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Multivalent Harmonic Functions Defined by Dziok-Srivastava Operator

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Abstract. Necessary and sufficient coefficient bounds and convolution condition for certain multivalent harmonic functions whose convolution with generalized hypergeometric functions is starlike of order γ are investigated. Results on extreme points, convex combination and distortion bounds using the coefficient condition are also obtained.

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1. Introduction

A complex-valued continuous function $f = u + iv$ in a complex domain $E \subset \mathbb{C}$ is said to be *harmonic* if both u and v are real harmonic in E. There is an interrelation between harmonic functions and analytic functions. If E is a simply connected domain, then $f = h + \bar{g}$ where h and g are analytic in E; the functions h and g are respectively called the analytic part and co-analytic part of f. The function $f = h + \bar{g}$ is said to be harmonic univalent in E if the mapping $z \to f(z)$ is orientation preserving, harmonic and univalent in E. This mapping is orientation preserving and locally univalent in E if and only if the Jacobian J_f of f given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$ is positive in E [16]. From the perspective of geometric functions theory, Clunie and Sheil-Small [10] initiated the study on these functions by introducing the class S_H consisting of normalized complex-valued harmonic univalent functions f defined on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. They gave necessary and sufficient conditions for f to be locally univalent and sense-preserving in $\mathbb{D}$. Coefficient bounds for functions in S_H were obtained. Since then, various subclasses of S_H were investigated by several authors [1, 5, 8, 9, 15, 19, 20, 21]. Note that the class S_H reduces to the class of normalized analytic univalent functions if the co-analytic part of f is identically to zero ($g \equiv 0$).

Multivalent harmonic functions in $\mathbb{D}$ were introduced by Duren, Hengartner and Laugesen [11] via the argument principle. In [2], the class of multivalent harmonic functions and the class $S_H^*(p,\gamma)$ of multivalent harmonic starlike functions of order γ, where $p \geq 1$,

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$0 \le \gamma < 1$ were discussed and studied. Motivated by [4], we introduce a class of multivalent harmonic functions starlike of order γ using the Dziok-Srivastava operator. Several related work using other linear operators can also be found in [3, 14, 26, 24].

Recall that the convolution of two analytic functions $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\psi(z) = \sum_{n=0}^{\infty} b_n z^n$ defined on $\mathbb{D}$ is the analytic function given by $\varphi(z) * \psi(z) = \sum_{n=0}^{\infty} a_n b_n z^n = \psi(z) * \varphi(z)$. Let $S_H(p)$ denote the class of multivalent harmonic functions $f = h + \bar{g}$ where

$$h(z) = z^p + \sum_{n=2}^{\infty} a_{n+p-1} z^{n+p-1}, \quad g(z) = \sum_{n=1}^{\infty} b_{n+p-1} z^{n+p-1} . \tag{1.1}$$

For $\alpha_i \in \mathbb{C}$ $(i = 1, 2, \dots, l)$ and $\beta_j \in \mathbb{C} \backslash \{0, -1, -2, \dots\}$ $(j = 1, 2, \dots, m)$, the generalized hypergeometric function ${}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z)$ is given by

$$ {}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_l)_n}{(\beta_1)_n \dots (\beta_m)_n n!} z^n $$
$$ (l \le m+1;\ l, m \in N_0 := N \cup \{0\}; z \in D) $$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of gamma function, by

$$ (\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1, & n = 0,\ \lambda \neq 0 \\ \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1), & n = 1, 2, 3, \dots \end{cases} $$

For an analytic function h of the form (1.1), Dziok and Srivastava [12] introduced the linear operator

$$ H_p^{l,m}[\alpha_1] h(z) = z^p \; {}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) * h(z) $$

which includes well known operators such as the Hohlov operator [13], Carlson-Shaffer operator [7], Ruscheweyh derivative operator [22], the generalized Bernardi-Libera-Livington integral operator [6], [17], [18] and the Srivastava-Owa fractional derivative operator [25]. For a harmonic function $f = h + \bar{g}$, with h and g given by (1.1), the Dziok-Srivastava operator is defined by

$$ H_p^{l,m}[\alpha_1] f(z) = H_p^{l,m}[\alpha_1] h(z) + \overline{H_p^{l,m}[\alpha_1] g(z)}, $$

where $H_p^{l,m}[\alpha_1] h(z) = z^p + \sum_{n=2}^{\infty} \phi_n a_{n+p-1} z^{n+p-1}$, $H_p^{l,m}[\alpha_1] g(z) = \sum_{n=1}^{\infty} \phi_n b_{n+p-1} z^{n+p-1}$ and

$$ \phi_n = \frac{(\alpha_1)_{n-1} \cdots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_m)_{n-1} (n-1)!}, \tag{1.2} $$

$\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_m$ are positive real numbers such that $l \le m+1$.

