quad i^2 = -1. \tag{2.11}$$

The point $(u_+,0)$ *is thus a focus (of the linearized system).*

We aim at showing the existence of a traveling wave corresponding to a Lax shocks by means of a saddle-to-attractor connection. However, as seen by Proposition 2.1, *linearization does not work* to show that $(u_+,0)$ is asymptotically stable. This is a contrary to the case in [31], where linearization shows that $(u_+,0)$ is asymptotically stable. To establish a saddle-to-attractor connection, we will prove in the next section that $(u_+,0)$ is asymptotically stable using LaSalle's invariance principle.

3. Existence of traveling waves

3.1. Estimate of attraction domain of the attracting equilibrium

In this section we will establish a sharp estimate the the domain of attraction of the attracting equilibrium $(u_+,0)$.

Now let us consider the differential equations

$$\begin{aligned} \frac{du(y)}{dy} &= b(u(y))v(y), \\ \frac{dv(y)}{dy} &= \gamma a(u(y))(-Q(u(y),v(y)) + h(u(y))), \quad -\infty < y < +\infty, \end{aligned} \tag{3.1}$$

where

$$Q(u,v) = R(u,b(u)v), \quad h(u) = -s(u-u_-) + f(u) - f(u_-).$$

The function Q then has the same property as the function R:

$$Q(u,v)v > 0, \quad v \neq 0$$

for any u. We define a Lyapunov-type function

$$L(u,v) = \gamma \int_u^{u^+} \frac{a(\xi)}{b(\xi)} h(\xi)d\xi + \frac{v^2}{2}. \tag{3.2}$$

Then we have

$$L(u_+,0) = 0, \quad \nabla L(u,v) = \left\langle -\frac{\gamma a(u)h(u)}{b(u)}, v \right\rangle.$$

And therefore

$$\begin{aligned} \dot{L}(u,v) &= \nabla L(u,v) \cdot \left\langle \frac{du}{dy}, \frac{dv}{dy} \right\rangle \\ &= \left\langle -\frac{\gamma a(u)h(u)}{b(u)}, v \right\rangle \langle b(u)v, \gamma a(u)(-Q(u,v) + h(u)) \rangle \\ &= -\gamma a(u)Q(u,v)v \langle 0, \quad \text{for} \quad v \neq 0. \end{aligned} \tag{3.3}$$

For definitiveness, we assume that

$$u_+ < u_-,$$

without restriction. We assume that there are two values $p <_+ < q \leq u_-$ such that

$$\int_p^q \frac{a(\xi)}{b(\xi)} h(\xi) d\xi > 0. \tag{3.4}$$

The condition (3.4) is fulfilled when for example the function f is differentiable at $u = u_+$ and the shock satisfies either the Lax shock inequality

$$f'(u_+) < s$$

or the strict Oleinik entropy condition

$$\frac{f(u) - u(u_+)}{u - u_+} < s, \quad \text{for} \quad u_+ + \theta' > u > u_+,$$

for some small $\theta' > 0$. Indeed, by continuity, there is a positive number $\theta > 0$ such that

$$\frac{f(u) - u(u_+)}{u - u_+} < s, \quad \text{for} \quad u_+ - \theta < u < u_+. \tag{3.5}$$

It follows from (3.5) that

$$\int_u^{u_+} \frac{a(\xi)}{b(\xi)} h(\xi) d\xi > 0, \quad u_+ - \theta < u < u_+.$$

Set $p = u_+ - \theta$, from the last inequality, by continuity, there exists some $q > u_+$ so that

$$\int_p^q \frac{a(\xi)}{b(\xi)} h(\xi) d\xi \geq 0,$$

which establishes (3.4).

