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$a(x)$-Convex Functions and Their Inequalities

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Abstract. In this paper $a(x)$-convex functions are defined and Jensen, Hermite-Hadamard, Lah-Ribarić and other inequalities for them are derived. Related analogous of the Lagrange and the Cauchy mean value theorems are also obtained and means of the Cauchy type are generated.

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1. Introduction

Convex functions are very important in the theory of inequalities. The third chapter of the classical book of Hardy, Littlewood and Pólya [3] is devoted to the theory of convex functions. Let us recall the definition of convex functions.

Definition 1.1. *Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is called* convex *if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{1.1}$$

for all points $x, y \in I$ and all $\lambda \in [0,1]$. It is called strictly convex *if the inequality* (1.1) *holds strictly whenever x and y are distinct points and $\lambda \in (0,1)$.*
If $-f$ is convex (respectively, strictly convex), we say that f is concave *(respectively,* strictly concave*). If f is both convex and concave, f is said to be* affine.

The geometric characterization depends upon the idea of a support line. The following result can be seen in [11].

Theorem 1.1.

(a) *$f\colon (a,b) \to \mathbb{R}$ is convex if and only if there is at least one line of support for f at each $x_0 \in (a,b)$, i.e.,*

$$f(x) \geq f(x_0) + \lambda(x - x_0),\ \textit{for all } x \in (a,b),$$

where λ depends on x_0 and is given by $\lambda = f'(x_0)$ when f' exists, and $\lambda \in [f'_-(x_0), f'_+(x_0)]$ when $f'_-(x_0) \neq f'_+(x_0)$.

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(b) *$f\colon (a,b) \to \mathbb{R}$ is convex if the function $f(x) - f(x_0) - \lambda(x - x_0)$ (the difference between the function and its support) is decreasing for $x < x_0$ and increasing for $x > x_0$.*

In this paper we define $a(x)$-convex functions and derive analogous to the line of support. Furthermore, we derive Jensen, Hermite-Hadamard, Lah-Ribarić and other inequalities for $a(x)$-convex functions. We continue with analogous of the Lagrange and the Cauchy mean value theorems and obtain means of the Cauchy type. Furthermore, we give generalization of the Jensen inequality for functions convex with respect to given function, derive analogous of the Lagrange and Cauchy mean value theorems for them and obtain means of Cauchy type.

The paper is organized as follows. In Section 2 we define $a(x)$-convex functions and give some inequalities for them. In Section 3 we derive analogues of the Lagrange mean value theorem and the Cauchy mean value theorem. In Section 4 we prove the exponential convexity of a function defined as the difference between the left-hand and the right-hand side of the Jensen inequality for $a(x)$-convex functions and give means of Cauchy type. In Section 5 we observe functions convex with respect to given function.

2. Definition and properties

Definition 2.1. *Let f, a be real functions defined on interval $I \subseteq \mathbb{R}$ such that f is differentiabile and af' integrable. Function f is called* $a(x)$-convex *on interval I if for every $x, y \in I$*

$$(y-x)(f'(y)-f'(x)) \geq (y-x)\int_x^y a(t)f'(t)dt \tag{2.1}$$

holds. Function f is called $a(x)$-concave *if the inequality in (2.1) is reversed.*

Remark 2.1. Notice that for $a(x) = 0$, f is convex.

Remark 2.2. If $x \neq y$, (2.1) is equivallent to

$$\frac{f'(y)-f'(x)}{y-x} \geq \frac{1}{y-x}\int_x^y a(t)f'(t)dt.$$

If f is $a(x)$-convex, $-f$ is $a(x)$-concave. So we will only give properties of $a(x)$-convex functions, because they are the same for $a(x)$-concave functions. Properties of $a(x)$-convex functions:

(1) Let f and g be $a(x)$-convex functions. Then $f+g$ is $a(x)$-convex.
(2) If f is $a(x)$-convex function and λ is non-negative real number, then λf is $a(x)$-convex function.

In applications we often use $a(x)$-convexity criteria given in the following theorem.

Theorem 2.1. *If f'' is a continuous function and af' an integrable function on interval I, f is $a(x)$-convex on interval I if and only if $f''(x) - a(x)f'(x) \geq 0$.*

Proof. Let f be $a(x)$-convex function, then for $x, y \in I$ such that $x \neq y$ we have

$$\frac{f'(y)-f'(x)}{y-x} \geq \frac{1}{y-x}\int_x^y a(t)f'(t)dt.$$

Taking a limit when $y \to x$ we get $f''(x) \geq a(x)f'(x)$.

Conversely, let $f''(t) \geq a(t)f'(t)$ for all $t \in I$. For $x, y \in I$ such that $x < y$ we have that

$$\int_x^y f''(t)dt \geq \int_x^y a(t)f'(t)dt. \tag{2.2}$$

Since f'' is continuous, we have that for every $[x,y] \subset I$, $\int_x^y f''(t)dt = f'(y) - f'(x)$. Furthermore, since $x < y$ we can multiply the inequality (2.2) by $y - x$ and get

$$(y-x)(f'(y) - f'(x)) \geq (y-x)\int_x^y a(t)f'(t)dt.$$

Hence, f is $a(x)$-convex function. So the proof is completed. ∎

Theorem 2.2. *A function f is $a(x)$-convex on interval I if and only if*

$$f(y) - f(x) - f'(x)(y-x) \geq \int_x^y (y-t)a(t)f'(t)dt \tag{2.3}$$

for all $x, y \in I$.

Proof. First, suppose that $s > x$. Then (2.1) is equivalent to

$$f'(s) - f'(x) \geq \int_x^s a(t)f'(t)dt.$$

For every $[x,y] \subseteq I$ we have

$$\int_x^y f'(s)ds - \int_x^y f'(x)ds \geq \int_x^y \int_x^s a(t)f'(t)dtds$$

i.e.

$$f(y) - f(x) - f'(x)(y-x) \geq \int_x^y (y-t)a(t)f'(t)dt.$$

Analogous, for $s < x$ and $[x,y] \in I$, we obtain (2.3). Since we have equivalence in each step, the proof is completed. ∎

Notice that Theorem 2.2 is analogous to existence of the line of support for convex functions.

Furthermore, we give the following generalization of the Jensen inequality.

Theorem 2.3. *Let $f : I \to \mathbb{R}$ be $a(x)$-convex function, $x_i \in I$ and $p_i \in [0,1]$, $i = 1, \ldots, n$ such that $\sum_{i=1}^n p_i = 1$ and let $\bar{x} = \sum_{i=1}^n p_i x_i$. Then the inequality*

$$\sum_{i=1}^n p_i f(x_i) - f(\bar{x}) \geq \sum_{i=1}^n p_i \int_{\bar{x}}^{x_i} (x_i - t)a(t)f'(t)dt. \tag{2.4}$$

holds.

Proof. Apply (2.3) with $y = x_i$ and $x = \bar{x} = \sum_{i=1}^{n} p_i x_i, p_i \in [0,1], i = 1,\ldots,n, \sum_{i=1}^{n} p_i = 1$. Then we obtain

$$f(x_i) - f(\bar{x}) - f'(\bar{x})(x_i - \bar{x}) \geq \int_{\bar{x}}^{x_i} (x_i - t)a(t)f'(t)dt. \tag{2.5}$$

If we multiply (2.5) by p_i and sum over i from 1 to n, we deduce (2.4). ∎

Remark 2.3. As a special case of (2.4) we obtain the following inequality

$$pf(x) + qf(y) - f(px + qy) \geq p\int_{\bar{x}}^{x} (x - t)a(t)f'(t)dt + q\int_{\bar{x}}^{y} (y - t)a(t)f'(t)dt, \tag{2.6}$$

where $p, q \in [0,1]$, $p + q = 1, \bar{x} = px + qy$.

Corollary 2.1. *Let $f : I \to \mathbb{R}$ be $a(x)$-convex function $x_1, x_2, x_3 \in I$, $x_1 \leq x_2 \leq x_3$. Then the inequality*

$$\begin{aligned}&(x_3 - x_2)f(x_1) + (x_1 - x_3)f(x_2) + (x_2 - x_1)f(x_3)\\ &\geq (x_2 - x_1)\int_{x_2}^{x_3} (x_3 - t)a(t)f'(t)dt - (x_3 - x_2)\int_{x_1}^{x_2} (x_1 - t)a(t)f'(t)dt\end{aligned} \tag{2.7}$$

holds.

