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The Radius of Convexity and a Sufficient Condition for Starlike Mappings

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Abstract. An estimate is provided for the radius of convexity of starlike mappings in the Euclidean unit ball B^n, and a sufficient condition is obtained for starlike mappings, which give the n-dimensional versions of the corresponding results of one complex variable.

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1. Introduction

The holomorphic functions of one complex variable which map the unit disk onto starlike or convex domains have been extensively studied. These functions are easily characterized by simple analytic or geometric conditions. In moving to higher dimensions, several difficulties arise. In the case of one complex variable, the following well known theorems had been established.

Theorem 1.1. [4] *Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is a holomorphic function on the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ in $\mathbb{C}$. If $\sum_{n=2}^{\infty} n|a_n| \le 1$, then f is a starlike function on the unit disk U in $\mathbb{C}$.*

Theorem 1.2. [4] *Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is a holomorphic function on the unit disk U in $\mathbb{C}$. If $\sum_{n=2}^{\infty} n^2|a_n| \le 1$, then f is a convex function in U.*

Theorem 1.3. [4] *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is a univalent starlike function on the unit disk U in $\mathbb{C}$, then $f(z)$ is convex in $|z| < r_0$ with $r_0 = 2 - \sqrt{3}$.*

In 1999, Roper and Suffridge [11] provided a sufficient condition for a normalized biholomorphic convex mapping on the open Euclidean unit ball in $\mathbb{C}^n$, which was the n-dimensional version of Theorem 1.2. After that, Zhu [15] provided a brief proof of the

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theorem of Roper and Suffridge. Liu and Zhu [7, 8] gave some sufficient conditions for a normalized biholomorphic convex mapping on some Reinhardt domains in $\mathbb{C}^n$.

The objective of this paper is to establish the n-dimensional versions for Theorem 1.1 and Theorem 1.3. Now we recall some definitions and notations.

Suppose that C^n is the space of n complex variables $z=(z_1,z_2,\cdots,z_n)$ with the usual inner product $\langle z,w\rangle=\sum_{j=1}^n z_j\overline{w_j}$ and the Euclidean norm $\|z\|=\sqrt{\langle z,z\rangle}$. Let $H(B^n)$ be the class of mappings $f(z)=(f_1(z),\cdots,f_n(z)), z=(z_1,\cdots,z_n)\in C^n$, which are holomorphic on the unit ball $B^n=\{z\in C^n:\|z\|<1\}$ with values in C^n. If $f\in H(B^n)$, we say that f is normalized if $f(0)=0$ and $Df(0)=I_n$, where I_n is the identity operator on C^n. A mapping $f\in H(B^n)$ is said to be locally biholomorphic on B^n if f has a local inverse at each point $z\in B^n$ or, equivalently, if the first Fréchet derivative

$$Df(z)=\left(\frac{\partial f_j(z)}{\partial z_k}\right)_{1\le j,k\le n}$$

is nonsingular at each point in B^n.

If $f\in H(B^n)$ is a normalized mapping, then $f(z)$ has the Taylor series expansion

$$f(z)=z+\sum_{k=2}^{\infty}A_k(z^k),\ z\in B^n,$$

where $A_k(z^k)=(1/k!)D^kf(0)(z^k)$, and $D^kf(0)(z^k)$ is the k-th order Fréchet derivative of f at $z=0$.

Let $H_m(B^n)$ denote the subclass of $H(B^n)$ consisting of mappings f, which are local biholomorphic and $f(z)=z+\sum_{k=m+1}^{\infty}(1/k!)D^kf(0)(z^k)$.

Let $S^*(B^n)$ and $K(B^n)$ be the subclasses of $H_1(B^n)$ consisting respectively of starlike and convex mappings on B^n. Then $f\in S^*(B^n)$ if and only if f is locally biholomorphic such that

$$\mathrm{Re}\langle Df(z)^{-1}f(z),z\rangle>0$$

for all $z\in B^n\backslash\{0\}$(see [6, Theorem 1]).

A mapping $f\in H_1(B^n)$(see, [1, 6]) is called starlike of order $\alpha\in(0,1)$ on B^n if

$$\left|\frac{1}{\|z\|^2}\langle Df(z)^{-1}f(z),z\rangle-\frac{1}{2\alpha}\right|<\frac{1}{2\alpha}\quad \text{for all}\quad z\in B^n\backslash\{0\}.$$

Let $S^*(\alpha,B^n)$ denote the class of starlike mappings of order α on B^n for $0<\alpha<1$ and let $S^*(0,B^n)\equiv S^*(B^n)$. It is evident that $S^*(\alpha,B^n)\subset S^*(B^n)$ for $0\le\alpha<1$. Let $S_m^*(\alpha,B^n)\equiv S^*(\alpha,B^n)\cap H_m(B^n)$ for $0\le\alpha<1$.