Let us take a sufficiently large number M so that

$$M^2 > |s| + \max_{u \in [p,q]} \frac{a(u)}{b(u)} \mathrm{Lip}_{[p,q]} f, \tag{3.6}$$

where $\mathrm{Lip}_{[p,q]} f$ denotes the Lipschitz constant of the function f on the interval $[p,q]$. From (3.4) and (3.6) we now define the set

$$\begin{aligned} G = &\{(u,v) \in \boldsymbol{R}^2 | (u - u_+)^2 + \frac{1}{M^2} v^2 \leq |u_+ - q|^2, \quad u \geq u_+\} \\ &\cup \{(u,v) \in \boldsymbol{R}^2 | (u - u_+)^2 + \frac{|u_+ - p|^2}{(M|u_+ - q|)^2} v^2 \leq |u_+ - p|^2, \quad u \leq u_+\}, \end{aligned} \tag{3.7}$$

(see Figure 1).

Lemma 3.1. *Let G be the set defined by* (3.7) *and ∂G denotes its boundary. For sufficiently large M, we have*

$$\min_{(u,v) \in \partial G} L(u,v) = L(q,0). \tag{3.8}$$

Proof. On the semi-ellipse $\partial G, u \geq u_+$, one has

$$v^2 = M^2(|u_+ - q|^2 - (u - u_+)^2.$$

Substituting v from the last equation to the expression of L, we have

$$\begin{aligned} L(u,v)\Big|_{(u,v) \in \partial G, u \geq u_+} &= \int_u^{u_+} \frac{a(\xi)}{b(\xi)} h(\xi) d\xi + \frac{M^2}{2} (|u_+ - q|^2 - (u - u_+)^2 \\ &:= g(u), \quad u \in [u_+, q]. \end{aligned}$$

A straightforward calculation gives

$$\begin{aligned}\frac{dg(u)}{du} &= -\frac{a(u)}{b(u)}h(u) - M^2(u-u_+)\\ &= -(u-u_+)\Big(M^2 + \frac{a(u)}{b(u)}\big(\frac{f(u)-f(u_+)}{u-u_+} - s\big)\Big) < 0, \quad u \in (u_+, q)\end{aligned}$$

where the last inequality follows from (3.6). The function g is therefore strictly decreasing for $u \in [u_+, q]$ and attains its minimum on this interval at the end-point $u = q$. This yields

$$\min_{(u,v)\in\,\partial G, u\ge u_+} L(u,v) = L(q,0).$$

Similarly, it holds that

$$\min_{(u,v)\in\,\partial G, u\le u_+} L(u,v) = L(p,0).$$

Moreover, it is derived from (3.4) that

$$L(p,0) \ge L(q,0)$$

so that (3.8) is established. The proof of Lemma 3.1 is complete. ▮

Lemma 3.2. *For any positive number* $0 < \beta < L(q,0)$*, the set*

$$\Omega_\beta := \{(u,v) \in G | L(u,v) \le \beta\} \tag{3.9}$$

(see Figure 1) is a compact set, positively invariant with respect to (3.1)*, and has the point* $(u_+, 0)$ *as an interior point.*

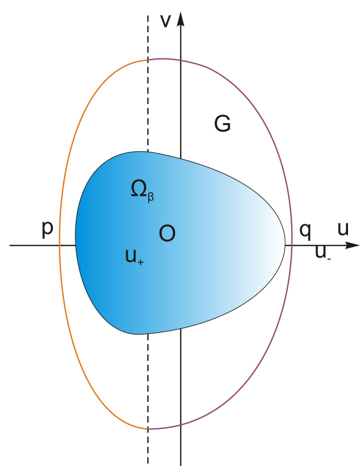

Figure 1. The set Ω_β defined by (3.9)

Proof. Since Ω_β is a subset of the bounded set G, the continuity then implies that Ω_β is a compact set.

We will show that the set Ω_β is in the interior of G. In deed, assume the contrary, then there is a point $(u_0, u_0) \in \Omega_\beta$ which lies on the boundary of G. Then, in view of Lemma 3.1 and by definition of minimum, we have

$$L(u_0, v_0) \geq L(q, 0) > \beta \geq L(u_0, v_0)$$

which is a contradiction. Thus, the closed curve $L(u, v) = \beta$ lies entirely in the interior of G. Moreover, the semi-negativity of the derivative along trajectories of (3.1) of L yields

$$\frac{dL(u(y), v(y))}{dy} \leq 0.$$

Thus,

$$L(u(y), v(y)) \leq L(u(0), v(0)) \leq \beta, \quad \forall y > 0,$$

which shows that any trajectory starting in Ω_β cannot cross the closed curve $L(u, v) = \beta$. Therefore, the compact set Ω_β is positively invariant with respect to (3.1). This completes the proof of Lemma 3.2. ∎

In what follows, we will show that trajectories of (3.1) starting in Ω_β exist for all $y > 0$. Moreover, these trajectories will converge to the node $(w_+, 0)$ as $y \to +\infty$.