Denote by $S_H^*(p, \alpha_1, \gamma)$, the class of multivalent harmonic functions satisfying

$$ \operatorname{Re} \left(\frac{z \left(H_p^{l,m}[\alpha_1] h(z) \right)' - \overline{z \left(H_p^{l,m}[\alpha_1] g(z) \right)'}}{\left(H_p^{l,m}[\alpha_1] h(z) \right) + \overline{\left(H_p^{l,m}[\alpha_1] g(z) \right)}} \right) \ge p\gamma \tag{1.3} $$

for $p \ge 1, 0 \le \gamma < 1, |z| = r < 1$. Note that $S_H^*(1, \alpha_1, \gamma) \equiv S_H^*(\alpha_1, \gamma)$ is the class defined in [4]. In the case of $l = m+1$ and $\alpha_2 = \beta_1, \dots, \alpha_l = \beta_m$, $S_H^*(p, 1, \gamma) \equiv S_H^*(p, \gamma)$ is investigated in [2] and $S_H^*(1, 1, \gamma) \equiv S_H^*(\gamma)$ is the class introduced by Jahangiri [15]. Further

$T_H^*(p,\alpha_1,\gamma),\ p \geq 1$ denotes the class of functions $f = h + \bar{g} \in S_H^*(p,\alpha_1,\gamma)$ where h and g are functions of the form

$$h(z) = z^p - \sum_{n=2}^{\infty} |a_{n+p-1}| z^{n+p-1} \ , \ g(z) = \sum_{n=1}^{\infty} |b_{n+p-1}| z^{n+p-1} \ . \tag{1.4}$$

2. Main results

Necessary coefficient conditions for the harmonic starlike functions and harmonic convex functions can be found in [10] and [23]. Now we derive sufficient coefficient bound for the class $S_H^*(p,\alpha_1,\gamma)$.

Theorem 2.1. *Let $f = h + \bar{g}$ be given by* (1.1) *and $\prod_{i=1}^{l} (\alpha_i)_{n-1} \geq \prod_{j=1}^{m} (\beta_j)_{n-1} (n-1)!$. If*

$$\sum_{n=2}^{\infty} \left(\frac{n + p\,(1-\gamma) - 1}{p\,(1-\gamma)} |a_{n+p-1}| + \frac{n + p\,(1+\gamma) - 1}{p\,(1-\gamma)} |b_{n+p-1}| \right) |\phi_n| \leq 1 - \frac{1+\gamma}{1-\gamma} |b_p| \tag{2.1}$$

where $|b_p| < (1-\gamma)/(1+\gamma)$, $0 \leq \gamma < 1$ and ϕ_n is given by (1.2)*, then the harmonic function f is orientation preserving in $\mathbb{D}$ and $f \in S_H^*(p,\alpha_1,\gamma)$.*

Proof. The inequality $|h'(z)| \geq |g'(z)|$ is enough to show that f is orientation preserving. Note that

$$\begin{aligned}
|h'(z)| &\geq p\,|z|^{p-1} - \sum_{n=2}^{\infty} (n+p-1) |a_{n+p-1}| |z|^{n+p-2} \\
&= p|z|^{p-1} \left(1 - \sum_{n=2}^{\infty} \frac{(n+p-1)}{p} |a_{n+p-1}| |z|^{n-1} \right) \\
&\geq p|z|^{p-1} \left(1 - \sum_{n=2}^{\infty} \frac{(n+p-1)}{p} |a_{n+p-1}| \right) \\
&\geq |z|^{p-1} \left(1 - \sum_{n=2}^{\infty} \frac{(n + p\,(1-\gamma) - 1)}{p\,(1-\gamma)} |\phi_n| |a_{n+p-1}| \right)
\end{aligned}$$

By hypothesis, since $|\phi_n| \geq 1$ and by (2.1),

$$\begin{aligned}
|h'(z)| &\geq |z|^{p-1} \left(\frac{1+\gamma}{1-\gamma} |b_p| + \sum_{n=2}^{\infty} \frac{(n + p\,(1+\gamma) - 1)}{p\,(1-\gamma)} |\phi_n| |b_{n+p-1}| \right) \\
&= |z|^{p-1} \left(\sum_{n=1}^{\infty} \frac{(n + p\,(1+\gamma) - 1)}{p\,(1-\gamma)} |\phi_n| |b_{n+p-1}| \right) \\
&\geq |z|^{p-1} \left(\sum_{n=1}^{\infty} (n+p-1) |b_{n+p-1}| \right) \\
&\geq |z|^{p-1} \left(\sum_{n=1}^{\infty} (n+p-1) |b_{n+p-1}| |z|^{n-1} \right) \\
&= \sum_{n=1}^{\infty} (n+p-1) |b_{n+p-1}| |z|^{n+p-2} = |g'(z)|
\end{aligned}$$

Thus, f is orientation preserving in $\mathbb{D}$.

