BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **35**(3) (2012), 717–726

Subclasses of Multivalent Harmonic Mappings Defined by Convolution

[1]K. G. SUBRAMANIAN, [2]B. ADOLF STEPHEN AND [3]S. K. LEE
[1]School of Computer Sciences, Universiti Sains Malaysia, 11800 USM Penang, Malaysia
[2]Department of Mathematics, Madras Christian College, Chennai 600059, India
[3]School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM Penang, Malaysia
[1]kgs@cs.usm.my, [2]adolfmcc2003@yahoo.co.in, [3]sklee@cs.usm.my

Abstract. A new subclass of multivalent harmonic functions defined by convolution is introduced in this paper. The subclass generates known subclasses of multivalent harmonic functions, and thus provides a unified treatment in the study of these subclasses. Sufficient coefficient conditions are obtained that are also shown to be necessary when the functions have negative coefficients. Growth estimates and extreme points are also determined. In addition conditions for starlikeness of the Dziok-Srivastava linear operator involving the generalized hypergeometric functions are discussed.

2010 Mathematics Subject Classification: Primary 30C45; Secondary 30C50

Keywords and phrases: Harmonic functions, convolution, fully starlike and fully convex harmonic maps, Dziok-Srivastava linear operator.

1. Introduction

Harmonic mappings are important in the study of minimal surfaces due to their role [17] in parametrizing minimal surfaces. Although harmonic mappings need not be analytic, they have been studied from the perspective of geometric function theory as generalizations of conformal mappings. The seminal works of Clunie and Sheil-Small [16] as well as Sheil-Small [29] showed that while certain classical results for conformal mappings have analogues for harmonic mappings, many other basic questions remain unsolved.

A multivalent harmonic function f in a simply connected domain can be expressed in the form $f = h + \overline{g}$, where both h and g are analytic. The function h is called the analytic part while g is the co-analytic part of f. Denote by $S_H^0(m)$, $m \geq 1$, the class of multivalent harmonic functions $f = h + \overline{g}$, where

$$h(z) = z^m + \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1} \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_{n+m-1} z^{n+m-1}, z \in U,$$

normalized by the conditions $h(0) = h'(0) = \cdots = h^{(m-1)}(0) = h^{(m)}(0) - m! = 0$ and $g(0) = g'(0) = \cdots = g^m(0) = 0$.

Communicated by V. Ravichandran.
Received: August 24, 2011; *Revised:* November 10, 2011.

Convexity and starlikeness are well-known hereditary properties of analytic univalent functions f in the unit disk U. In other words, if f maps U onto a convex domain, then the image of every subdisk $|z| < r < 1$ is also a convex domain. Likewise, if f with $f(0) = 0$ maps U onto a domain starlike with respect to the origin, the image of every subdisk $|z| < r < 1$ is also a domain starlike with respect to the origin. However these hereditary properties do not extend to harmonic mappings. Chuaqui *et al.* [15] considered starlike and convex harmonic mappings that do inherit the hereditary properties. They called such functions fully starlike and fully convex harmonic mappings respectively.

Various subclasses of harmonic univalent functions have been introduced and studied by several authors [2, 7, 10–12, 14, 19–24, 26, 27, 31, 33]. In particular, the investigation by Silverman [31] gives a sufficient coefficient condition for harmonic functions $f = h + \overline{g}$ to belong to the well-known classes ${S_H^*}^0$ and K_H^0. The proof of the coefficient condition in [31] for the class ${S_H^*}^0$ rests on showing that $\partial/\partial\theta(argf(re^{i\theta})) > 0, 0 < r < 1, 0 \leq \theta < 2\pi$. The latter condition is equivalent to the analytic description in [15] for fully starlike harmonic functions. Therefore the coefficient condition for ${S_H^*}^0$ obtained in [31] is sufficient for functions to be fully starlike. It is known that fully starlike harmonic functions need not be univalent. Thus the coefficient bounds in [31] also provides a sufficient condition for univalence of fully starlike functions. Fully convex harmonic functions on the other hand are known to be univalent. In this instance, the coefficient bound for convexity obtained by Silverman [31] will not only be sufficient for fully convex harmonic functions, but in the case of harmonic functions $f = h + \overline{g}$ with negative coefficients, the coefficient condition will also be necessary. Thus the coefficient condition obtained in [31] is necessary and sufficient for fully convex harmonic functions with negative coefficients.

Several authors [3–6, 13, 20, 25, 28] have investigated various subclasses of multivalent harmonic functions. In this work, we introduce a new subclass of multivalent harmonic functions defined by convolution.