Theorem 3.1. *Given* $(u_0, v_0) \in \Omega_\beta$. *The initial-value problem for* (3.1) *with initial condition* $(u(0), v(0)) = (u_0, v_0)$ *admits a unique global solution* $(u(y), v(y))$ *for all* $y \geq 0$. *Moreover, this trajectory converges to* $(u_+, 0)$ *as* $y \to +\infty$, *i.e.,*

$$\lim_{y \to +\infty} (u(y), v(y)) = (u_+, 0).$$

This means that the equilibrium point $(u_+, 0)$ *is asymptotically stable.*

Proof. As seen by Lemma 3.2, the set Ω_β defined by (3.4) is a compact set and is positively invariant with respect to (3.1). This simply means that any solution of (3.1) starting in Ω_β lies entirely in Ω_β. It follows from the existence theory of differential equations that there is a unique solution starting in Ω_β defined for all $y \geq 0$ (see [14], Theorem 3.3, for example). Denote

$$E = \{(u, v) \in \Omega_\beta | \dot{L}(u, v) = 0\}$$

Then, it follows from (3.3) that

$$\begin{aligned} E &= \{(u, v) \in \Omega_\beta | \dot{L}(u, v) = -\gamma a(u) Q(u, v) v = 0\} \\ &= \{(u, v) \in \Omega_\beta | v = 0\}. \end{aligned} \tag{3.10}$$

Applying LaSalle's invariance principle (see [14], Theorem 4.4), wa can see that every trajectory of (3.1) starting in Ω_β approaches the largest invariant set M of the set E as $y \to \infty$. Therefore, it is sufficient to show that the set M contains only one point $(u_+, 0)$. This can be done by showing that no solution can stay in E, other than the trivial solution $(u(y), v(y)) \equiv (u_+, 0)$. Indeed, let $(u(y), v(y))$ be a solution that stays in E. Then,

$$\frac{du(y)}{dy} = b(u) v(y) \equiv 0,$$

which yields

$$u \equiv u_+.$$

That is,

$$(u(y), v(y)) \equiv (u_+, 0).$$

Thus, LaSalle's invariance principle implies that every trajectory of (3.1) starting in Ω_β converges to the equilibrium point $(u_+,0)$ as $y\to\infty$. This completes the proof. ∎

3.2. Saddle-to-attractor connection and existence of traveling waves

In this section will establish existence result. First we consider the stable trajectories issuing from the saddle point $(u_-,0)$ at $-\infty$. Assume that we can always have a number $p=p(u_-)$ such that

$$\int_{p(u_-)}^{u_-} h(v)dv > 0. \tag{3.11}$$

Then u_- plays the role for q in (3.9). The union of the sets $\Omega_\beta, 0<\beta<L(u_-,0)$ can give a reasonable estimate for the attraction domain. In fact, we set

$$\Omega = \cup_{0<\beta<L(u_-,0)}\Omega_\beta. \tag{3.12}$$

Then, a straightforward calculation shows that

$$\begin{aligned}\Omega &= \{(w,z)\in \boldsymbol{R}^2 | \quad L(u,v)-L(u_-,0)<0\}\\ &= \left\{(u,v)\in \boldsymbol{R}^2 | \quad \gamma\int_u^{u_-}\frac{a(\xi)}{b(\xi)}h(\xi)d\xi+\frac{v^2}{2}<0\right\}\end{aligned} \tag{3.13}$$

Due to the continuity, we can draw some conclusions of the estimate Ω of the attraction domain in the following lemma. See Figure 2.