Next, we prove $f \in S_H^*(p, \alpha_1, \gamma)$ by establishing condition (1.3). First, let

$$w(z) = \frac{z\left(H_p^{l,m}[\alpha_1]h(z)\right)' - \overline{z\left(H_p^{l,m}[\alpha_1]g(z)\right)'}}{\left(H_p^{l,m}[\alpha_1]h(z)\right) + \overline{\left(H_p^{l,m}[\alpha_1]g(z)\right)}} = \frac{A(z)}{B(z)},$$

where

$$A(z) = z\left(H_p^{l,m}[\alpha_1]h(z)\right)' - \overline{z\left(H_p^{l,m}[\alpha_1]g(z)\right)'}, \quad B(z) = \left(H_p^{l,m}[\alpha_1]h(z)\right) + \overline{\left(H_p^{l,m}[\alpha_1]g(z)\right)}.$$

Now,

$$\begin{aligned}
&|A(z) + p\,(1-\gamma)B(z)| - |A(z) - p\,(1+\gamma)B(z)| \\
&\geq (2p - p\gamma)|z^p| - \sum_{n=2}^{\infty}(n+2p-p\gamma-1)|\phi_n a_{n+p-1}z^{n+p-1}| \\
&\quad - \sum_{n=1}^{\infty}(n+p\gamma-1)\overline{|\phi_n b_{n+p-1}z^{n+p-1}|} - p\gamma|z^p| \\
&\quad - \sum_{n=2}^{\infty}(n-p\gamma-1)|\phi_n a_{n+p-1}z^{n+p-1}| - \sum_{n=1}^{\infty}(n+2p+p\gamma-1)\overline{|\phi_n b_{n+p-1}z^{n+p-1}|} \\
&= 2p\,(1-\gamma)|z^p| - \sum_{n=2}^{\infty}(2n+2p-2p\gamma-2)|\phi_n||a_{n+p-1}||z^{n+p-1}| \\
&\quad - \sum_{n=1}^{\infty}(2n+2p+2p\gamma-2)|\overline{\phi}_n||\overline{b}_{n+p-1}||\overline{z}^{n+p-1}| \\
&= 2p\,(1-\gamma)|z^p|\left(1 - \sum_{n=2}^{\infty}\frac{(n+p-p\gamma-1)}{p\,(1-\gamma)}|\phi_n||a_{n+p-1}||z^{n-1}|\right. \\
&\quad \left. - \sum_{n=1}^{\infty}\frac{(n+p+p\gamma-1)}{p\,(1-\gamma)}|\phi_n||b_{n+p-1}||z^{n-1}|\right) \\
&\geq 2p\,(1-\gamma)|z^p|\left(1 - \sum_{n=2}^{\infty}\frac{(n+p-p\gamma-1)}{p\,(1-\gamma)}|\phi_n||a_{n+p-1}|\right. \\
&\quad \left. - \sum_{n=1}^{\infty}\frac{(n+p+p\gamma-1)}{p\,(1-\gamma)}|\phi_n||b_{n+p-1}|\right) \\
&= 2p\,(1-\gamma)|z^p|\left(1 - \frac{1+\gamma}{1-\gamma}|b_p| - \left(\sum_{n=2}^{\infty}\left[\frac{(n+p-p\gamma-1)}{p\,(1-\gamma)}|a_{n+p-1}|\right.\right.\right. \\
&\quad \left.\left.\left. + \frac{(n+p+p\gamma-1)}{p\,(1-\gamma)}|b_{n+p-1}|\right]|\phi_n|\right)\right)
\end{aligned}$$

The last expression is non-negative by (2.1). Since Re $w \geq p\gamma$ if and only if $|A(z) + p\,(1-\gamma)B(z)| \geq |A(z) - p\,(1+\gamma)B(z)|$, $f \in S_H^*(p, \alpha_1, \gamma)$.

For $\sum_{n=1}^{\infty}(|x_{n+p-1}| + |\bar{y}_{n+p-1}|) = 1$ and $x_p = 0$, the function

(2.2)

$$f_1(z) = z^p + \sum_{n=2}^{\infty}\frac{p\,(1-\gamma)}{[n+p(1-\gamma)-1]|\phi_n|}x_{n+p-1}z^{n+p-1} + \sum_{n=1}^{\infty}\frac{p(1-\gamma)}{[n+p(1+\gamma)-1]|\phi_n|}\bar{y}_{n+p-1}\bar{z}^{n+p-1}$$

shows equality in the coefficient bound given by (2.1). For the function f_1 defined in (2.2), the coefficients are

$$a_{n+p-1} = \frac{p\,(1-\gamma)}{[n+p(1-\gamma)-1]|\phi_n|}x_{n+p-1} \quad \text{and} \quad b_{n+p-1} = \frac{p(1-\gamma)}{[n+p(1+\gamma)-1]|\phi_n|}\bar{y}_{n+p-1},$$

and since condition (2.1) holds, this implies $f_1 \in S_H^*(p,\alpha_1,\gamma)$.