Proof. Apply (2.6) with $x = x_1$, $px + qy = x_2, y = x_3$, $p + q = 1$. ∎

Furthermore, we obtain the Lah-Ribarić inequality for $a(x)$-convex functions.

Corollary 2.2. *Let $f : I \to \mathbb{R}$ be $a(x)$-convex function, $x_i \in [m, M] \subseteq I$ for each $i \in \{1, \ldots n\}$ and $p_i \in [0,1]$ such that $\sum_{i=1}^{n} p_i = 1$. Then*

$$\begin{aligned}\sum_{i=1}^{n} p_i f(x_i) \leq \frac{M - \bar{x}}{M - m} f(m) + \frac{\bar{x} - m}{M - m} f(M) + \frac{1}{M - m} \times\\ \times \sum_{i=1}^{n} p_i \left[(M - x_i)\int_{m}^{x_i} (m - t)a(t)f'(t)dt - (x_i - m)\int_{x_i}^{M} (M - t)a(t)f'(t)dt \right].\end{aligned}$$

(2.8)

Proof. Apply (2.7) with $x_1 = m, x_2 = x_i$, $\sum_{i=1}^{n} p_i x_i = \bar{x}$, $x_3 = M$. Multiply the result by p_i and sum over i from 1 to n, then we obtain (2.8). ∎

Next, we obtain the Hermite inequality for $a(x)$-convex functions.

Corollary 2.3. *Let $f : [c,d] \to \mathbb{R}$ be $a(x)$-convex function. Then the inequality*

$$\frac{1}{d - c}\int_{c}^{d} f(t)dt \leq \frac{f(c) + f(d)}{2} + \frac{1}{2(d - c)}\int_{c}^{d} (d - t)(c - t)a(t)f'(t)dt \tag{2.9}$$

holds.

Proof. Apply (2.7) with $c \le x \le d$. Then we obtain

$$\begin{aligned} f(x) \le {} & \frac{f(d)-f(c)}{d-c}(x-c) + f(c) + \frac{d-x}{d-c}\int_c^x (c-t)a(t)f'(t)dt \\ & - \frac{x-c}{d-c}\int_x^d (d-t)a(t)f'(t)dt. \end{aligned} \tag{2.10}$$

Integrating (2.10) over $[c,d]$ and multiplying with $1/(d-c)$ we get

$$\begin{aligned} \frac{1}{d-c}\int_c^d f(t)dt \le {} & \frac{f(c)+f(d)}{2} + \frac{1}{(d-c)^2} \times \\ & \times \left[\int_c^d (d-x)\int_c^x (c-t)a(t)f'(t)dt\,dx - \int_c^d (x-c)\int_x^d (d-t)a(t)f'(t)dt\,dx\right]. \end{aligned} \tag{2.11}$$

Applying the Fubini theorem to the right-hand side of (2.11) we obtain (2.9) and the proof is complete. ∎

Let us consider integral Jensen's inequality for $a(x)$-convex functions.

Theorem 2.4. *Let $f : [c,d] \to \mathbb{R}$ be $a(x)$-convex function, g be a function such that $img \subseteq [c,d]$, $f \circ g$ is integrabile on $[c,d]$ and $p : [c,d] \to \mathbb{R}$ is non-negative function such that $\int_c^d p(s)ds > 0$. Then the inequality*

$$\frac{\int_c^d p(s)f(g(s))\,ds}{\int_c^d p(s)ds} - f(\overline{g}) \ge \frac{1}{\int_c^d p(s)ds}\int_c^d p(s)\int_{\overline{g}}^{g(s)} (g(s)-t)a(t)f'(t)dt\,ds \tag{2.12}$$

holds, where

$$\overline{g} = \frac{\int_c^d p(s)g(s)\,ds}{\int_c^d p(s)ds}.$$

Proof. Apply (2.3) with

$$y = g(s),\ x = \overline{g} = \frac{\int_c^d p(s)g(s)\,ds}{\int_c^d p(s)ds}.$$

Then we obtain

$$f(g(s)) - f(\overline{g}) - f'(\overline{g})(g(s)-\overline{g}) \ge \int_{\overline{g}}^{g(s)} (g(s)-t)a(t)f'(t)dt \tag{2.13}$$

Multiplying (2.13) with $p(s)$ and then integrating over $[c,d]$ we get

$$\int_c^d p(s)f(g(s))ds - f(\overline{g})\int_c^d p(s)ds \geq \int_c^d p(s)\int_{\overline{g}}^{g(s)}(g(s)-t)a(t)f'(t)dt\,ds. \tag{2.14}$$

Multiplying (2.14) by $[\int_c^d p(s)ds]^{-1}$ we obtain (2.12). ∎

As a special case of (2.12) for $g(s)=s$ and $p(s)=1$ we obtain the Hadamard inequality for $a(x)$-convex functions.

Corollary 2.4. *Let $f:[c,d]\to\mathbb{R}$ be $a(x)$-convex function. Then the inequality*

$$\frac{1}{d-c}\int_c^d f(s)ds - f\left(\frac{c+d}{2}\right) \geq \frac{1}{d-c}\int_c^d\int_{\frac{c+d}{2}}^{s}(s-t)a(t)f'(t)dt\,ds$$

holds.

Remark 2.4. Combining results from Corollary 2.3 and 2.4 the following result holds for $a(x)$-convex functions. Let $f:[c,d]\to\mathbb{R}$ be $a(x)$-convex function. Then the inequality

$$\begin{aligned}
&f\left(\frac{c+d}{2}\right)+\frac{1}{d-c}\int_c^d\int_{\frac{c+d}{2}}^{s}(s-t)a(t)f'(t)dt\,ds\\
&\leq \frac{1}{d-c}\int_c^d f(t)dt \leq \frac{f(c)+f(d)}{2}+\frac{1}{2(d-c)}\int_c^d (d-t)(c-t)a(t)f'(t)dt
\end{aligned}$$

holds.

Theorem 2.5. *Let I be an interval in $\mathbb{R}$. A function $f:I\to\mathbb{R}$ is $a(x)$-convex if and only if the function $F:I\to\mathbb{R}$ defined by*

$$F(s)=f(s)-\int_{x_0}^{s}(s-t)a(t)f'(t)dt \tag{2.15}$$

is convex, where $x_0\in I$.

Proof. If f is $a(x)$-convex function, then (2.4) holds, that is

$$\begin{aligned}
\sum_{i=1}^n p_i f(x_i) - f(\overline{x}) &\geq \sum_{i=1}^n p_i\int_{\overline{x}}^{x_i}(x_i-t)a(t)f'(t)dt\\
&= \sum_{i=1}^n p_i\int_{x_0}^{x_i}(x_i-t)a(t)f'(t)dt - \sum_{i=1}^n p_i\int_{x_0}^{\overline{x}}(x_i-t)a(t)f'(t)dt.
\end{aligned} \tag{2.16}$$

(2.16) is equivalent to

$$\sum_{i=1}^n p_i f(x_i) - \sum_{i=1}^n p_i\int_{x_0}^{x_i}(x_i-t)a(t)f'(t)dt \geq f(\overline{x}) - \sum_{i=1}^n p_i\int_{x_0}^{\overline{x}}(x_i-t)a(t)f'(t)dt. \tag{2.17}$$

Let the function F be defined by (2.15). Left-hand side of (2.17) is equal to $\sum_{i=1}^{n} p_i F(x_i)$, while the second term on the right-hand sides becomes

$$\sum_{i=1}^{n} p_i \int_{x_0}^{\bar{x}} (x_i - t)a(t)f'(t)dt = \int_{x_0}^{\bar{x}} \sum_{i=1}^{n} p_i (x_i - t)a(t)f'(t)dt$$
$$= \int_{x_0}^{\bar{x}} (\bar{x} - t)a(t)f'(t)dt,$$

so we get

$$\sum_{i=1}^{n} p_i F(x_i) - F(\bar{x}) \geq 0$$

and the function F is convex function.

Conversely, let F be a convex function. Then

$$F(y) - F(x) - F'(x)(y - x) \geq 0$$

for $x, y \in I$. After a simple calculation we obtain that (2.3) holds, that is f is $a(x)$-convex. ∎

Remark 2.5. We can apply the function F defined by (2.15) to inequalities for convex functions and obtain inequalities for $a(x)$-convex functions, but here we omit the details.

Remark 2.6. Using similar method as in Corollary 2.2, 2.3 and 2.4 we can obtain more inequalities for $a(x)$-convex functions like for example Petrović and Giaccardi inequality, but here we omit the details.