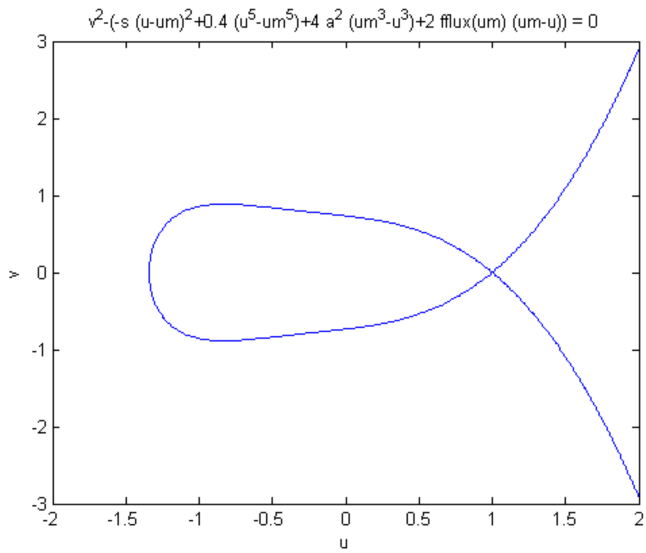

Figure 2. Attraction domain: The bounded set enclosed by the curve is the set Ω given by (3.13) which gives an estimate of the attraction domain of the attracting equilibrium

Lemma 3.3. *Under the condition* (3.11)*, the attraction domain of the node* $(u_+,0)$ *of* (3.1) *contains the set* Ω *defined by* (3.12)*. The set* Ω *is open, connected and has the point* $(u_-,0)$ *on its boundary.*

We now consider the stable trajectories leaving the saddle point $(u_-,0)$. Since the stable trajectories are tangent to the eigenvector $< 1,\lambda_2(u_-,0) >$, (see (2.10)), one of them leaves the saddle point $(u_-,0)$ in the quadrant

$$Q_1 = \{(u,v)|u < u_-, v < 0\}$$

and the other leaves the saddle point in the quadrant

$$Q_2 = \{(u,v)|u > u_-, v > 0\}.$$

Since we need the stable trajectory that goes beyond the attracting equilibrium, we will study only the stable trajectory that leaves the saddle point in the quadrant Q_1.

Multiply both sides of the second equation of (3.1) by $b(u)\frac{du}{dy} = v$ and integrating from $(-\infty,y)$, we get

$$\int_{-\infty}^{y} v\frac{dv}{dy}dy = \int_{-\infty}^{y} \frac{du}{dy}\frac{\gamma a(u(y))}{b(u(y))}(-Q(u(y),v(y))+h(u(y))))dy$$

or

$$\frac{v^2}{2} = \int_{u_-}^{u} \frac{\gamma a(\xi)}{b(\xi)}(-Q(\xi,v)+h(\xi))d\xi.$$

Since (u,v) is Q_1, $v<0$ and thus $Q(.,v)<0$. Thus, we have

$$0 \le \frac{v^2}{2} < \int_{u_-}^{u} \frac{\gamma a(\xi)}{b(\xi)}h(\xi)d\xi$$

or

$$\int_{u}^{u_-} \frac{\gamma a(\xi)}{b(\xi)}h(\xi)d\xi + \frac{v^2}{2} < 0.$$

The last inequality shows that the stable trajectory leaving the saddle point enters the attraction domain of the attracting equilibrium:

$$(u(y),v(y)) \in \Omega,$$

which establishes a saddle-to-attractor connection.

The above argument leads us to the following main theorem for the existence of traveling waves.

Theorem 3.2. *Assume that the flux function f is locally Lipschitz, and that there is a shock wave of* (1.2) *connecting the left-hand state u_- and the right-hand state u_+ with the shock speed s satisfying the Oleinik entropy condition. In addition, assume that the condition* (3.11) *holds. Then, there exists a traveling wave of* (1.1) *connecting the states u_-,u_+.*

4. The case of regular diffusion and numerical illustration

Parts of the analysis in the previous section can be applied, with slight modifications, to a conservation law with regular diffusion and dispersion. In this section, we will extend our analysis to improve our earlier result in [31]. Precisely, we consider the following diffusive-dispersive conservation law

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = \varepsilon\partial_{xx}u(x,t) + \delta\partial_{xxx}u(x,t), \quad x \in \boldsymbol{R}, t > 0, \tag{4.1}$$

where the constants $\varepsilon > 0, \delta > 0$ represent the diffusion and dispersion coefficients, respectively.