A mapping $f\in H(B^n)$ is a convex mapping on B^n, written $f\in K(B^n)$, if and only if for any $z=(z_1,\cdots,z_n)\in B^n$ and $b=(b_1,\cdots,b_n)\in C^n$ such that $\mathrm{Re}\langle b,z\rangle=0$, we have(see, [2])

$$\| b \|^2-\mathrm{Re}\langle Df(z)^{-1}D^2f(z)(b,b),z\rangle\ge 0. \tag{1.1}$$

Definition 1.1. *Let $\mathscr{F}$ be a nonempty subset of $H_1(B^n)$. The number $r_c(\mathscr{F})$, called the radius of convexity of $\mathscr{F}$, is the biggest positive number $r\in(0,1]$ such that each mapping $f(z)\in\mathscr{F}$ is convex on $B_r^n=\{z\in C^n:\|z\|<r\}$. The number $r^*(\mathscr{F})$, called the radius of starlikeness of $\mathscr{F}$, is the biggest positive number $r\in(0,1]$ such that each mapping $f(z)\in\mathscr{F}$ is starlike on $B_r^n=\{z\in C^n:\|z\|<r\}$.*

It is well known that the radius $r_c(H_1(B^n))$ of convexity of $H_1(B^n)$ doesn't exist when $n\ge 2$, and the radius of convexity of starlike mappings on the unit polydisc and four classes of classical domains also don't exist. However, there exist the radius $r_c(S^*(B^n))$ of convexity of starlike mappings on Euclidean unit ball B^n(see, [10, 12, 13]), it is conjectured in [13] that $r_c(S^*(B^n)) = 2-\sqrt{3}$.

In order to derive our main results, we need the following lemmas.

Lemma 1.1. [9] *Let* $\alpha \in [0,1)$ *and*

$$M_m(\alpha) = \begin{cases} \frac{(m+1)\sqrt{1-2\alpha}}{\sqrt{(m+1)^2+1-2\alpha}}, & 0 \le \alpha \le \frac{1}{m+3}, \\ \frac{(m+1)(1-\alpha)}{m+1+\alpha}, & \frac{1}{m+3} < \alpha < 1. \end{cases} \tag{1.2}$$

If $f \in H_m(B^n)$ *satisfies the following inequality*

$$\|Df(z) - I_n\| \le M_m(\alpha)$$

for all $z \in B^n$*, then* $f \in S_m^*(\alpha, B^n)$.

Lemma 1.2. [4] *Let* $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ *be a normalized starlike mapping on* B^n*. Then*

$$\|A_k\| \le \frac{e^2}{4}(k+1)^2,\ k \ge 2.$$

Lemma 1.3. [11] *Let* $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ *be a holomorphic mapping on* B^n*. If* $\sum_{k=2}^{\infty} k^2\|A_k\| \le 1$*, then* $f(z)$ *is a convex mapping on* B^n.

2. Main results

Theorem 2.1. *Let* $f(z) = z + \sum_{k=m+1}^{\infty} A_k(z^k)$ *be a holomorphic mapping on* B^n*. If*

$$\sum_{k=m+1}^{\infty} (k-\alpha)\|A_k\| \le N_m(\alpha), \tag{2.1}$$

where $N_m(\alpha)$ *is defined by*

$$N_m(\alpha) = \begin{cases} \frac{(m+1-\alpha)\sqrt{1-2\alpha}}{\sqrt{(m+1)^2+1-2\alpha}}, & 0 \le \alpha \le \frac{1}{m+3}, \\ \frac{(m+1-\alpha)(1-\alpha)}{m+1+\alpha}, & \frac{1}{m+3} < \alpha < 1, \end{cases} \tag{2.2}$$

then $f \in S_m^*(\alpha, B^n)$.