3. Mean value theorems

Lemma 3.1. *Let I be an open interval. Let a be an integrabile function and $h \in C^2(I)$ be such that $h'' - ah'$ is bounded by integrabile functions M and m, that is, $m(x) \leq h''(x) - a(x)h'(x) \leq M(x)$, for every $x \in I$. Then functions Φ_1, Φ_2 defined by*

$$\Phi_1(x) = R_1(x) - h(x),$$
$$\Phi_2(x) = h(x) - R_2(x),$$

where

$$R_1(x) = \int \left(e^{\int a(x)dx} \int M(x)\, e^{-\int a(x)dx} dx \right) dx,$$
$$R_2(x) = \int \left(e^{\int a(x)dx} \int m(x)\, e^{-\int a(x)dx} dx \right) dx,$$

are $a(x)$-convex.

Proof. Since $h''(x) - a(x)h'(x) \leq M(x)$ and $R_1''(x) - a(x)R_1'(x) = M(x)$ we have,

$$\Phi_1''(x) - a(x)\Phi_1'(x) = R_1''(x) - a(x)R_1'(x) - (h''(x) - a(x)h'(x)) \geq 0.$$

So Φ_1 is $a(x)$-convex function. In the same way, since $h''(x) - a(x)h'(x) \geq m(x)$ and $R_2''(x) - a(x)R_2'(x) = m(x)$ we have,

$$\Phi_2''(x) - a(x)\Phi_2'(x) = h''(x) - a(x)h'(x) - (R_2''(x) - a(x)R_2'(x)) \geq 0.$$

So Φ_2 is $a(x)$-convex function. ∎

Now we will state and prove the Lagrange-type mean value theorems. These theorems are consequences of the Cauchy mean value theorem but we will prove them by using $a(x)$-convex functions from Lemma 3.1.

Theorem 3.1. *Let a, h'' be continuous and g be a positive and continuous function on compact interval $I \subseteq \mathbb{R}$. Let $x_i \in I$ and $p_i \in [0,1]$, $i = 1,\ldots,n$ such that $\sum_{i=1}^{n} p_i = 1$ and let $\bar{x} = \sum_{i=1}^{n} p_i x_i$. Then there exists $\eta \in I$ such that*

$$\begin{aligned} &\sum_{i=1}^{n} p_i h(x_i) - h(\bar{x}) - \sum_{i=1}^{n} p_i \int_{\bar{x}}^{x_i} (x_i - t) a(t) h'(t) dt \\ &\qquad = \frac{h''(\eta) - a(\eta) h'(\eta)}{g(\eta)} \sum_{i=1}^{n} p_i \int_{\bar{x}}^{x_i} (x_i - t) g(t) dt. \end{aligned} \tag{3.1}$$

Proof. Since $\frac{h''-ah'}{g}$ is continuous on I, there exists

$$m = \min_{t \in I} \left(\frac{h''(t) - a(t)h'(t)}{g(t)} \right) \text{ and } M = \max_{t \in I} \left(\frac{h''(t) - a(t)h'(t)}{g(t)} \right).$$

Applying (2.4) on functions Φ_1 and Φ_2 from Lemma 3.1, with $M(x) = Mg(x)$, $m(x) = mg(x)$, the following inequalities hold:

$$\sum_{i=1}^{n} p_i \Phi_1(x_i) - \Phi_1(\bar{x}) \geq \sum_{i=1}^{n} p_i \int_{\bar{x}}^{x_i} (x_i - t) a(t) \Phi_1'(t) dt,$$

$$\sum_{i=1}^{n} p_i \Phi_2(x_i) - \Phi_2(\bar{x}) \geq \sum_{i=1}^{n} p_i \int_{\bar{x}}^{x_i} (x_i - t) a(t) \Phi_2'(t) dt.$$

It follows that

$$\begin{aligned} m &\left(\sum_{i=1}^{n} p_i R_3(x_i) - R_3(\bar{x}) - \sum_{i=1}^{n} p_i \int_{\bar{x}}^{x_i} (x_i - t) a(t) R_3'(t) dt \right) \\ &\leq \sum_{i=1}^{n} p_i h(x_i) - h(\bar{x}) - \sum_{i=1}^{n} p_i \int_{\bar{x}}^{x_i} (x_i - t) a(t) h'(t) dt \\ &\leq M \left(\sum_{i=1}^{n} p_i R_3(x_i) - R_3(\bar{x}) - \sum_{i=1}^{n} p_i \int_{\bar{x}}^{x_i} (x_i - t) a(t) R_3'(t) dt \right), \end{aligned}$$

where

$$R_3(x) = \int \left(e^{\int a(x)dx} \int g(x) e^{-\int a(x)dx} dx \right) dx.$$

Therefore, there exists $\eta \in I$ such that

$$\sum_{i=1}^{n} p_i h(x_i) - h(\overline{x}) - \sum_{i=1}^{n} p_i \int_{\overline{x}}^{x_i} (x_i - t)a(t)h'(t)dt$$
$$= \frac{h''(\eta) - a(\eta)h'(\eta)}{g(\eta)} \left[\sum_{i=1}^{n} p_i R_3(x_i) - R_3(\overline{x}) - \sum_{i=1}^{n} p_i \int_{\overline{x}}^{x_i} (x_i - t)a(t)R_3'(t)dt \right].$$

Since $R_3''(x) - a(x)R_3'(x) = g(x)$, we have $a(x)R_3'(x) = R_3''(x) - g(x)$. So

$$\sum_{i=1}^{n} p_i R_3(x_i) - R_3(\overline{x}) - \sum_{i=1}^{n} p_i \int_{\overline{x}}^{x_i} (x_i - t)a(t)R_3'(t)dt = \sum_{i=1}^{n} p_i \int_{\overline{x}}^{x_i} (x_i - t)g(t)dt.$$

Hence, there exists $\eta \in I$ such that (3.1) holds. ∎

Theorem 3.2. *Let a, h'' be continuous and g be a positive and continuous function on compact interval $I \subseteq \mathbb{R}$. Let $x_i \in [m,M] \subseteq I$ and $p_i \in [0,1]$, $i = 1,\ldots,n$ such that $\sum_{i=1}^{n} p_i = 1$ and let $\overline{x} = \sum_{i=1}^{n} p_i x_i$. Then there exists $\eta \in I$ such that*

$$\sum_{i=1}^{n} p_i h(x_i) - \frac{M-\overline{x}}{M-m} h(m) - \frac{\overline{x}-m}{M-m} h(M)$$
$$- \frac{1}{M-m} \sum_{i=1}^{n} p_i \left[(M-x_i) \int_{m}^{x_i} (m-t)a(t)h'(t)dt - (x_i - m) \int_{x_i}^{M} (M-t)a(t)h'(t)dt \right]$$
$$= \frac{h''(\eta) - a(\eta)h'(\eta)}{g(\eta)} \times$$
$$\times \frac{1}{M-m} \sum_{i=1}^{n} p_i \left[(M-x_i) \int_{m}^{x_i} (m-t)g(t)dt - (x_i - m) \int_{x_i}^{M} (M-t)g(t)dt \right].$$

Proof. Similar to the proof of Theorem 3.1, apply Lah-Ribarić inequality from (2.8) to functions Φ_1 and Φ_2 from Lemma 3.1. ∎

Theorem 3.3. *Let a, h'' be continuous and g be a positive and continuous function on $[c,d] \subseteq \mathbb{R}$. Then there exists $\eta \in [c,d]$ such that*

$$\frac{1}{d-c} \int_{c}^{d} h(t)dt - \frac{h(c)+h(d)}{2} - \frac{1}{2(d-c)} \int_{c}^{d} (d-t)(c-t)a(t)h'(t)dt$$
$$= \frac{h''(\eta) - a(\eta)h'(\eta)}{g(\eta)} \cdot \frac{1}{2(d-c)} \int_{c}^{d} (d-t)(c-t)g(t)dt.$$

Proof. Similar to the proof of Theorem 3.1, apply Hermite inequality from (2.9) to functions Φ_1 and Φ_2 from Lemma 3.1. ∎

Theorem 3.4. *Let a, h'' be continuous and G be a positive and continuous function on $[c,d] \subseteq \mathbb{R}$. Let $p : [c,d] \to \mathbb{R}$ be non-negative function such that $\int_c^d p(s)ds > 0$. Let g be a function such that*

$$\operatorname{Im} g \subseteq [c,d], \; \overline{g} = \frac{\int_c^d p(s)g(s)ds}{\int_c^d p(s)ds}.$$