In [31], we consider a *traveling wave* of (4.1) to be a smooth solution $u = u(y)$ depending on the re-scaled variable

$$y := \alpha \frac{x - st}{\varepsilon} = \frac{x - st}{\sqrt{\delta}}, \quad \alpha = \varepsilon / \sqrt{\delta}, \tag{4.2}$$

for some constant speed s, and satisfies the boundary conditions

$$\begin{aligned} &\lim_{y \to \pm\infty} u(y) = u_{\pm}, \\ &\lim_{y \to \pm\infty} \frac{du}{dy} = \lim_{y \to \pm\infty} \frac{d^2u}{dy^2} = 0. \end{aligned} \tag{4.3}$$

Arguing similarly as in Section 2, by setting

$$v = \frac{du}{dy}$$

we find that (u, v) satisfies the following system of first-order ordinary differential equations

$$\begin{aligned} \frac{du(y)}{dy} &= v(y), \\ \frac{dv(y)}{dy} &= -\alpha v(y) - s(u(y) - u_-) + f(u(y)) - f(u_-). \end{aligned} \tag{4.4}$$

It was shown in [31] that the point $(u_+, 0)$ is an asymptotically attracting equilibrium, and the point $(u_-, 0)$ is a saddle point.

The Lyapunov function for the attracting equilibrium $(u_+, 0)$ is given by

$$L(u, v) = \int_u^{u_+} h(v)dv + \frac{1}{2}v^2,$$

where

$$h(u) = -s(u - u_-) + f(u) - f(u_-).$$

Assume that the condition (3.11) holds. Then, the following result was established.

Proposition 4.1. *[31, Prop. 3.3] The domain of attraction of the attracting equilibrium* $(u_+, 0)$ *includes the set* Ω *defined by*

$$\Omega = \{(u, v) \in \boldsymbol{R}^2 |\ L(u, v) < L(u_-, 0)\}. \tag{4.5}$$

Arguing similarly as in Section 3, we find that the stable trajectory of the saddle point $(u_-, 0)$ getting down from $(u_-, 0)$ at $-\infty$ eventually enters the set Ω. This establishes the following theorem.

Theorem 4.1. *Assume that the flux function f is locally Lipschitz, and that there is a shock wave of* (1.3) *connecting the left-hand state u_- and the right-hand state u_+ with the shock speed s satisfying the Oleinik entropy condition. In addition, assume that the condition* (3.11) *holds. Then, there exists a unique traveling wave of* (4.1) *connecting the states* u_-, u_+.

Next, we will present some numerical illustration for the traveling wave of (4.1) where

$$f(u) = u^4 - 6a^2u^2, \quad a = \text{constant}, a > 0. \tag{4.6}$$

The function f admits two inflection points $u = \pm a$, see Figure 3.

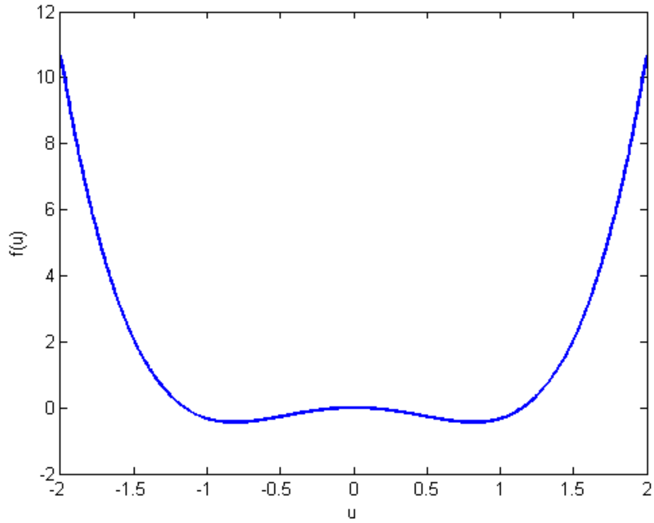

Figure 3. The flux function given by (4.6)

In the following experiment we take

$$\varepsilon = 10^{-2}, \quad \delta = 10^{-3}, \quad a = 1/3, \quad u_- = 1, \quad u_+ = 0. \tag{4.7}$$

The estimate of the attraction domain is shown by Figure 4.