To show that the converse need not be true, consider the function

$$f(z) = z^p + \frac{p(1-\gamma)}{[1+p(1-\gamma)]\phi_2}z^{p+1} + \frac{\gamma-1}{2(1+\gamma)}\bar{z}^p.$$

It can be shown that

$$\operatorname{Re}\left(\frac{z\left[z^p + \frac{p(1-\gamma)}{[1+p(1-\gamma)]}z^{p+1}\right]' - \bar{z}\left[\frac{(\gamma-1)}{2(1+\gamma)}\bar{z}^p\right]'}{z^p + \frac{p(1-\gamma)}{[1+p(1-\gamma)]}z^{p+1} + \frac{(\gamma-1)}{2(1+\gamma)}\bar{z}^p}\right) \geq p\gamma, \quad (p \geq 1, 0 \leq \gamma < 1)$$

thus $f \in S_p^*(p,\alpha_1,\gamma)$ but

$$\begin{aligned}&\sum_{n=2}^{\infty}\frac{n+p(1-\gamma)-1}{p(1-\gamma)}|a_{n+p-1}||\phi_n| + \sum_{n=1}^{\infty}\frac{n+p(1+\gamma)-1}{p(1-\gamma)}|b_{n+p-1}||\phi_n| \\ &= \frac{1+p(1-\gamma)}{p(1-\gamma)}\left|\frac{p(1-\gamma)}{[1+p(1-\gamma)]\phi_2}\right||\phi_2| + \frac{1+\gamma}{1-\gamma}\left|\frac{\gamma-1}{2(1+\gamma)}\right| > 1.\end{aligned}$$

∎

The next result provide a convolution condition for f to be in the class $S_H^*(p,\alpha_1,\gamma)$.

Theorem 2.2. *$f \in S_H^*(p,\alpha_1,\gamma)$ if and only if*

$$\begin{aligned}&H_p^{l,m}[\alpha_1]h(z) * \left[\frac{2p(1-\gamma)z^p + (\xi-2p+2p\gamma+1)z^{p+1}}{(1-z)^2}\right] \\ &- \overline{H_p^{l,m}[\alpha_1]g(z)} * \left[\frac{2p(\xi+\gamma)\bar{z}^p + (\xi-2p\xi-2p\gamma+1)\bar{z}^{p+1}}{(1-\bar{z})^2}\right] \neq 0, \quad |\xi| = 1, z \in D.\end{aligned}$$

Proof. A necessary and sufficient condition for $f \in S_H^*(p,\alpha_1,\gamma)$ is given by (1.3) and we have

$$\operatorname{Re}\left(\frac{1}{p(1-\gamma)}\left[\frac{z\left(H_p^{l,m}[\alpha_1]h(z)\right)' - \overline{z\left(H_p^{l,m}[\alpha_1]g(z)\right)'}}{\left(H_p^{l,m}[\alpha_1]h(z)\right) + \overline{\left(H_p^{l,m}[\alpha_1]g(z)\right)}} - p\gamma\right]\right) \geq 0.$$

Since

$$\begin{aligned}&\frac{1}{p(1-\gamma)}\left[\frac{z\left(H_p^{l,m}[\alpha_1]h(z)\right)' - \overline{z\left(H_p^{l,m}[\alpha_1]g(z)\right)'}}{\left(H_p^{l,m}[\alpha_1]h(z)\right) + \overline{\left(H_p^{l,m}[\alpha_1]g(z)\right)}} - p\gamma\right] \\ &= \frac{1}{p(1-\gamma)}\left[\frac{p + \sum_{n=2}^{\infty}(n+p-1)\phi_n a_{n+p-1}z^{n-1} - \frac{\bar{z}^p}{z^p}\sum_{n=1}^{\infty}(n+p-1)\overline{\phi_n b_{n+p-1}z^{n-1}}}{1 + \sum_{n=2}^{\infty}\phi_n a_{n+p-1}z^{n-1} + \frac{\bar{z}^p}{z^p}\sum_{n=1}^{\infty}\overline{\phi_n b_{n+p-1}z^{n-1}}} - p\gamma\right] \\ &= 1\end{aligned}$$

at $z = 0$, the above required condition is equivalent to

$$\frac{1}{p(1-\gamma)}\left[\frac{z\left(H_p^{l,m}[\alpha_1]h(z)\right)' - \overline{z\left(H_p^{l,m}[\alpha_1]g(z)\right)'}}{\left(H_p^{l,m}[\alpha_1]h(z)\right) + \overline{\left(H_p^{l,m}[\alpha_1]g(z)\right)}} - p\gamma\right] \neq \frac{\xi-1}{\xi+1}, \tag{2.3}$$