Definition 1.1. *Let σ be a real constant with $|\sigma| = 1$ and $0 \leq \alpha < 1$. Let $\Phi_\sigma(z) = \phi_1(z) + \sigma\overline{\phi_2(z)}$ be a given multivalent harmonic function in $U = \{z : |z| < 1\}$, where ϕ_1 and ϕ_2 are of the form*

$$\phi_1(z) = z^m + \sum_{n=2}^{\infty} A_{n+m-1} z^{n+m-1} \quad \text{and} \quad \phi_2(z) = z^m + \sum_{n=2}^{\infty} B_{n+m-1} z^{n+m-1}. \tag{1.1}$$

A multivalent harmonic function $f = h + \overline{g} \in S_H^0(m)$ where

$$h(z) = z^m + \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1}, \quad g(z) = \sum_{n=2}^{\infty} b_{n+m-1} z^{n+m-1}, \tag{1.2}$$

belongs to the class $S_H^0(\Phi_\sigma, m, \alpha)$ if $\Phi_\sigma * f \in S_H^0(m)$ satisfies the inequality

$$\operatorname{Re}\left\{ \frac{z(h*\phi_1)'(z) - \sigma\overline{z(g*\phi_2)'(z)}}{(h*\phi_1)(z) + \sigma\overline{(g*\phi_2)(z)}} \right\} > m\alpha, \quad (z \in U). \tag{1.3}$$

Here $*$ is the convolution operator given by

$$\begin{aligned}(\Phi_\sigma * f)(z) &= (\phi_1 + \sigma\overline{\phi_2}) * (h + \overline{g})(z)\\ &= z^m + \sum_{n=2}^{\infty} a_{n+m-1} A_{n+m-1} z^{n+m-1} + \sigma\overline{\sum_{n=2}^{\infty} b_{n+m-1} B_{n+m-1} z^{n+m-1}}.\end{aligned}$$

The subclasses $S_H^0(m,\alpha)$ of multivalent harmonic starlike functions and $K_H^0(m,\alpha)$ of multivalent harmonic convex functions investigated by Ahuja and Jahangiri [5] can in fact be expressed in the form

$$S_H^0(\Phi_1,m,\alpha)=S_H^0(m,\alpha) \quad \text{and} \quad S_H^0(\Phi_{-1},m,\alpha)=K_H^0(m,\alpha) \tag{1.4}$$

with

$$\Phi_1(z)=\frac{z^m}{1-z}+\frac{\overline{z}^m}{1-\overline{z}}$$

and

$$\Phi_{-1}(z)=\frac{1}{m}\Big[\frac{z^m}{(1-z)^2}-\frac{(1-m)z^m}{(1-z)}\Big]-\frac{1}{m}\Big[\frac{\overline{z}^m}{(1-\overline{z})^2}-\frac{(1-m)\overline{z}^m}{(1-\overline{z})}\Big],$$

respectively. It is clear then that the subclass $S_H^0(\Phi_\sigma,m,\alpha)$ can generate known subclasses of multivalent harmonic mappings, and provides a unified treatment in the study of these subclasses.

In Section 2 of this paper, a necessary and sufficient convolution condition is obtained for $S_H^0(\Phi_\sigma,m,\alpha)$ which as an application yields a sufficient coefficient condition for the class. In Section 3 an appropriate general class $TS_H^0(\Phi_\sigma,m,\alpha)$ of harmonic functions with negative coefficients is defined, and necessary and sufficient coefficient conditions are obtained. Section 3 is also devoted to determining growth estimates and extreme points for the class. In Section 4, conditions for starlikeness of the Dziok-Srivastava linear operator involving the generalized hypergeometric functions are discussed. Since many operators can be expressed in terms of the hypergeometric functions, the results obtained here will be useful for such operators.

2. Main results

We now derive a convolution characterization for functions in the class $S_H^0(\Phi_\sigma,m,\alpha)$.

Theorem 2.1. *Let $f=h+\overline{g}\in S_H^0(m)$. Then $f\in S_H^0(\Phi_\sigma,m,\alpha)$ if and only if*

$$(h*\phi_1)*\left[\frac{z^m+\frac{x+1-2m+2m\alpha}{2m-2m\alpha}z^{m+1}}{(1-z)^2}\right]-\sigma\overline{(g*\phi_2)}*\left[\frac{\frac{x+\alpha}{1-\alpha}\overline{z}^m-\frac{(2m-1)x-1+2m\alpha}{2m-2m\alpha}\overline{z}^{m+1}}{(1-\overline{z})^2}\right]\neq 0,$$

where $|x|=1, 0<|z|<1$.

Proof. A necessary and sufficient condition for $f=h+\overline{g}$ to be in the class $S_H^0(\Phi_\sigma,m,\alpha)$, with h and g of the form (1.2), is given by (1.3). Since

$$\frac{z(h*\phi_1)'(z)-\sigma\overline{z(g*\phi_2)'(z)}}{(h*\phi_1)(z)+\sigma\overline{(g*\phi_2)(z)}}=m$$

at $z=0$, the condition (1.3) is equivalent to

(2.1)
$$\frac{1}{m(1-\alpha)}\left\{\frac{z(h*\phi_1)'(z)-\sigma\overline{z(g*\phi_2)'(z)}}{(h*\phi_1)(z)+\sigma\overline{(g*\phi_2)(z)}}-m\alpha\right\}\neq\frac{x-1}{x+1}; \quad |x|=1, x\neq-1, 0<|z|<1.$$