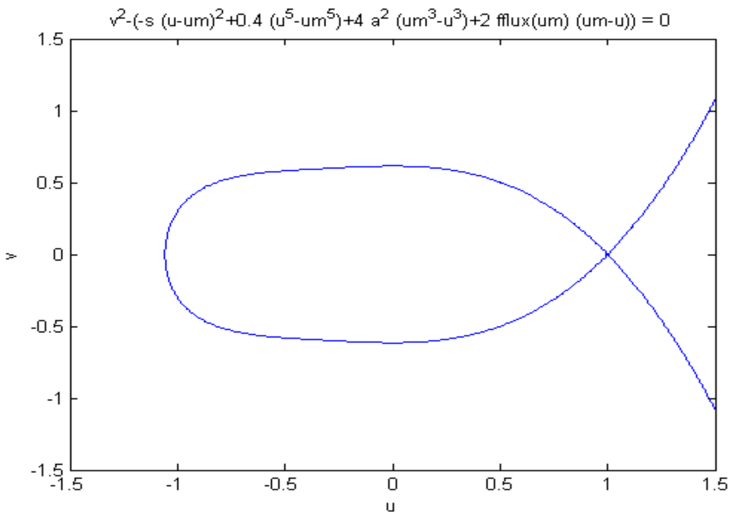

Figure 4. Estimate of the attraction domain of the attracting equilibrium $(0,0)$ of (4.4), (4.6) and (4.7)

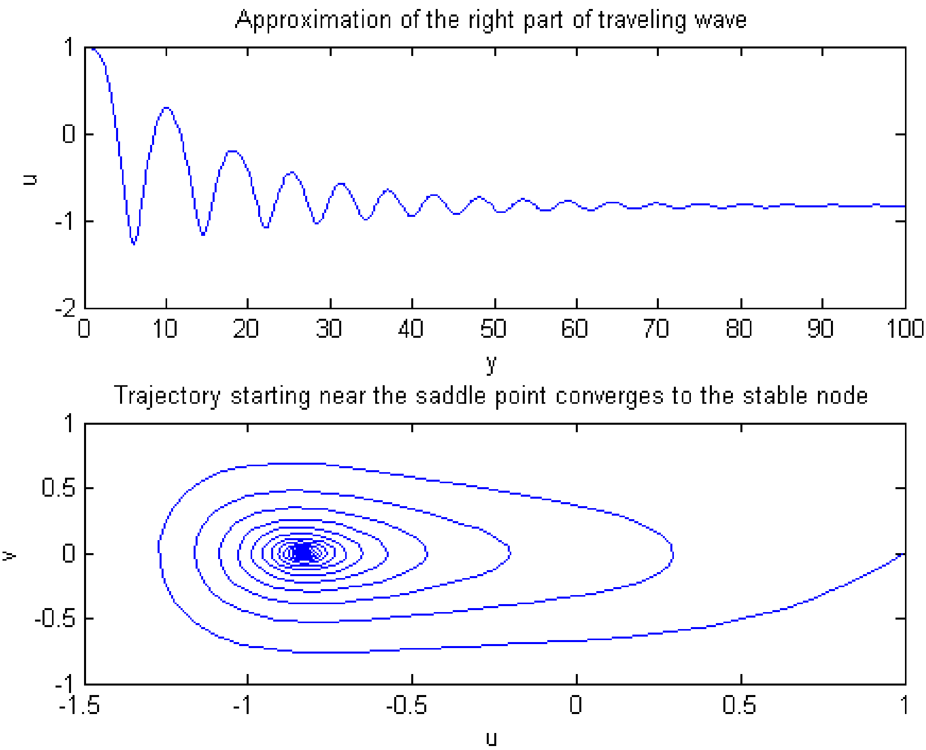

Figure 5. Illustration of the right part of the traveling wave in (y,u)-plane (above) and the curve $\mathscr{C}$ in the (u,v)-plane (below).