$$|\xi| = 1,\ \xi \neq -1,\ 0 < |z| < 1.$$

Simple algebraic manipulation in (2.3) yields

$$\begin{aligned}
0 \neq\ & (\xi+1)\left(z\left(H_p^{l,m}[\alpha_1]h(z)\right)' - \overline{z\left(H_p^{l,m}[\alpha_1]g(z)\right)'} - p\gamma H_p^{l,m}[\alpha_1]h(z) - p\gamma\overline{H_p^{l,m}[\alpha_1]g(z)}\right)\\
& -\ (\xi-1)p(1-\gamma)H_p^{l,m}[\alpha_1]h(z) - (\xi-1)p(1-\gamma)\overline{H_p^{l,m}[\alpha_1]g(z)}\\
=\ & H_p^{l,m}[\alpha_1]h(z) * \left((\xi+1)\left(\frac{z^p}{(1-z)^2} - \frac{(1-p)z^p}{(1-z)}\right) - \frac{(2p\gamma+p\xi-p)z^p}{(1-z)}\right)\\
& -\ \overline{H_p^{l,m}[\alpha_1]g(z)} * \left(\overline{(\bar{\xi}+1)\left(\frac{z^p}{(1-z)^2} - \frac{(1-p)z^p}{(1-z)}\right)} + \overline{\frac{(2p\gamma+p\bar{\xi}-p)z^p}{(1-z)}}\right)\\
=\ & H_p^{l,m}[\alpha_1]h(z) * \left[\frac{2p(1-\gamma)z^p + (\xi-2p+2p\gamma+1)z^{p+1}}{(1-z)^2}\right]\\
& -\ \overline{H_p^{l,m}[\alpha_1]g(z)} * \left[\frac{2p(\xi+\gamma)\bar{z}^p + (\xi-2p\xi-2p\gamma+1)\bar{z}^{p+1}}{(1-\bar{z})^2}\right].
\end{aligned}$$

∎

The coefficient bound for class $T_H^*(p,\alpha_1,\gamma)$ is determined in the following theorem. Furthermore, we use the coefficient condition to obtain extreme points, convex combination and distortion bounds.

Theorem 2.3. *Let $f = h + \bar{g}$ be given by* (1.4). *Then $f \in T_H^*(p,\alpha_1,\gamma)$ if and only if*

$$\sum_{n=2}^{\infty}\left(\frac{n+p\,(1-\gamma)-1}{p\,(1-\gamma)}|a_{n+p-1}| + \frac{n+p\,(1+\gamma)-1}{p\,(1-\gamma)}|b_{n+p-1}|\right)|\phi_n| \leq 1 - \frac{1+\gamma}{1-\gamma}|b_p| \tag{2.4}$$

where $|b_p| < (1-\gamma)/(1+\gamma)$, $0 \leq \gamma < 1$ and ϕ_n is given by (1.2).

Proof. Since $T_H^*(p,\alpha_1,\gamma) \subset S_H^*(p,\alpha_1,\gamma)$, sufficiency part follows from Theorem 2.1. To prove the necessity part, suppose that $f \in T_H^*(p,\alpha_1,\gamma)$. Then we obtain

$$\mathrm{Re}\left(\frac{pz^p - \sum_{n=2}^{\infty}(n+p-1)|a_{n+p-1}|\phi_n z^{n+p-1} - \sum_{n=1}^{\infty}(n+p-1)|\bar{b}_{n+p-1}|\bar{\phi}_n\bar{z}^{n+p-1}}{z^p - \sum_{n=2}^{\infty}|a_{n+p-1}|\phi_n z^{n+p-1} + \sum_{n=1}^{\infty}|\bar{b}_{n+p-1}|\bar{\phi}_n\bar{z}^{n+p-1}}\right) \geq p\gamma,$$

and the result follows by letting $r \to 1^-$ along real axis. ∎

Let clco $T_H^*(p,\alpha_1,\gamma)$ denote the closed convex hull of $T_H^*(p,\alpha_1,\gamma)$. Now we determine the extreme points of clco $T_H^*(p,\alpha_1,\gamma)$.

Theorem 2.4. *Let f be given by* (1.4). *Then $f \in$ clco $T_H^*(p,\alpha_1,\gamma)$ if and only if f can be expressed in the form*

$$f = \sum_{n=1}^{\infty}\left(X_{n+p-1}h_{n+p-1} + Y_{n+p-1}g_{n+p-1}\right), \tag{2.5}$$

where

$$h_p(z) = z^p,\ h_{n+p-1}(z) = z^p - \frac{p(1-\gamma)}{[n+p(1-\gamma)-1]|\phi_n|} z^{n+p-1} \quad (n = 2, 3, \ldots),$$

$$g_{n+p-1}(z) = z^p + \frac{p(1-\gamma)}{[n+p(1+\gamma)-1]|\phi_n|} \overline{z}^{n+p-1} \quad (n = 1, 2, 3, \ldots),$$

ϕ_n is given by (1.2)*, and $\sum_{n=1}^{\infty}(X_{n+p-1}+Y_{n+p-1}) = 1$, with $X_{n+p-1} \geq 0, Y_{n+p-1} \geq 0$. In particular, the extreme points of $T_H^*(p, \alpha_1, \gamma)$ are h_{n+p-1} and g_{n+p-1}.*

Proof. Let f be of the form (2.5), then we have

$$\begin{aligned} f(z) &= X_p h_p(z) + \sum_{n=2}^{\infty} X_{n+p-1}\left(z^p - \frac{p(1-\gamma)}{[n+p(1-\gamma)-1]\,|\phi_n|} z^{n+p-1}\right) \\ &\quad + \sum_{n=1}^{\infty} Y_{n+p-1}\left(z^p + \frac{p(1-\gamma)}{[n+p(1+\gamma)-1]\,|\phi_n|} \overline{z}^{n+p-1}\right) \\ &= z^p - \sum_{n=2}^{\infty} \frac{p(1-\gamma)}{[n+p(1-\gamma)-1]\,|\phi_n|} X_{n+p-1} z^{n+p-1} \\ &\quad + \sum_{n=1}^{\infty} \frac{p(1-\gamma)}{[n+p(1+\gamma)-1]\,|\phi_n|} Y_{n+p-1} \overline{z}^{n+p-1}. \end{aligned} \tag{2.6}$$