Proof. Let $q(z) = f(z) - z$. Then $q(z) = \sum_{k=m+1}^{\infty} A_k(z^k) \in H(B^n)$ and

$$Dq(z) = \sum_{k=m+1}^{\infty} kA_k(z^{k-1}, \cdot), \tag{2.3}$$

it follows from (2.1), (2.2) and the above equation that

$$\begin{aligned} \|Df(z) - I_n\| = \|Dq(z)\| &= \sup_{\|u\|\le 1}\{\|Dq(z)(u)\|\} \\ &= \sup_{\|u\|\le 1}\Big\|\sum_{k=m+1}^{\infty} kA_k(z^{k-1}, u)\| \\ &\le \sum_{k=m+1}^{\infty} k\|A_k\| \end{aligned} \tag{2.4}$$

$$\leq \frac{m+1}{m+1-\alpha}\sum_{k=m+1}^{\infty}(k-\alpha)\|A_k\|$$
$$\leq \frac{m+1}{m+1-\alpha}N_m(\alpha) = M_m(\alpha) < 1,$$

where $M_m(\alpha)$ is defined by (1.2). This implies that $Df(z) = I_n - (I_n - Df(z))$ is an invertible linear operator (see[14, p.192]), thus $f \in H_m(B^n)$. Hence by Lemma 1.1, we obtain that $f \in S_m^*(\alpha, B^n)$, and the proof is complete. ∎

Setting $\alpha = 0, m = 1$ in Theorem 2.1, we have the following corollary.

Corollary 2.1. *Let $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ be a holomorphic mapping on B^n. If*

$$\sum_{k=2}^{\infty} k\|A_k\| \leq \frac{2}{\sqrt{5}} \approx 0.89443, \tag{2.5}$$

then $f(z)$ is a starlike mapping on B^n.

Remark 2.1. It is evident that Corollary 2.1 is the n-dimensional version of Theorem 1.1.

Theorem 2.2. *Suppose that $\alpha \in (0,1)$. Let $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ be holomorphic mapping on B^n. If*

$$\sum_{k=2}^{\infty}(k-\alpha)\|A_k\| \leq \frac{1-|1-2\alpha|}{2}, \tag{2.6}$$

then $f \in S^*(\alpha, B^n)$.

Proof. First, it follows from (2.6) that

$$\sum_{k=2}^{\infty} k\|A_k\| \leq \frac{2}{2-\alpha}\sum_{k=2}^{\infty}(k-\alpha)\|A_k\|$$
$$\leq \frac{1-|1-2\alpha|}{2-\alpha} < 1.$$

Next, by (2.3), we obtain

$$\|I_n - Df(z)\| = \left\| -\sum_{k=2}^{\infty} kA_k(z^{k-1}, \cdot)\right\| \leq \sum_{k=2}^{\infty} k\|A_k\| < 1, z \in B^n,$$

which implies that $Df(z) = I_n - (I_n - Df(z))$ is an invertible linear operator (see[14, p.192]), and

$$\|Df(z)^{-1}\| \leq \frac{1}{1-\|I_n - Df(z)\|} \leq \frac{1}{1-\sum_{k=2}^{\infty} k\|A_k\|}, \quad z \in B^n.$$

For $z \in B^n \backslash \{0\}$, from (2.6), we obtain

$$\left|\frac{2\alpha}{\|z\|^2}\langle Df(z)^{-1}f(z), z\rangle - 1\right| = \left|\frac{1}{\|z\|^2}\langle Df(z)^{-1}(2\alpha f(z) - Df(z)(z)), z\rangle\right|$$
$$\leq \frac{1}{\|z\|}\|Df(z)^{-1}\|\|2\alpha f(z) - Df(z)(z)\|$$
$$\leq \frac{1}{\|z\|} \cdot \frac{|1-2\alpha| \cdot \|z\| + \sum_{k=2}^{\infty}(k-2\alpha)\|A_k\|\|z\|^k}{1-\sum_{k=2}^{\infty} k\|A_k\|}$$
$$< \frac{|1-2\alpha| + \sum_{k=2}^{\infty}(k-2\alpha)\|A_k\|}{1-\sum_{k=2}^{\infty} k\|A_k\|} \leq 1,$$

it follows that $f \in S^*(\alpha, B^n)$. This completes the proof. ∎

Notice that

$$\frac{(2-\alpha)(1-\alpha)}{2+\alpha} \geq \alpha \Leftrightarrow \alpha \leq \frac{2}{5},$$

by means of Theorems 2.1 and 2.2, we obtain the following corollary.