Then there exists $\eta \in [c,d]$ such that

$$\frac{\int_c^d p(s)h(g(s))\,ds}{\int_c^d p(s)ds} - h(\overline{g}) - \frac{1}{\int_c^d p(s)ds}\int_c^d p(s)\int_{\overline{g}}^{g(s)} (g(s)-t)a(t)h'(t)dt\,ds$$
$$= \frac{h''(\eta)-a(\eta)h'(\eta)}{G(\eta)} \frac{1}{\int_c^d p(s)ds}\int_c^d p(s)\int_{\overline{g}}^{g(s)} (g(s)-t)G(t)dtds.$$

Proof. Similar to the proof of Theorem 3.1, apply integral Jensen inequality from (2.12) to functions Φ_1 and Φ_2 from Lemma 3.1. ∎

Theorem 3.5. *Let a, h'' be continuous and g be a positive and continuous function on $[c,d] \subseteq \mathbb{R}$. Then there exists $\eta \in [c,d]$ such that*

$$\frac{1}{d-c}\int_c^d h(s)ds - h\left(\frac{c+d}{2}\right) - \frac{1}{d-c}\int_c^d \int_{\frac{c+d}{2}}^{s} (s-t)a(t)h'(t)dtds$$
$$= \frac{h''(\eta)-a(\eta)h'(\eta)}{g(\eta)} \frac{1}{d-c}\int_c^d \int_{\frac{c+bd}{2}}^{s} (s-t)g(t)dtds.$$

Proof. Similar to the proof of Theorem 3.1, apply Hadamard inequality from (2.4) to functions Φ_1 and Φ_2 from Lemma 3.1. ∎

Now we will state and prove the Cauchy-type mean value theorems.

Theorem 3.6. *Let I be a compact interval in $\mathbb{R}$. Let $x_i \in I$ and $p_i \in [0,1]$, $i = 1,\dots,n$ such that $\sum_{i=1}^n p_i = 1$ and let $\overline{x} = \sum_{i=1}^n p_i x_i$. Let $h_1, h_2 \in C^2(I)$ and let a be a continuous function such that*

$$\sum_{i=1}^n p_i h_2(x_i) - h_2(\overline{x}) - \sum_{i=1}^n p_i \int_{\overline{x}}^{x_i} (x_i - t)a(t)h_2'(t)dt \neq 0 \tag{3.2}$$

Then there exists $\eta \in I$ *such that*

$$\frac{h_1''(\eta)-a(\eta)h_1'(\eta)}{h_2''(\eta)-a(\eta)h_2'(\eta)} = \frac{\sum\limits_{i=1}^{n} p_i h_1(x_i) - h_1(\overline{x}) - \sum\limits_{i=1}^{n} p_i \int\limits_{\overline{x}}^{x_i} (x_i-t)a(t)h_1'(t)dt}{\sum\limits_{i=1}^{n} p_i h_2(x_i) - h_2(\overline{x}) - \sum\limits_{i=1}^{n} p_i \int\limits_{\overline{x}}^{x_i} (x_i-t)a(t)h_2'(t)dt}. \tag{3.3}$$

Proof. Let

$$c_1 = \sum_{i=1}^{n} p_i h_2(x_i) - h_2(\overline{x}) - \sum_{i=1}^{n} p_i \int\limits_{\overline{x}}^{x_i} (x_i-t)a(t)h_2'(t)dt,$$

$$c_2 = \sum_{i=1}^{n} p_i h_1(x_i) - h_1(\overline{x}) - \sum_{i=1}^{n} p_i \int\limits_{\overline{x}}^{x_i} (x_i-t)a(t)h_1'(t)dt.$$

Now we apply (3.1) to the function $c_1h_1 - c_2h_2$. The following equality holds:

$$\begin{aligned} & c_1\left[\sum_{i=1}^{n} p_i h_1(x_i) - h_1(\overline{x}) - \sum_{i=1}^{n} p_i \int\limits_{\overline{x}}^{x_i} (x_i-t)a(t)h_1'(t)dt\right] \\ & - c_2\left[\sum_{i=1}^{n} p_i h_2(x_i) - h_2(\overline{x}) - \sum_{i=1}^{n} p_i \int\limits_{\overline{x}}^{x_i} (x_i-t)a(t)h_2'(t)dt\right] \\ & = \frac{c_1h_1''(\xi) - c_2h_2''(\xi) - a(\xi)\left(c_1h_1'(\xi) - c_2h_2'(\xi)\right)}{g(\xi)} \times \\ & \times \left[\sum_{i=1}^{n} p_i \int\limits_{\overline{x}}^{x_i} (x_i-t)g(t)dt\right]. \end{aligned} \tag{3.4}$$

It is easy to see that the left-hand side of (3.4) is equal to 0, so the right-hand side should also be equal to 0. From (3.2) we get that the right-hand side in (3.1) is not equal to 0, so the part in square brackets on the right-hand side of (3.4) is not equal to 0. For the right-hand side in (3.4) to be equal to 0 it follows that $c_1h_1''(\xi) - c_2h_2''(\xi) - a(\xi)\left(c_1h_1'(\xi) - c_2h_2'(\xi)\right) = 0$. After a short calculation, it is easy to see that (3.3) follows from $c_1(h_1''(\xi) - a(\xi)h_1'(\xi)) - c_2(h_2''(\xi) - a(\xi)h_2'(\xi)) = 0$, so the proof is completed. ∎

Theorem 3.7. *Let* I *be a compact interval in* $\mathbb{R}$. *Let* $x_i \in [m,M] \subseteq I$ *and* $p_i \in [0,1]$, $i = 1,\ldots,n$ *such that* $\sum\limits_{i=1}^{n} p_i = 1$ *and let* $\overline{x} = \sum\limits_{i=1}^{n} p_i x_i$. *Let*

$$\begin{aligned} L(h) = & \sum_{i=1}^{n} p_i h(x_i) - \frac{M-\overline{x}}{M-m}h(m) - \frac{\overline{x}-m}{M-m}h(M) - \frac{1}{M-m} \times \\ & \times \sum_{i=1}^{n} p_i \left[(M-x_i)\int\limits_{m}^{x_i}(m-t)a(t)h'(t)dt - (x_i-m)\int\limits_{x_i}^{M}(M-t)a(t)h'(t)dt\right]. \end{aligned}$$

Let $h_1, h_2 \in C^2(I)$ *and let* a *be a continuous function such that*

$$L(h_2) \neq 0.$$

Then there exists $\eta \in I$ *such that*

$$\frac{h_1''(\eta)-a(\eta)h_1'(\eta)}{h_2''(\eta)-a(\eta)h_2'(\eta)}=\frac{L(h_1)}{L(h_2)}.$$

Proof. Similar to the proof of Theorem 3.6. ∎

Theorem 3.8. *Let* $[c,d]$ *be an interval in* $\mathbb{R}$. *Let* $h_1,h_2 \in C^2([c,d])$ *and let a be a continuous function such that*

$$\frac{1}{d-c}\int_c^d h_2(t)dt-\frac{h_2(c)+h_2(d)}{2}-\frac{1}{2(d-c)}\int_c^d (d-t)(c-t)a(t)h_2'(t)dt \neq 0.$$

Then there exists $\eta \in [c,d]$ *such that*

$$\frac{h_1''(\eta)-a(\eta)h_1'(\eta)}{h_2''(\eta)-a(\eta)h_2'(\eta)}=\frac{\frac{1}{d-c}\int_c^d h_1(t)dt-\frac{h_1(c)+h_1(d)}{2}-\frac{1}{2(d-c)}\int_c^d(d-t)(c-t)a(t)h_1'(t)dt}{\frac{1}{d-c}\int_c^d h_2(t)dt-\frac{h_2(c)+h_2(d)}{2}-\frac{1}{2(d-c)}\int_c^d(d-t)(c-t)a(t)h_2'(t)dt}.$$

Proof. Similar to the proof of Theorem 3.6. ∎

Theorem 3.9. *Let* $[c,d]$ *be an interval in* $\mathbb{R}$. *Let* $p:[c,d]\to\mathbb{R}$ *be non-negative function such that* $\int_c^d p(s)ds>0$ *and let g be a function such that*

$$\operatorname{Im} g \subseteq [c,d],\ \overline{g}=\frac{\int_c^d p(s)g(s)ds}{\int_c^d p(s)ds}.$$