Furthermore, let

$$|a_{n+p-1}| = \frac{p(1-\gamma)}{[n+p(1-\gamma)-1]\,|\phi_n|} X_{n+p-1} \quad \text{and} \quad |b_{n+p-1}| = \frac{p(1-\gamma)}{[n+p(1+\gamma)-1]\,|\phi_n|} Y_{n+p-1}.$$

Then

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{[n+p\,(1-\gamma)-1]\,|\phi_n|}{p\,(1-\gamma)}\left(\frac{p(1-\gamma)}{[n+p(1-\gamma)-1]\,|\phi_n|} X_{n+p-1}\right) \\ &\quad + \sum_{n=1}^{\infty} \frac{[n+p\,(1+\gamma)-1]\,|\phi_n|}{p\,(1-\gamma)}\left(\frac{p(1-\gamma)}{[n+p(1+\gamma)-1]\,|\phi_n|} Y_{n+p-1}\right) \\ &= \sum_{n=2}^{\infty} X_{n+p-1} + \sum_{n=1}^{\infty} Y_{n+p-1} = 1 - X_p \leq 1. \end{aligned}$$

Thus $f \in \operatorname{clco}\ T_H^*(p, \alpha_1, \gamma)$.

Conversely, suppose that $f \in \operatorname{clco}\ T_H^*(p, \alpha_1, \gamma)$. Set

$$X_{n+p-1} = \frac{[n+p\,(1-\gamma)-1]|\phi_n||a_{n+p-1}|}{p\,(1-\gamma)} \quad (n = 2, 3, \ldots),$$

$$Y_{n+p-1} = \frac{[n+p\,(1+\gamma)-1]|\phi_n||b_{n+p-1}|}{p\,(1-\gamma)} \quad (n = 1, 2, \ldots),$$

and define $X_p = 1 - \sum_{n=2}^{\infty} X_{n+p-1} - \sum_{n=1}^{\infty} Y_{n+p-1}$. Then,

$$\begin{aligned} f(z) &= z^p - \sum_{n=2}^{\infty} |a_{n+p-1}| z^{n+p-1} + \sum_{n=1}^{\infty} |b_{n+p-1}| \overline{z}^{n+p-1} \\ &= z^p - \sum_{n=2}^{\infty} \frac{p\,(1-\gamma) X_{n+p-1}}{[n+p\,(1-\gamma)-1]|\phi_n|} z^{n+p-1} + \sum_{n=1}^{\infty} \frac{p\,(1-\gamma) Y_{n+p-1}}{[n+p\,(1+\gamma)-1]|\phi_n|} \overline{z}^{n+p-1} \end{aligned}$$

$$= X_p z^p + \sum_{n=2}^{\infty} X_{n+p-1}\left(z^p - \frac{p\,(1-\gamma)}{[n+p\,(1-\gamma)-1]|\phi_n|} z^{n+p-1}\right)$$
$$+ \sum_{n=1}^{\infty} Y_{n+p-1}\left(z^p + \frac{p\,(1-\gamma)}{[n+p\,(1+\gamma)-1]|\phi_n|} \overline{z}^{n+p-1}\right)$$
$$= \sum_{n=1}^{\infty} \left(X_{n+p-1} h_{n+p-1} + Y_{n+p-1} g_{n+p-1}\right)$$

as required. ∎

Theorem 2.5. *The class $T_H^*(p, \alpha_1, \gamma)$ is closed under convex combination.*

Proof. For $i = 1,2,3,...$, suppose that $f_i(z) \in T_H^*(p, \alpha_1, \gamma)$, where f_i is given by

$$f_i(z) = z^p - \sum_{n=2}^{\infty} |a_{i,n+p-1}| z^{n+p-1} + \sum_{n=1}^{\infty} |b_{i,n+p-1}| \overline{z}^{n+p-1} \;.$$

By Theorem 2.3,

$$\sum_{n=2}^{\infty} \frac{n+p\,(1-\gamma)-1}{p\,(1-\gamma)} |\phi_n||a_{i,n+p-1}| + \sum_{n=1}^{\infty} \frac{n+p\,(1+\gamma)-1}{p\,(1-\gamma)} |\phi_n||b_{i,n+p-1}| \le 1. \tag{2.7}$$

For $\sum_{i=1}^{\infty} t_i = 1,\ 0 \le t_i \le 1$, the convex combination of f_i may be written as,

$$\sum_{i=1}^{\infty} t_i f_i(z) = z^p - \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i |a_{i,n+p-1}|\right) z^{n+p-1} + \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i |b_{i,n+p-1}|\right) \overline{z}^{n+p-1} \;.$$