Corollary 2.2. *Suppose that $\alpha \in [0,1)$. Let $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ be a holomorphic mapping on B^n. If*

$$\sum_{k=2}^{\infty} (k-\alpha)\|A_k\| \leq B(\alpha),$$

where $B(\alpha)$ is defined by

$$B(\alpha) = \begin{cases} \frac{(2-\alpha)\sqrt{1-2\alpha}}{\sqrt{5-2\alpha}}, & 0 \leq \alpha \leq \frac{1}{4}, \\ \frac{(2-\alpha)(1-\alpha)}{2+\alpha}, & \frac{1}{4} < \alpha \leq \frac{2}{5}, \\ \alpha, & \frac{2}{5} < \alpha < \frac{1}{2}, \\ 1-\alpha, & \frac{1}{2} \leq \alpha < 1, \end{cases} \tag{2.7}$$

then $f \in S^*(\alpha, B^n)$.

Remark 2.2. Setting $\alpha = 1/2$ in Theorem 2.2 or Corollary 2.2, we get Theorem 2.4 in [3].

Theorem 2.3. *Suppose that $n \geq 2$. If $f \in S^*(B^n)$, then f is convex on $B^n_{r_1}$, where $r_1 \approx 0.012$ is the root in $(0,1)$ of the equation*

$$e^2\left[\frac{r^2+4r+1}{(1-r)^5} - 1\right] = 1. \tag{2.8}$$

Hence $r_c(S^*(B^n)) \in \left[r_1, \sqrt{3}/9\right]$.

Proof. Since $f \in S^*(B^n)$, we have

$$f(z) = z + \sum_{k=2}^{\infty} A_k(z^k),$$

and by Lemma 1.2, we obtain that

$$\|A_k\| \leq \frac{e^2}{4}(k+1)^2,\ k \geq 2. \tag{2.9}$$

Let $F(z) = (1/r)f(rz) = z + \sum_{k=2}^{\infty} \beta_k(z^k)$ for $z \in \overline{B}^n$ and $r \in (0,1)$, then $\beta_k = r^{k-1}A_k, k \geq 2$. In view of (2.9), direct computation yields that

$$\sum_{k=2}^{\infty} k^2\|\beta_k\| \leq \sum_{k=2}^{\infty} k^2 \frac{e^2}{4}(k+1)^2 \cdot r^{k-1} = e^2\left[\frac{r^2+4r+1}{(1-r)^5} - 1\right] \leq 1$$

for $r \in (0, r_1]$, where r_1 is the root in $(0,1)$ of the equation (2.8). Hence by Lemma 1.3, we obtain that F is convex on B^n, thus f is convex on $B^n_{r_1}$, and $r_c(S^*(B^n)) \geq r_1$.

Finally, we verify that $r_c(S^*(B^n)) \leq \sqrt{3}/9$. To this end, let

$$f_0(z) = \left(z_1 + \frac{3\sqrt{3}}{2} z_n^2, z_2, \cdots, z_n, \right)$$

it is easy to prove that $f_0(z) \in S^*(B^n)$ (refer to the proof of Example 5 in [11]).

Set

$$F_0(z) = \frac{1}{r} f_0(rz) = \left(z_1 + \frac{3\sqrt{3}}{2} r z_n^2, z_2, \cdots, z_n \right)$$

for $z \in \overline{B}^n$ and $r \in (0,1)$. Straightforward computations yield

$$(DF_0(z))^{-1} D^2 F_0(z)(b,b) = \left(3\sqrt{3} r \overline{z_1} b_n^2, 0, \cdots, 0 \right),\ z \in B^n,\ b = (b_1, b_2, \cdots, b_n) \in B^n.$$

Therefore for $0 < r \leq \sqrt{3}/9$, we obtain

$$\begin{aligned} \|b\|^2 - \mathrm{Re}\langle (DF_0(z))^{-1} D^2 F_0(z)(b,b), z \rangle &= \|b\|^2 - \mathrm{Re}(3\sqrt{3} r \overline{z_1} b_n^2) \\ &\geq \|b\|^2 - 3\sqrt{3} r |z_1| |b_n|^2 \\ &\geq \|b\|^2 (1 - 3\sqrt{3} r) \geq 0 \end{aligned}$$

for all $z = (z_1, z_2, \cdots, z_n) \in B^n$, $b \in C^n$ and $\mathrm{Re}\langle z, b \rangle = 0$. Hence in view of (1.1) we conclude that $F_0(z)$ is convex in B^n for $0 < r \leq \sqrt{3}/9$.