Let $h_1,h_2 \in C^2([c,d])$ *and let a be a continuous function such that*

$$\frac{\int_c^d p(s)h_2(g(s))\,ds}{\int_c^d p(s)ds}-h_2(\overline{g})-\frac{1}{\int_c^d p(s)ds}\int_c^d p(s)\int_{\overline{g}}^{g(s)}(g(s)-t)a(t)h_2'(t)dt\,ds \neq 0$$

Then there exists $\eta \in [c,d]$ *such that*

$$\frac{h_1''(\eta)-a(\eta)h_1'(\eta)}{h_2''(\eta)-a(\eta)h_2'(\eta)}=\frac{\frac{\int_c^d p(s)h_1(g(s))\,ds}{\int_c^d p(s)ds}-h_1(\overline{g})-\frac{1}{\int_c^d p(s)ds}\int_c^d p(s)\int_{\overline{g}}^{g(s)}(g(s)-t)a(t)h_1'(t)dt\,ds}{\frac{\int_c^d p(s)h_2(g(s))\,ds}{\int_c^d p(s)ds}-h_2(\overline{g})-\frac{1}{\int_c^d p(s)ds}\int_c^d p(s)\int_{\overline{g}}^{g(s)}(g(s)-t)a(t)h_2'(t)dt\,ds}.$$

Proof. Similar to the proof of Theorem 3.6. ∎

Theorem 3.10. *Let $[c,d]$ be an interval in $\mathbb{R}$. Let $h_1, h_2 \in C^2([c,d])$ and let a be a continuous function such that*

$$\frac{1}{d-c}\int_c^d h_2(s)ds - h_2\left(\frac{c+d}{2}\right) - \frac{1}{d-c}\int_c^d \int_{\frac{c+d}{2}}^s (s-t)a(t)h_2'(t)dtds \neq 0.$$

Then there exists $\eta \in [c,d]$ such that

$$\frac{h_1''(\eta) - a(\eta)h_1'(\eta)}{h_2''(\eta) - a(\eta)h_2'(\eta)} = \frac{\frac{1}{d-c}\int_c^d h_1(s)ds - h_1\left(\frac{c+d}{2}\right) - \frac{1}{d-c}\int_c^d \int_{\frac{c+d}{2}}^s (s-t)a(t)h_1'(t)dtds}{\frac{1}{d-c}\int_c^d h_2(s)ds - h_2\left(\frac{c+d}{2}\right) - \frac{1}{d-c}\int_c^d \int_{\frac{c+d}{2}}^s (s-t)a(t)h_2'(t)dtds}.$$

Proof. Similar to the proof of Theorem 3.6. ∎

4. Exponential convexity and means of Cauchy type

First we recall some basic facts about convexity, log-convexity and log-convexity in the Jensen sense (see e.g. [3, 8, 11]).

Definition 4.1. *Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is* convex in the Jensen sense *on an interval I if for each $a, b \in I$*

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}$$

holds.

We recall that for a continuous function f, convexity and convexity in the Jensen sense are equivalent properties.

Definition 4.2. *A positive function $f : I \to (0,\infty)$ is said to be* logarithmically convex *if* $\log f$ *is convex function on I. For such function f, we shortly say f is log-convex. A positive function $f : I \to (0,\infty)$ is* log-convex in the Jensen sense *if for each $a, b \in I$*

$$f^2\left(\frac{a+b}{2}\right) \leq f(a)f(b)$$

holds, i.e., if $\log f$ *is convex in the Jensen sense.*

Lemma 4.1. *Let positive function $f : I \to (0,\infty)$ be log-convex and let $a_1, a_2, b_1, b_2 \in I$ be such that $a_1 \leq b_1$, $a_2 \leq b_2$ and $a_1 \neq a_2$, $b_1 \neq b_2$. Then the following inequality is valid*

$$\left[\frac{f(a_2)}{f(a_1)}\right]^{\frac{1}{a_2-a_1}} \leq \left[\frac{f(b_2)}{f(b_1)}\right]^{\frac{1}{b_2-b_1}}.$$

Next we recall some basic facts about exponential convexity (see e.g. [2, 6, 7]).

Definition 4.3. *A function $h : (a,b) \to \mathbb{R}$ is* exponentially convex *if it is continuous and*

$$\sum_{i,j=1}^{n} t_i t_j h(x_i + x_j) \geq 0,$$

holds for all $n \in \mathbb{N}$ and all choices $t_i, x_i \in \mathbb{R}$, $i = 1, \ldots, n$ such that $x_i + x_j \in (a,b)$, $1 \leq i, j \leq n$.

Lemma 4.2. *Let $h:(a,b)\to\mathbb{R}$. The following statements are equivalent:*

(i) *h is exponentially convex,*
(ii) *h is continuous and*

$$\sum_{i,j=1}^{n} t_i t_j h\left(\frac{x_i+x_j}{2}\right)\geq 0,$$

for every $n\in\mathbb{N}$, $t_i\in\mathbb{R}$ and every $x_i\in(a,b)$, $1\leq i\leq n$.

Remark 4.1. Condition from Lemma 4.2, part (ii) is equivalent with positive semi-definiteness of matrices

$$\left[h\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{n}, \tag{4.1}$$

for all $n\in\mathbb{N}$.

Remark 4.2. Note that for $n=2$ from (4.1) we get

$$h(x_1)h(x_2)-h^2\left(\frac{x_1+x_2}{2}\right)\geq 0,$$

hence, exponentially convex function is log-convex in the Jensen sense, and, being continuous, it is also log-convex function.

Lemma 4.3. *Let $q\in\mathbb{R}$. Then the function φ_q defined by*

$$\varphi_q(x)=\int\left(e^{\int a(x)dx}\int x^{q-2}e^{-\int a(x)dx}dx\right)dx \tag{4.2}$$

is $a(x)$-convex for $x>0$.

Proof. Since $\varphi_q''(x)-a(x)\varphi_q'(x)=x^{q-2}\geq 0$, $x>0$, therefore $\varphi_q(x)$ is $a(x)$-convex function for $x>0$. ∎

For $q\in\mathbb{R}$, let the function ξ be defined as follows:

$$\xi(q)=\begin{cases}\frac{\sum_{i=1}^{n}p_ix_i^q-\bar{x}^q}{q(q-1)}, & q\neq 0,1;\\ \ln\bar{x}-\sum_{i=1}^{n}p_i\ln x_i, & q=0;\\ \sum_{i=1}^{n}p_ix_i\ln x_i-\bar{x}\ln\bar{x}, & q=1,\end{cases} \tag{4.3}$$

where $p_i\in[0,1]$, $i=1,\dots,n$, $\sum_{i=1}^{n}p_i=1$ and $\bar{x}=\sum_{i=1}^{n}p_ix_i$. Obviously, we have that $\xi(q)>0$ for all $q\in\mathbb{R}$.

Lemma 4.4. *Let $q\in\mathbb{R}$, let the function φ_q be defined by (4.2) for mutually different numbers $x_i>0$, $i=1,\dots,n$ and let the function ξ be defined by (4.3). Then*

$$\sum_{i=1}^{n}p_i\varphi_q(x_i)-\varphi_q(\bar{x})-\sum_{i=1}^{n}p_i\int_{\bar{x}}^{x_i}(x_i-t)a(t)\varphi_q'(t)dt=\xi(q) \tag{4.4}$$

holds, where $p_i\in[0,1]$, $i=1,\dots,n$, $\sum_{i=1}^{n}p_i=1$ and $\bar{x}=\sum_{i=1}^{n}p_ix_i$.