The trajectory $\mathscr{C}$ of (4.4), (4.6) and (4.7) starting at a point $(u_0,v_0) = (0.99,-0.001)$ near the saddle point $(1,0)$ and located inside the attraction domain converges to the attracting equilibrium. The right part of the traveling wave and the stable trajectory in the phase domain are shown by Figure 5. Here, we draw the trajectory for increasing time and we can see that for sufficiently large time, the trajectory approaches the attracting equilibrium.

Acknowledgement. The author would like to thank Professor P. G. LeFloch for many helpful suggestions.

References

[1] N. Bedjaoui, F. Coquel, C. Chalons and P.G. LeFloch, Non-monotonic traveling waves in van der Waals fluids, *Anal. Appl. (Singap.)* **3** (2005), no. 4, 419–446.

[2] N. Bedjaoui and P. G. Lefloch, Diffusive-dispersive traveling waves and kinetic relations. III. An hyperbolic model of elastodynamics, *Ann. Univ. Ferrara Sez. VII (N.S.)* **47** (2001), 117–144.

[3] N. Bedjaoui and P. G. LeFloch, Diffusive-dispersive traveling waves and kinetic relations. I. Nonconvex hyperbolic conservation laws, *J. Differential Equations* **178** (2002), no. 2, 574–607.

[4] N. Bedjaoui and P. G. LeFloch, Diffusive-dispersive travelling waves and kinetic relations. II. A hyperbolic-elliptic model of phase-transition dynamics, *Proc. Roy. Soc. Edinburgh Sect. A* **132** (2002), no. 3, 545–565.

[5] N. Bedjaoui and P. G. LeFloch, Diffusive-dispersive travelling waves and kinetic relations. V. Singular diffusion and nonlinear dispersion, *Proc. Roy. Soc. Edinburgh Sect. A* **134** (2004), no. 5, 815–843.

[6] J. L. Bona and M. E. Schonbek, Travelling-wave solutions to the Korteweg-de Vries-Burgers equation, *Proc. Roy. Soc. Edinburgh Sect. A* **101** (1985), no. 3–4, 207–226.

[7] J. Correia, P. G. LeFloch and Mai Duc Thanh, Hyperbolic systems of conservation laws with Lipschitz continuous flux-functions: the Riemann problem, *Bol. Soc. Brasil. Mat. (N.S.)* **32** (2001), no. 3, 271–301.

[8] H. Fan, A vanishing viscosity approach on the dynamics of phase transitions in van der Waals fluids, *J. Differential Equations* **103** (1993), no. 1, 179–204.

[9] H. Fan, Traveling waves, Riemann problems and computations of a model of the dynamics of liquid/vapor phase transitions, *J. Differential Equations* **150** (1998), no. 2, 385–437.

[10] R. Hagan and M. Slemrod, The viscosity-capillarity criterion for shocks and phase transitions, *Arch. Rational Mech. Anal.* **83** (1983), no. 4, 333–361.

[11] B. T. Hayes and P. G. LeFloch, Non-classical shocks and kinetic relations: Scalar conservation laws, *Arch. Rational Mech. Anal.* **139** (1997), no. 1, 1–56.

[12] B. T. Hayes and P. G. LeFloch, Nonclassical shocks and kinetic relations: strictly hyperbolic systems, *SIAM J. Math. Anal.* **31** (2000), no. 5, 941–991 (electronic).

[13] D. Jacobs, B. McKinney and M. Shearer, Travelling wave solutions of the modified Korteweg-de Vries-Burgers equation, *J. Differential Equations* **116** (1995), no. 2, 448–467.

[14] H. K. Khalil, *Nonlinear Systems*, Prentice Hall, New Jersey, 2002.