If $r > \sqrt{3}/9$, we may find $z_1 \in U$ such that $\overline{z_1} r > \sqrt{3}/9$. Moreover, if $z = (z_1, 0, \cdots, 0), b = (0, \cdots, 0, 1)$, then $\mathrm{Re}\langle z, b \rangle = 0$ and

$$\|b\|^2 - \mathrm{Re}\langle (DF_0(z))^{-1} D^2 F_0(z)(b,b), z \rangle = 1 - 3\sqrt{3} r \overline{z_1} < 0.$$

So $F_0(z)$ is convex in B^n if and only if $0 < r \leq \sqrt{3}/9$, and thus $f_0(z)$ is convex in B_r^n if and only if $0 < r \leq \sqrt{3}/9$. Hence $r_c(S^*(B^n)) \leq \sqrt{3}/9$, and the proof is complete. ∎

Remark 2.3. From Theorem 2.3, we know that $r_c(S^*(B^n)) \leq \sqrt{3}/9 \approx 0.19245 < 2 - \sqrt{3}$, and the conjecture proposed in [13] that $r_c(S^*(B^n)) = 2 - \sqrt{3}$ is not right when $n \geq 2$ (see also [5, Example 3.5]).

Now we consider the radius problems associated with the following sets (see[3]):

$$\mathscr{F}_n = \left\{ f(z) = z + \sum_{k=2}^{\infty} A_k(z^k) \in H(B^n) : \|A_k\| \leq 1 \right\}$$

and

$$\mathscr{G}_n = \left\{ f(z) = z + \sum_{k=2}^{\infty} A_k(z^k) \in H(B^n) : \|A_k\| \leq k \right\}.$$

In [3], the following result was obtained.

Theorem 2.4. [3]

(1) *If $f \in \mathscr{G}_n$, then f is starlike on $B^n_{r_1(\mathscr{G}_n)}$ and $r^*(\mathscr{G}_n) \in [r_1(\mathscr{G}_n), r_u(\mathscr{G}_n)]$, where $r_1(\mathscr{G}_n) =$* 0.12038... *is the unique solution in* $(0,1)$ *of the equation*

$$2r^3 - 6r^2 + 9r - 1 = 0,$$

and $r_u(\mathscr{G}_n) = 0.16487...$ is the unique solution in $(0,1)$ of the equation

$$1 + r = 2(1 - r)^3.$$

(2) *If $f \in \mathscr{F}_n$, then f is starlike on $B^n_{r_2(\mathscr{F}_n)}$ and $r^*(\mathscr{F}_n) \in [r_2(\mathscr{F}_n), r_u(\mathscr{F}_n)]$, where $r_u(\mathscr{F}_n) = 1 - 1/\sqrt{2} = 0.29289...$ and $r_2(\mathscr{F}_n) = 0.21922...$ is the unique solution in $(0,1)$ of the equation*

$$2r^2 - 5r + 1 = 0.$$

Theorem 2.5. *Let $\alpha \in [0,1)$ and $B(\alpha)$ is defined by (2.7).*

(1) *If $f \in \mathscr{G}_n$, then f is starlike of order α on $B^n_{r_3(\mathscr{G}_n,\alpha)}$, where $r_3(\mathscr{G}_n,\alpha)$ is the unique solution in (0, 1) of the equation*

$$(B(\alpha)+1-\alpha)r^3 - 3(B(\alpha)+1-\alpha)r^2 + (3B(\alpha)+4-2\alpha)r - B(\alpha) = 0. \tag{2.10}$$

(2) *If $f \in \mathscr{F}_n$, then f is starlike of order α on $B^n_{r_4(\mathscr{F}_n,\alpha)}$, where*

$$r_4(\mathscr{F}_n,\alpha) = 1 + \frac{\alpha - \sqrt{(\alpha-2)^2 + 4B(\alpha)}}{2(B(\alpha)-\alpha+1)}. \tag{2.11}$$

Proof. (1) If $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k) \in \mathscr{G}_n$, let $h(z) = f(rz)/r$ for $z \in B^n, r \in (0,1)$, then we have $h(z) = z + \sum_{k=2}^{\infty} r^{k-1} A_k(z^k)$. Since

$$\sum_{k=2}^{\infty} (k-\alpha) r^{k-1} \|A_k\| \le \sum_{k=2}^{\infty} k(k-\alpha) r^{k-1} = \frac{1+r}{(1-r)^3} - 1 - \alpha \frac{2r-r^2}{(1-r)^2} \le B(\alpha)$$

if and only if

$$(B(\alpha)+1-\alpha)r^3 - 3(B(\alpha)+1-\alpha)r^2 + (3B(\alpha)+4-2\alpha)r - B(\alpha) \le 0.$$

From Corollary 2.2, we obtain $h \in S^*(\alpha, B^n)$, which implies that f is starlike of order α on $B^n_{r_3(\mathscr{G}_n,\alpha)}$.