Proof. Since $\varphi_q''(x) - a(x)\varphi_q'(x) = x^{q-2}$, we have $a(x)\varphi_q'(x) = \varphi_q''(x) - x^{q-2}$, so

$$\int_{\bar{x}}^{x_i} (x_i - t)a(t)\varphi_q'(t)dt = \int_{\bar{x}}^{x_i} (x_i - t)(\varphi_q''(t) - t^{q-2})dt.$$

Hence

$$\sum_{i=1}^n p_i\varphi_q(x_i) - \varphi_q(\bar{x}) - \sum_{i=1}^n p_i \int_{\bar{x}}^{x_i} (x_i - t)a(t)\varphi_q'(t)dt = \sum_{i=1}^n p_i\varphi_q(\bar{x}) - \varphi_q(\bar{x})$$

$$-\sum_{i=1}^n p_i \begin{cases} (\bar{x} - x_i)\varphi_q'(\bar{x}) + \varphi_q(x_i) - \varphi_q(\bar{x}) - x_i\frac{x_i^{q-1} - \bar{x}^{q-1}}{q-1} + \frac{x_i^q - \bar{x}^q}{q}, & q \neq 0,1; \\ (\bar{x} - x_i)\varphi_q'(\bar{x}) + \varphi_q(x_i) - \varphi_q(\bar{x}) + x_i\left(\frac{1}{x_i} - \frac{1}{\bar{x}}\right) + \ln x_i - \ln\bar{x}, & q = 0; \\ (\bar{x} - x_i)\varphi_q'(\bar{x}) + \varphi_q(x_i) - \varphi_q(\bar{x}) - x_i(\ln x_i - \ln\bar{x}) + x_i - \bar{x}, & q = 1; \end{cases}$$

$$= \xi(q)$$ ∎

In the following theorem we explore some properties of the mapping $q \mapsto \xi(q)$ (see also [1] and [12]).

Theorem 4.1. *Let $q \in \mathbb{R}$ and let the function ξ be defined by (4.3) for mutually different numbers $x_i > 0$, $i = 1, \ldots, n$. Let $p_i \in [0,1]$, $i = 1, \ldots, n$, $\sum_{i=1}^n p_i = 1$ and $\bar{x} = \sum_{i=1}^n p_i x_i$. Then*

- (i) *the function $q \mapsto \xi(q)$ is continuous on $\mathbb{R}$,*
- (ii) *for every $m \in \mathbb{N}$ and $q_j \in \mathbb{R}$, $j = 1, \ldots, m$, the matrix $[\xi((q_j + q_k)/2)]_{j,k=1}^m$ is a positive semi-definite matrix. Particularly*

$$\det\left[\xi\left(\frac{q_j + q_k}{2}\right)\right]_{j,k=1}^m \geq 0;$$

- (iii) *the function $q \mapsto \xi(q)$ is exponentially convex,*
- (iv) *the function $q \mapsto \xi(q)$ is log-convex.*

Proof. (i) In order to prove that the function $q \to \xi(q)$ is continuous on $\mathbb{R}$, we need to verify that $\lim_{q\to 0}\xi(q) = \xi(0)$ and $\lim_{q\to 1}\xi(q) = \xi(1)$. Both is obtained by a simple calculation. Hence, ξ is continuous on $\mathbb{R}$.

(ii) Let $m \in \mathbb{N}$, $t_j \in \mathbb{R}$, $q_j \in \mathbb{R}$, $j = 1, 2, \ldots, m$. Denote $q_{jk} = (q_j + q_k)/2$. Let φ_q be defined by (4.2). Consider the function $f : \mathbb{R}^+ \to \mathbb{R}$,

$$f(x) = \sum_{j,k=1}^m t_j t_k \varphi_{q_{jk}}(x).$$

Then

$$f''(x) - a(x)f'(x) = \sum_{j,k=1}^m t_j t_k \varphi_{q_{jk}}''(x) - a(x)\sum_{j,k=1}^m t_j t_k \varphi_{q_{jk}}'(x)$$

$$= \sum_{j,k=1}^m t_j t_k(\varphi_{q_{jk}}''(x) - a(x)\varphi_{q_{jk}}'(x)) = \sum_{j,k=1}^m t_j t_k x^{q_{jk}-2}$$

$$= \left(\sum_{j=1}^m t_j x^{(q_j-2)/2}\right)^2 \geq 0$$

Hence, f is $a(x)$-convex function.

Now we can apply (2.4) to the function f defined above, and obtain

$$\sum_{j,k=1}^{m} t_j t_k \left(\sum_{i=1}^{n} p_i \varphi_{q_{jk}}(x_i) - \varphi_{q_{jk}}(\bar{x}) - \sum_{i=1}^{n} p_i \int_{\bar{x}}^{x_i} (x_i - t) a(t) \varphi'_{q_{jk}}(t) dt \right) \geq 0.$$

Now, from (4.4) it follows that

$$\sum_{j,k=1}^{m} t_j t_k \xi(q_{jk}) \geq 0.$$

Therefore, the matrix $[\xi((q_j+q_k)/2)]_{j,k=1}^{m}$ is positive semi-definite.

(iii) Follows form (i), (ii) and Lemma 4.2.

(iv) Follows from (iii) and Remark 4.2. ∎

Theorem 3.6 enables us to define various types of means, because if the function $(h_1'' - ah_1')/(h_2'' - ah_2')$ has inverse, from (3.3) we have

$$\eta = \left(\frac{h_1'' - ah_1'}{h_2'' - ah_2'} \right)^{-1} \left(\frac{\sum_{i=1}^{n} p_i h_1(x_i) - h_1(\bar{x}) - \sum_{i=1}^{n} p_i \int_{\bar{x}}^{x_i} (x_i - t) a(t) h_1'(t) dt}{\sum_{i=1}^{n} p_i h_2(x_i) - h_2(\bar{x}) - \sum_{i=1}^{n} p_i \int_{\bar{x}}^{x_i} (x_i - t) a(t) h_2'(t) dt} \right), \quad \eta \in I.$$

Let us observe differential equations $h_1''(\eta) - a(\eta)h_1'(\eta) = \eta^{q-2}$ and $h_2''(\eta) - a(\eta)h_2'(\eta) = \eta^{s-2}$. Then from (3.3) we get

$$\eta = \left(\frac{\sum_{i=1}^{n} p_i h_1(x_i) - h_1(\bar{x}) - \sum_{i=1}^{n} p_i \int_{\bar{x}}^{x_i} (x_i - t) a(t) h_1'(t) dt}{\sum_{i=1}^{n} p_i h_2(x_i) - h_2(\bar{x}) - \sum_{i=1}^{n} p_i \int_{\bar{x}}^{x_i} (x_i - t) a(t) h_2'(t) dt} \right)^{\frac{1}{q-s}}.$$

From (4.4) we have

$$\eta = \left\{ \frac{s(s-1)}{q(q-1)} \cdot \frac{\sum_{i=1}^{n} p_i x_i^q - \bar{x}^q}{\sum_{i=1}^{n} p_i x_i^s - \bar{x}^s} \right\}^{\frac{1}{q-s}}.$$

Hence, we have mean

$$M(\mathbf{x}; q, s) = \left\{ \frac{s(s-1)}{q(q-1)} \cdot \frac{\sum_{i=1}^{n} p_i x_i^q - \bar{x}^q}{\sum_{i=1}^{n} p_i x_i^s - \bar{x}^s} \right\}^{\frac{1}{q-s}}, \tag{4.5}$$

where $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is n-tuple of mutually different numbers greater than zero, $q \neq s$, $q, s \neq 0, 1$. We have

$$M(\mathbf{x}; q, s) = \left(\frac{\xi(q)}{\xi(s)} \right)^{\frac{1}{q-s}},$$

where ξ is defined by (4.3). All continuous extensions of (4.5) are now obvious but the case $q = s$:

$$M(\mathbf{x};q,q) = Exp\left(\frac{\sum_{i=1}^{n} p_i x_i^q \ln x_i - \overline{x}^q \ln \overline{x}}{\sum_{i=1}^{n} p_i x_i^p - \overline{x}^q} + \frac{1-2q}{q(q-1)}\right), q \neq 0,1.$$

Remark 4.3. For $x_1,x_2,x_3 \in I$ such that $x_1 < x_3$, $x_2 = p_1x_1 + p_2x_2$ and $p_1 = (x_3-x_2)/(x_3-x_1)$, $p_2 = (x_2-x_1)/(x_3-x_1)$ from (4.5) we obtain Generalized Stolarsky mean (for $n=2$)

$$E(x_1,x_2,x_3;q,s) = \left\{\frac{[x_1,x_2,x_3;h_q]}{[x_1,x_2,x_3;h_s]}\right\}^{\frac{1}{q-s}}$$

where $h_q(x) = x^q/(q(q-1))$, $h_s(x) = x^s/(s(s-1))$, $q \neq s$, $q,s \neq 0,1$ and $[x_1,x_2,x_3;h_q]$ is divided difference of the function h_q on the knots x_1,x_2,x_3. Generalized Stolarsky means of order (p,q) $(p \neq q)$ of $\mathbf{x} = (x_0,x_1,\ldots,x_n)$ are introduced in [5]. All continuous extensions of Generalized Stolarsky means are known (see [5],[13]) and their monotonicity in both parameters is proven (see [4]).