[15] D. Kröner, P. G. LeFloch and M.-D. Thanh, The minimum entropy principle for compressible fluid flows in a nozzle with discontinuous cross-section, *M2AN Math. Model. Numer. Anal.* **42** (2008), no. 3, 425–442.

[16] P. Lax, Shock waves and entropy, in *Contributions to Nonlinear Functional Analysis (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1971)*, 603–634, Academic Press, New York.

[17] P. G. LeFloch, An introduction to nonclassical shocks of systems of conservation laws, in *An Introduction to Recent Developments in Theory and Numerics for Conservation Laws (Freiburg/Littenweiler, 1997)*, 28–72, Lect. Notes Comput. Sci. Eng., 5 Springer, Berlin.

[18] P. G. LeFloch, *Hyperbolic Systems of Conservation Laws*, Lectures in Mathematics ETH Zürich, Birkhäuser, Basel, 2002.

[19] P. G. LeFloch and M. D. Thanh, Nonclassical Riemann solvers and kinetic relations. III. A nonconvex hyperbolic model for van der Waals fluids, *Electron. J. Differential Equations* **2000**, No. 72, 19 pp. (electronic).

[20] P. G. LeFloch and M. D. Thanh, Nonclassical Riemann solvers and kinetic relations. I. A nonconvex hyperbolic model of phase transitions, *Z. Angew. Math. Phys.* **52** (2001), no. 4, 597–619.

[21] P. G. LeFloch and M. D. Thanh, Non-classical Riemann solvers and kinetic relations. II. An hyperbolic-elliptic model of phase-transition dynamics, *Proc. Roy. Soc. Edinburgh Sect. A* **132** (2002), no. 1, 181–219.

[22] P. G. LeFloch and M. D. Thanh, Properties of Rankine-Hugoniot curves for van der Waals fluids, *Japan J. Indust. Appl. Math.* **20** (2003), no. 2, 211–238.

[23] P. G. LeFloch and Mai Duc Thanh, The Riemann problem for fluid flows in a nozzle with discontinuous cross-section, *Commun. Math. Sci.* **1** (2003), no. 4, 763–797.

[24] P. G. LeFloch and M. D. Thanh, The Riemann problem for the shallow water equations with discontinuous topography, *Commun. Math. Sci.* **5** (2007), no. 4, 865–885.

[25] T. P. Liu, The Riemann problem for general 2×2 conservation laws, *Trans. Amer. Math. Soc.* **199** (1974), 89–112.

[26] O. A. Oleĭnik, Construction of a generalized solution of the Cauchy problem for a quasi-linear equation of first order by the introduction of "vanishing viscosity", *Uspehi Mat. Nauk* **14** (1959), no. 2 (86), 159–164.

[27] M. Shearer and Y. Yang, The Riemann problem for a system of conservation laws of mixed type with a cubic nonlinearity, *Proc. Roy. Soc. Edinburgh Sect. A* **125** (1995), no. 4, 675–699.

[28] M. Slemrod, Admissibility criteria for propagating phase boundaries in a van der Waals fluid, *Arch. Rational Mech. Anal*. **81** (1983), no. 4, 301–315.

[29] M. Slemrod, A limiting "viscosity" approach to the Riemann problem for materials exhibiting change of phase, *Arch. Rational Mech. Anal.* **105** (1989), no. 4, 327–365.

[30] M. D. Thanh, The Riemann problem for a nonisentropic fluid in a nozzle with discontinuous cross-sectional area, *SIAM J. Appl. Math.* **69** (2009), no. 6, 1501–1519.

[31] M. D. Thanh, Global existence of diffusive-dispersive traveling waves for general flux functions, *Nonlinear Anal.* **72** (2010), no. 1, 231–239.

[32] M. D. Thanh, Attractor and traveling waves of a fluid with nonlinear diffusion and dispersion, *Nonlinear Anal.* **72** (2010), no. 6, 3136–3149.

[33] M. D. Thanh, Existence of traveling waves in elastodynamics with variable viscosity and capillarity, *Nonlinear Anal.* doi:10.1016/j.nonrwa.2010.06.010.