(2) If $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k) \in \mathscr{F}_n$, also let $h(z) = f(rz)/r$ for $z \in B^n, r \in (0,1)$, then we have $h(z) = z + \sum_{k=2}^{\infty} r^{k-1} A_k(z^k)$. Since

$$\sum_{k=2}^{\infty} (k-\alpha) r^{k-1} \|A_k\| \le \sum_{k=2}^{\infty} (k-\alpha) r^{k-1} = \frac{2r-r^2}{(1-r)^2} - \alpha \frac{r}{1-r} \le B(\alpha)$$

if and only if

$$(B(\alpha)-\alpha+1)r^2 - (2B(\alpha)+2-\alpha)r + B(\alpha) \ge 0.$$

Since $(B(\alpha)-\alpha+1)r^2 - (2B(\alpha)+2-\alpha)r + B(\alpha) \ge 0$ for $0 < r \le r_4(\mathscr{F}_n,\alpha)$, by Corollary 2.2, we obtain $h \in S^*(\alpha, B^n)$, which implies that f is starlike of order α on $B^n_{r_4(\mathscr{F}_n,\alpha)}$. This completes the proof. ∎

Remark 2.4. Let $\alpha = 0$ in Theorem 2.5, by a simple calculation, we have $r_3(\mathscr{G}_n, 0) = 0.1526...$ and $r_4(\mathscr{F}_n, 0) = 0.27345...$, which improves Theorem 2.4 or Theorem 4.2 in [3]. Hence

$$r_3(\mathscr{G}_n, 0) = 0.1526... \le r^*(\mathscr{G}_n) \le r_u(\mathscr{G}_n) = 0.16487...$$

and

$$r_4(\mathscr{F}_n, 0) = 1 - \sqrt{5 - 2\sqrt{5}} = 0.27345... \le r^*(\mathscr{F}_n) \le r_u(\mathscr{F}_n) = 1 - 1/\sqrt{2} = 0.29289....$$

3. Sufficient condition in complex Hilbert spaces

Suppose that X is a complex Hilbert space with product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$, and $B = \{z \in X : \|z\| < 1\}$ is the unit ball in X.

A mapping f is in $S_m^*(\alpha, B)$ if and only if $f(z) = z + \sum_{k=m+1}^{+\infty} A_k(z^k)$ is a locally biholomorphic mapping on B and satisfies the following inequalities

$$\left|\frac{1}{\|z\|^2}\langle Df(z)^{-1}f(z), z\rangle - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}, \quad z \in B\backslash\{0\}$$

for $0 < \alpha < 1$ and

$$\mathrm{Re}\langle Df(z)^{-1}f(z), z\rangle > 0, \quad z \in B\backslash\{0\}$$

for $\alpha = 0$. We call the biholomorphic mapping $f \in S_m^*(\alpha, B)$ starlike of order α.

Lemma 3.1. [9] *Let $\alpha \in [0,1)$ and $M_m(\alpha)$ is given by (1.2). If $f(z) = z + \sum_{k=m+1}^{+\infty} A_k(z^k)$ is a locally biholomorphic mapping on B and satisfies the following inequality*

$$\|Df(z) - I\| \leq M_m(\alpha)$$

for all $z \in B$, where I is the identity operator on X, then $f \in S_m^*(\alpha, B)$.

With the aid of Lemma 3.1, applying the similar method of Theorems 2.1 and 2.2, we may obtain the following theorem.

Theorem 3.1. *Let $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ be a holomorphic mapping on B. If*

$$\sum_{k=2}^{\infty}(k-\alpha)\|A_k\| \leq B(\alpha), \tag{3.1}$$

where $B(\alpha)$ is defined by (2.7), then $f \in S^*(\alpha, B)$.

Remark 3.1. Setting $\alpha = 1/2$ in Theorem 3.1, we get Theorem 5.3 in [3].

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