5. Functions convex with respect to given function

Definition 5.1. *A function $f: I \to \mathbb{R}$ is convex with respect to strictly monotone function $h: J \to \mathbb{R}$, $J \subseteq I$ if $f \circ h^{-1}$ is convex.*

Remark 5.1.

(a) If f and h are differentiable functions and h is strictly increasing, then $f \circ h^{-1}$ is convex if and only if f'/h' is increasing.

(b) If f and h are twice differentiable functions and h is strictly increasing, then $f \circ h^{-1}$ is convex if and only if $f''(x)h'(x) - f'(x)h''(x) \geq 0$.

There are many results concerning functions convex with respect to given function, i.e. one generalization of the Jensen inequality (see also [9] and [10]).

Theorem 5.1. *Let $f: [0,b] \to \mathbb{R}$ and let $H: [0,b] \to \mathbb{R}$ be an increasing function such that f is convex with respect to H. If $p_i \geq 0$, $i = 1,\ldots,n$ such that $P_n = \sum_{i=1}^{n} p_i > 0$ holds, then*

$$f\left(H^{-1}\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i H(a_i)\right)\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(a_i). \tag{5.1}$$

If f is concave with respect to H, then the reverse inequality in (5.1) *holds.*

Remark 5.2. If f is twice differentiable and $a(x)$-convex function, then for $a(x) = H''(x)/H'(x)$, where H is twice differentiable and strictly increasing, we obtain from $a(x)$-convexity criteria given in Theorem 2.1 that

$$f''(x) - \frac{H''(x)}{H'(x)} f'(x) \geq 0,$$

$$\frac{f''(x)H'(x) - f'(x)H''(x)}{H'(x)} \geq 0$$

or $f''(x)H'(x) - f'(x)H''(x) \geq 0$ and we conclude that f'/H' is increasing function, i.e. $x \mapsto f(H^{-1}(x))$ is convex function. We proved that if f is $H''(x)/H'(x)$-convex (H is twice differentiable and strictly increasing), then $x \mapsto f(H^{-1}(x))$ is convex function.

Let $\mathbf{a}$ be n-tuple of mutually different numbers $a_i > 0$, $i = 1,\ldots,n$ and let $p_i \geq 0$ be such that $P_n = \sum_{i=1}^{n} p_i > 0$ holds. Then for a strictly monotone continuous function H, the quasi-arithmetic mean is defined as follows:

$$M_H(\mathbf{a};\mathbf{p}) = H^{-1}\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i H(a_i)\right) \tag{5.2}$$

From (5.2) we can deduce the power mean of order $t \in \mathbb{R}$ which is defined as follows:

$$M_t(\mathbf{a};\mathbf{p}) = \begin{cases} \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i^t\right)^{\frac{1}{t}}, & t \neq 0; \\ Exp\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i \ln a_i\right), & t = 0. \end{cases} \tag{5.3}$$

Lemma 5.1. *Let I be an open interval. Let $f,H \in C^2(I)$ be such that $H'(x) > 0$ for every $x \in I$ and*

$$m \leq \frac{f''(x)H'(x) - f'(x)H''(x)}{(H'(x))^3} \leq M.$$

Then functions Φ_1 and Φ_2 defined by

$$\begin{aligned} \Phi_1(x) &= \frac{1}{2}MH^2(x) - f(x), \\ \Phi_2(x) &= f(x) - \frac{1}{2}mH^2(x), \end{aligned} \tag{5.4}$$

are convex functions with respect to H.

Proof. Set $h(x) = \Phi_1\left[H^{-1}(x)\right] = \frac{1}{2}Mx^2 - f\left[H^{-1}(x)\right]$. We have

$$h''(x) = M - \frac{H'(H^{-1}(x))\,f''(H^{-1}(x)) - f'(H^{-1}(x))\,H''(H^{-1}(x))))}{(H'(H^{-1}(x)))^3} \geq 0.$$

So Φ_1 is convex function with respect to H. Similarly, we get that Φ_2 is convex function with respect to H. ∎

Remark 5.3. Functions Φ_1 and Φ_2 in Lemma 5.1 were obtained from Lemma 3.1 for $a(x) = H''(x)/H'(x)$, $M(x) = M(H'(x))^2$ and $m(x) = m(H'(x))^2$. After some calculation we get $R_1(x) = MH^2(x)/2$ and $R_2(x) = mH^2(x)/2$. Hence, we get (5.4).

Theorem 5.2. *Let $f,H \in C^2(I)$, I compact interval in $\mathbb{R}$, be such that f is convex with respect to H. Let $H'(x) > 0$ for every $x \in I$ and let $p_i \geq 0$, $i = 1,\ldots,n$ such that $P_n = \sum_{i=1}^{n} p_i > 0$ holds. Then there exists $\eta \in I$ such that*

$$\begin{aligned} &\frac{1}{P_n}\sum_{i=1}^{n} p_i f(a_i) - f(M_H(\mathbf{a};\mathbf{p})) \\ &= \frac{f''(\eta)H'(\eta) - f'(\eta)H''(\eta)}{2(H'(\eta))^3}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i H^2(a_i) - (H(M_H(\mathbf{a};\mathbf{p})))^2\right]. \end{aligned} \tag{5.5}$$

Proof. Since $(f''H' - f'H'')/(H')^3$ is continuous on compact interval I, there exists

$$m = \min_{t\in I}\left(\frac{f''(t)H'(t) - f'(t)H''(t)}{(H'(t))^3}\right) \text{ and } M = \max_{t\in I}\left(\frac{f''(t)H'(t) - f'(t)H''(t)}{(H'(t))^3}\right).$$

Applying inequality (5.1) on functions Φ_1 and Φ_2 from Lemma 5.1, the following inequalities hold:

$$\Phi_1\left(M_H(\mathbf{a};\mathbf{p})\right) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i\Phi_1(a_i),$$

$$\Phi_2\left(M_H(\mathbf{a};\mathbf{p})\right) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i\Phi_2(a_i).$$

It follows that,

$$\begin{aligned}
&\frac{m}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_iH^2(a_i) - \left(H(M_H(\mathbf{a};\mathbf{p}))\right)^2\right] \\
&\le \frac{1}{P_n}\sum_{i=1}^{n} p_if(a_i) - f\left(M_H(\mathbf{a};\mathbf{p})\right) \\
&\le \frac{M}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_iH^2(a_i) - \left(H(M_H(\mathbf{a};\mathbf{p}))\right)^2\right].
\end{aligned}$$

Therefore, there exists $\eta \in I$ such that (5.5) holds. ∎

Theorem 5.3. *Let I be an interval in $\mathbb{R}$. Let $f_1, f_2, H \in C^2(I)$ be such that f_1 and f_2 are convex with respect to H. Let $H'(x) > 0$ for every $x \in I$ and let $p_i \ge 0$, $i = 1, \ldots, n$ such that $P_n = \sum_{i=1}^{n} p_i > 0$ holds. Let*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_if_2(a_i) - f_2\left(M_H(\mathbf{a};\mathbf{p})\right) \ne 0.$$

Then there exists $\eta \in I$ such that

$$\frac{f_1''(\eta)H'(\eta) - f_1'(\eta)H''(\eta)}{f_2''(\eta)H'(\eta) - f_2'(\eta)H''(\eta)} = \frac{\frac{1}{P_n}\sum_{i=1}^{n} p_if_1(a_i) - f_1\left(M_H(\mathbf{a};\mathbf{p})\right)}{\frac{1}{P_n}\sum_{i=1}^{n} p_if_2(a_i) - f_2\left(M_H(\mathbf{a};\mathbf{p})\right)}. \tag{5.6}$$

Proof. Similar to the proof of Theorem 3.6. ∎

Lemma 5.2. *Let $t \in \mathbb{R}$ and let $H \in C^2(I)$ be such that $H'(x) > 0$ for every $x \in I$. Then the function Ψ_t defined by*

$$\Psi_t(x) = \int\left(H'(x)\int\frac{x^{t-2}}{H'(x)}dx\right)dx \tag{5.7}$$

is convex with respect to H for $x > 0$.

Proof. Since

$$\Psi_t''(x) - \frac{H''(x)}{H'(x)}\Psi_t'(x) = x^{t-2} \ge 0,\ x > 0,$$

therefore $\Psi_t(x)$ is $H''(x)/H'(x)$-convex function for $x > 0$. Hence, $\Psi_t(x)$ is convex with respect to H. ∎

Remark 5.4. Function Ψ_t in Lemma 5.2 was obtained from Lemma 4.3 for $a(x) = H''(x)/H'(x)$.