Then, by (2.7)

$$\sum_{n=2}^{\infty} \frac{[n+p\,(1-\gamma)-1]|\phi_n|}{p\,(1-\gamma)} \left(\left|\sum_{i=1}^{\infty} t_i |a_{i,n+p-1}|\right|\right)$$
$$+ \sum_{n=1}^{\infty} \frac{[n+p\,(1+\gamma)-1]|\phi_n|}{p\,(1-\gamma)} \left(\left|\sum_{i=1}^{\infty} t_i |b_{i,n+p-1}|\right|\right)$$
$$= \sum_{i=1}^{\infty} t_i \left(\sum_{n=2}^{\infty} \frac{[n+p\,(1-\gamma)-1]|\phi_n|}{p\,(1-\gamma)} |a_{i,n+p-1}| + \sum_{n=1}^{\infty} \frac{[n+p\,(1+\gamma)-1]]|\phi_n|}{p\,(1-\gamma)} |b_{i,n+p-1}|\right)$$
$$\le \sum_{i=1}^{\infty} t_i = 1.$$

Hence, $\sum_{i=1}^{\infty} t_i f_i(z) \in T_H^*(p, \alpha_1, \gamma)$. ∎

In the last theorem below we give distortion inequalities for f in the class $T_H^*(p, \alpha_1, \gamma)$.

Theorem 2.6. *If $f \in T_H^*(p, \alpha_1, \gamma)$ with $\phi_n \ge \phi_2$, then for $|z| = r < 1$,*

$$|f(z)| \le (1+|b_p|)\, r^p + r^{p+1}\left(\frac{p\,(1-\gamma)}{[p\,(1-\gamma)+1]|\phi_2|} - \frac{p\,(1+\gamma)|b_p|}{[p\,(1-\gamma)+1]|\phi_2|}\right)$$

and

$$|f(z)| \ge (1-|b_p|)\, r^p - r^{p+1}\left(\frac{p\,(1-\gamma)}{[p\,(1-\gamma)+1]|\phi_2|} - \frac{p\,(1+\gamma)|b_p|}{[p\,(1-\gamma)+1]|\phi_2|}\right).$$

Proof. Since

$$\begin{aligned}
&\frac{p\,(1-\gamma)+1}{p\,(1-\gamma)}|\phi_2|\sum_{n=2}^{\infty}\left(|a_{n+p-1}|+|b_{n+p-1}|\right)\\
&\le \sum_{n=2}^{\infty}\frac{n+p\,(1-\gamma)-1}{p\,(1-\gamma)}\left(|a_{n+p-1}|+|b_{n+p-1}|\right)|\phi_n|\\
&\le \sum_{n=2}^{\infty}\left(\frac{n+p\,(1-\gamma)-1}{p\,(1-\gamma)}|a_{n+p-1}|+\frac{n+p\,(1+\gamma)-1}{p\,(1-\gamma)}|b_{n+p-1}|\right)|\phi_n|,
\end{aligned}$$

the result of Theorem 2.3 gives

$$\sum_{n=2}^{\infty}\left(|a_{n+p-1}|+|b_{n+p-1}|\right)\le\frac{p\,(1-\gamma)}{[p\,(1-\gamma)+1]|\phi_2|}\left(1-\frac{1+\gamma}{1-\gamma}|b_p|\right). \tag{2.8}$$

Next, again since $f\in T_H^*(p,\alpha_1,\gamma)$, we have from (2.8) and $|z|=r$ that

$$\begin{aligned}
|f(z)|&=\left|z^p-\sum_{n=2}^{\infty}|a_{n+p-1}|z^{n+p-1}+\sum_{n=1}^{\infty}|b_{n+p-1}|\bar{z}^{n+p-1}\right|\\
&\le |z^p|+\sum_{n=2}^{\infty}|a_{n+p-1}|\,|z|^{n+p-1}+\sum_{n=1}^{\infty}|b_{n+p-1}|\,|\bar{z}|^{n+p-1}\\
&=r^p+\sum_{n=2}^{\infty}|a_{n+p-1}|r^{n+p-1}+\sum_{n=1}^{\infty}|b_{n+p-1}|r^{n+p-1}\\
&\le (1+|b_p|)r^p+\left(\sum_{n=2}^{\infty}\left(|a_{n+p-1}|+|b_{n+p-1}|\right)\right)r^{p+1}\\
&\le (1+|b_p|)r^p+r^{p+1}\left(\frac{p\,(1-\gamma)}{[p\,(1-\gamma)+1]|\phi_2|}-\frac{p\,(1+\gamma)|b_p|}{[p\,(1-\gamma)+1]|\phi_2|}\right)
\end{aligned}$$

which gives the first result.

In a similar manner, we obtain the following lower bound.

$$\begin{aligned}
|f(z)|&\ge r^p-\sum_{n=2}^{\infty}|a_{n+p-1}|r^{n+p-1}-\sum_{n=1}^{\infty}|b_{n+p-1}|r^{n+p-1}\\
&=(1-|b_p|)r^p-\sum_{n=2}^{\infty}\left(|a_{n+p-1}|+|b_{n+p-1}|\right)r^{n+p-1}\\
&\ge (1-|b_p|)r^p-r^{p+1}\left(\frac{p\,(1-\gamma)}{[p\,(1-\gamma)+1]|\phi_2|}-\frac{p\,(1+\gamma)|b_p|}{[p\,(1-\gamma)+1]|\phi_2|}\right).
\end{aligned}$$

∎

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References

[1] O. P. Ahuja, Use of theory of conformal mappings in harmonic univalent mappings with directional convexity, *Bull. Malays. Math. Sci. Soc. (2)* **35** (2012), no. 3, 775–784.