Remark 5.5. Additionaly taking $H(x) = x^q$ for $x > 0$, $q \in \mathbb{R}$ in Lemma 5.2 we get

$$\psi_t(x) = \begin{cases} \frac{x^t}{t(t-q)} & t \neq 0, q; \\ -\frac{\ln x}{q} & t = 0, q \neq 0; \\ \frac{1}{t}\left(x^t \ln x - \frac{x^t}{t}\right) & t = q. \end{cases} \tag{5.8}$$

Lemma 5.3. *Let $t \in \mathbb{R}$ and let the function Ψ_t be defined by (5.7). Let* **a** *be n-tuple of mutually different numbers $a_i > 0, i = 1, \ldots, n$ and let* **p** *be a real n-tuple such that $P_n = \sum_{i=1}^{n} p_i > 0$ holds. Then*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i \Psi_t(a_i) - \Psi_t(M_H(\mathbf{a};\mathbf{p})) = \Psi_t(M_{\Psi_t}(\mathbf{a};\mathbf{p})) - \Psi_t(M_H(\mathbf{a};\mathbf{p})). \tag{5.9}$$

Proof. Applying (5.2) to the left-hand side of (5.9) we get the right-hand side of (5.9). ∎

Let us define a right-hand side in (5.9) as

$$\rho(t) = \Psi_t(M_{\Psi_t}(\mathbf{a};\mathbf{p})) - \Psi_t(M_H(\mathbf{a};\mathbf{p})). \tag{5.10}$$

In the following theorem we explore some properties of the mapping $t \mapsto \rho(t)$.

Theorem 5.4. *Let $t \in \mathbb{R}$ and let the function ρ be defined by (5.10) for mutually different numbers $a_i > 0, i = 1, \ldots, n$ and for $p_i \geq 0$ such that $P_n = \sum_{i=1}^{n} p_i > 0$ holds. Assume that the function $t \mapsto \rho(t)$ is continuous. Then*

(i) *for every $m \in \mathbb{N}$ and $t_j \in \mathbb{R}$, $j = \ldots, m$, the matrix $[\rho(t_j + t_k/2)]_{j,k=1}^{m}$ is a positive semi-definite matrix. Particularly*

$$\det\left[\rho\left(\frac{t_j + t_k}{2}\right)\right]_{j,k=1}^{m} \geq 0;$$

(ii) *the function $t \mapsto \rho(t)$ is exponentially convex,*

(iii) *the function $t \mapsto \rho(t)$ is log-convex.*

Proof. Similar to the proof of Theorem 4.1. ∎

Lemma 5.4. *Let $t \in \mathbb{R}$ and let the function ψ_t be defined by (5.8). Let* **a** *be n-tuple of mutually different numbers $a_i > 0, i = 1, \ldots, n$ and let* **p** *be a real n-tuple such that $P_n = \sum_{i=1}^{n} p_i > 0$. Then*

$$\begin{aligned} &\frac{1}{P_n}\sum_{i=1}^{n} p_i \psi_t(a_i) - \psi_t(M_H(\mathbf{a};\mathbf{p})) \\ &= \begin{cases} \frac{1}{t(t-q)}\left[M_t^t(\mathbf{a};\mathbf{p}) - M_q^t(\mathbf{a};\mathbf{p})\right], & t \neq 0, q; \\ \frac{1}{q}\left[\ln M_q(\mathbf{a};\mathbf{p}) - \ln M_0(\mathbf{a};\mathbf{p})\right], & t = 0, q \neq 0; \\ \frac{1}{q}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i^q \ln a_i - M_q^q(\mathbf{a};\mathbf{p}) \ln M_q(\mathbf{a};\mathbf{p})\right], & t = q \neq 0. \end{cases} \end{aligned} \tag{5.11}$$

Proof. Applying (5.8) and (5.3) to the left-hand side of (5.11) we get the right-hand side of (5.11). ∎

Let us define the right-hand side in (5.11) as

$$\zeta(t) = \begin{cases} \frac{1}{t(t-q)}\left[M_t^t(\mathbf{a};\mathbf{p}) - M_q^t(\mathbf{a};\mathbf{p})\right], & t \neq 0, q; \\ \frac{1}{q}\left[\ln M_q(\mathbf{a};\mathbf{p}) - \ln M_0(\mathbf{a};\mathbf{p})\right], & t = 0; \\ \frac{1}{q}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i^q \ln a_i - M_q^q(\mathbf{a};\mathbf{p}) \ln M_q(\mathbf{a};\mathbf{p})\right], & t = q. \end{cases} \tag{5.12}$$

Obviously, we have that $\zeta(t) > 0$ for all $t \in \mathbb{R}$.

In the following theorem we explore some properties of the mapping $t \mapsto \zeta(t)$.

Theorem 5.5. *Let $t \in \mathbb{R}$ and let the function ζ be defined by (5.12) for mutually different numbers $a_i > 0$, $i = 1, \ldots, n$ and for $p_i \geq 0$ such that $P_n = \sum_{i=1}^{n} p_i > 0$ holds. Then*

(i) *the function $t \mapsto \zeta(t)$ is continuous on $\mathbb{R}$,*

(ii) *for every $m \in \mathbb{N}$ and $t_j \in \mathbb{R}$, $j = 1, \ldots, m$, the matrix $[\zeta(t_j + t_k/2)]_{j,k=1}^m$ is a positive semi-definite matrix. Particularly*

$$\det\left[\zeta\left(\frac{t_j + t_k}{2}\right)\right]_{j,k=1}^m \geq 0;$$

(iii) *the function $t \mapsto \zeta(t)$ is exponentially convex,*

(iv) *the function $t \mapsto \zeta(t)$ is log-convex.*

Proof. Similar to the proof of Theorem 4.1. ∎

Theorem 5.3 enables us to define various types of means, because if the function $(f_1''H' - f_1'H'')/(f_2''H' - f_2'H'')$ has inverse, from (5.6) and (5.9) we have

$$\eta = \left(\frac{f_1''H' - f_1'H''}{f_2''H' - f_2'H''}\right)^{-1}\left(\frac{f_1(M_{f_1}(\mathbf{a};\mathbf{p})) - f_1(M_H(\mathbf{a};\mathbf{p}))}{f_2(M_{f_2}(\mathbf{a};\mathbf{p})) - f_2(M_H(\mathbf{a};\mathbf{p}))}\right), \eta \in I.$$

Let us observe differential equations

$$f_1''(\eta) - \frac{H''(\eta)}{H'(\eta)} f_1'(\eta) = \eta^{t-2} \text{ and } f_2''(\eta) - \frac{H''(\eta)}{H'(\eta)} f_2'(\eta) = \eta^{s-2}.$$

Then from (5.6) and (5.10) we have mean

$$M_{t,s}(\mathbf{a};\mathbf{p}) = \left\{\frac{f_1(M_{f_1}(\mathbf{a};\mathbf{p})) - f_1(M_H(\mathbf{a};\mathbf{p}))}{f_2(M_{f_2}(\mathbf{a};\mathbf{p})) - f_2(M_H(\mathbf{a};\mathbf{p}))}\right\}^{\frac{1}{t-s}}.$$

Additionaly taking $H(x) = x^q$, then from (5.6) and (5.11) we have mean

$$M_{t,s}^q(\mathbf{a};\mathbf{p}) = \left\{\frac{s(s-q)}{t(t-q)} \frac{M_t^t(\mathbf{a};\mathbf{p}) - M_q^t(\mathbf{a};\mathbf{p})}{M_s^s(\mathbf{a};\mathbf{p}) - M_q^s(\mathbf{a};\mathbf{p})}\right\}^{\frac{1}{t-s}} \tag{5.13}$$

where $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ is n-tuple of mutually different numbers greater that zero and $p_i \geq 0$ such that $P_n = \sum_{i=1}^{n} p_i > 0$ holds, $t \neq s \neq q \neq t$ and $t, s \neq 0$. Hence, we have

$$M_{t,s}^{q}(\mathbf{a};\mathbf{p}) = \left(\frac{\zeta(t)}{\zeta(s)} \right)^{\frac{1}{t-s}},$$

where ζ is defined by (5.12). All continuous extensions of (5.13) can be found in [1], so here we omit the details. Monotonicity of these means can also be found in [1].

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