[2] O. P. Ahuja and J. M. Jahangiri, Multivalent harmonic starlike functions, *Ann. Univ. Mariae Curie-Skłodowska Sect. A* **55** (2001), 1–13.

[3] O. P. Ahuja, S. Joshi and N. Sangle, Multivalent harmonic uniformly starlike functions, *Kyungpook Math. J.* **49** (2009), no. 3, 545–555.

[4] H. A. Al-Kharsani and R. A. Al-Khal, Univalent harmonic functions, *JIPAM. J. Inequal. Pure Appl. Math.* **8** (2007), no. 2, Article 59, 8 pp.

[5] K. Al-Shaqsi and M. Darus, On harmonic functions defined by derivative operator, *J. Inequal. Appl.* **2008**, Art. ID 263413, 10 pp.

[6] S. D. Bernardi, Convex and starlike univalent functions, *Trans. Amer. Math. Soc.* **135** (1969), 429–446.

[7] B. C. Carlson and D. B. Shaffer, Starlike and prestarlike hypergeometric functions, *SIAM J. Math. Anal.* **15** (1984), no. 4, 737–745.

[8] R. Chandrashekar, G. Murugusundaramoorthy, S. K. Lee and K. G. Subramanian, A class of complex-valued harmonic functions defined by Dziok-Srivastava operator, *Chamchuri J. Math.*, **1** (2009), no. 2, 31–42.

[9] Sh. Chen, S. Ponnusamy and X. Wang, Coefficient estimates and Landau-Bloch's constant for planar harmonic mappings, *Bull. Malays. Math. Sci. Soc. (2)* **34** (2011), no. 2, 255–265.

[10] J. Clunie and T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **9** (1984), 3–25.

[11] P. Duren, W. Hengartner and R. S. Laugesen, The argument principle for harmonic functions, *Amer. Math. Monthly* **103** (1996), no. 5, 411–415.

[12] J. Dziok and H. M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, *Appl. Math. Comput.* **103** (1999), no. 1, 1–13.

[13] E. Hohlov, Operators and operations in the class of univalent functions, *Izv. Vyssh. Uchebn. Zaved. Mat.* **10** (1978), 83–89.

[14] J. M. Jahangiri, B. Şeker and S. S. Eker, Sălăgean-type harmonic multivalent functions, *Acta Univ. Apulensis Math. Inform. No. 18* (2009), 233–244.

[15] J. M. Jahangiri, Harmonic functions starlike in the unit disk, *J. Math. Anal. Appl.* **235** (1999), no. 2, 470–477.

[16] H. Lewy, On the non-vanishing of the Jacobian in certain one-to-one mappings, *Bull. Amer. Math. Soc.* **42** (1936), no. 10, 689–692.

[17] R. J. Libera, Some classes of regular univalent functions, *Proc. Amer. Math. Soc.* **16** (1965), 755–758.

[18] A. E. Livingston, On the radius of univalence of certain analytic functions, *Proc. Amer. Math. Soc.* **17** (1966), 352–357.

[19] G. Murugusundaramoorthy, K. Vijaya and R. K. Raina, A subclass of harmonic functions with varying arguments defined by Dziok-Srivastava operator, *Arch. Math. (Brno)* **45** (2009), no. 1, 37–46.

[20] G. Murugusundaramoorthy, A class of Ruscheweyh-type harmonic univalent functions with varying arguments, *Southwest J. Pure Appl. Math.* **2003**, no. 2, 90–95.

[21] T. Rosy, B. A. Stephen, K. G. Subramanian and J. M. Jahangiri, Goodman-Rønning-type harmonic univalent functions, *Kyungpook Math. J.* **41** (2001), no. 1, 45–54.

[22] S. Ruscheweyh, New criteria for univalent functions, *Proc. Amer. Math. Soc.* **49** (1975), 109–115.

[23] T. Sheil-Small, Constants for planar harmonic mappings, *J. London Math. Soc. (2)* **42** (1990), no. 2, 237–248.

[24] P. Sharma and N. Khan, Harmonic multivalent functions involving a linear operator, *Int. J. Math. Anal. (Ruse)* **3** (2009), no. 5–8, 295–308.

[25] H. M. Srivastava and S. Owa, Some characterization and distortion theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators, and certain subclasses of analytic functions, *Nagoya Math. J.* **106** (1987), 1–28.

[26] K. G. Subramanian, B. A. Stephen and S. K. Lee, Subclasses of multivalent harmonic mappings defined by convolution, *Bull. Malays. Math. Sci. Soc. (2)* **35** (2012), no. 3, 717–726.