By a simple algebraic manipulation, (2.1) yields

$$\begin{aligned}0\neq{}&(x+1)[z(h*\phi_1)'(z)-\sigma\overline{z(g*\phi_2)'(z)}]-m\alpha(x+1)[(h*\phi_1)(z)+\sigma\overline{(g*\phi_2)(z)}]\\&-m(x-1)(1-\alpha)[(h*\phi_1)(z)+\sigma\overline{(g*\phi_2)(z)}]\end{aligned}$$

$$= (h*\phi_1)*\left[(x+1)\left(\frac{z^m}{(1-z)^2}-\frac{(1-m)z^m}{1-z}\right)-\frac{(xm+2m\alpha-m)z^m}{1-z}\right]$$
$$-\sigma\overline{(g*\phi_2)}*\overline{\left[(\bar{x}+1)\left(\frac{z^m}{(1-z)^2}-\frac{(1-m)z^m}{1-z}\right)+\frac{(\bar{x}m+2m\alpha-m)z^m}{1-z}\right]}.$$

The latter condition together with (1.3) for $x=-1$ establishes the result for all $|x|=1$. ∎

Sufficient coefficient conditions for the multivalent harmonic starlike functions and multivalent harmonic convex functions were obtained in [5]. Here an application of Theorem 2.1 yields a sufficient coefficient condition for multivalent harmonic functions to belong to the class $S_H^0(\Phi_\sigma,m,\alpha)$.

Theorem 2.2. *Let $f=h+\overline{g}\in S_H^0(m)$. Then $f\in S_H^0(\Phi_\sigma,m,\alpha)$ if*

$$\sum_{n=2}^{\infty}\frac{n+m(1-\alpha)-1}{m(1-\alpha)}|a_{n+m-1}||A_{n+m-1}|+\sum_{n=2}^{\infty}\frac{n+m(1+\alpha)-1}{m(1-\alpha)}|b_{n+m-1}||B_{n+m-1}|\le 1.$$

Proof. For h and g given by (1.2), Theorem 2.1 gives

$$\left|(h*\phi_1)*\left[\frac{z^m+\frac{x+1-2m+2m\alpha}{2m-2m\alpha}z^{m+1}}{(1-z)^2}\right]-\sigma\overline{(g*\phi_2)}*\left[\frac{\frac{x+\alpha}{1-\alpha}\overline{z}^m-\frac{(2m-1)x-1+2m\alpha}{2m-2m\alpha}\overline{z}^{m+1}}{(1-\overline{z})^2}\right]\right|$$
$$=\left|z^m+\sum_{n=2}^{\infty}\left[n+(n-1)\frac{x+1-2m+2m\alpha}{2m-2m\alpha}\right]a_{n+m-1}A_{n+m-1}z^{n+m-1}\right.$$
$$\left.-\sigma\sum_{n=2}^{\infty}\left[n\frac{x+\alpha}{1-\alpha}-(n-1)\frac{(2m-1)x-1+2m\alpha}{2m-2m\alpha}\right]\overline{b_{n+m-1}B_{n+m-1}z^{n+m-1}}\right|$$
$$>|z^m|\left[1-\sum_{n=2}^{\infty}\frac{n+m(1-\alpha)-1}{m(1-\alpha)}|a_{n+m-1}||A_{n+m-1}|\right.$$
$$\left.-\sum_{n=2}^{\infty}\frac{n+m(1+\alpha)-1}{m(1-\alpha)}|b_{n+m-1}||B_{n+m-1}|\right].$$

The last expression is non-negative by hypothesis, and hence by Theorem 2.1, it follows that $f\in S_H^0(\Phi_\sigma,m,\alpha)$. ∎

The sufficient coefficient conditions for the classes $S_H^0(m,\alpha)$ (and $K_H^0(m,\alpha)$) can be obtained from Theorem 2.2 on using the relations (1.4). These are given in the following two corollaries:

Corollary 2.1. [5] *Let $f=h+\overline{g}\in S_H^0(m)$. Then $f\in S_H^0(m,\alpha)$ if*

$$\sum_{n=2}^{\infty}\frac{n+m(1-\alpha)-1}{m(1-\alpha)}|a_{n+m-1}|+\sum_{n=2}^{\infty}\frac{n+m(1+\alpha)-1}{m(1-\alpha)}|b_{n+m-1}|\le 1.$$

Corollary 2.2. [5] *Let $f=h+\overline{g}\in S_H^0(m)$. Then $f\in K_H^0(m,\alpha)$ if*

$$\sum_{n=2}^{\infty}\frac{(n+m-1)(n+m(1-\alpha)-1)}{m^2(1-\alpha)}|a_{n+m-1}|$$
$$+\sum_{n=2}^{\infty}\frac{(n+m-1)(n+m(1+\alpha)-1)}{m^2(1-\alpha)}|b_{n+m-1}|\le 1.$$

3. Harmonic maps with negative coefficients

Following the work of Silverman [30], several subclasses of analytic functions with negative coefficients have been introduced and studied. A unified class of analytic p-valent functions with negative coefficients defined by convolution was introduced in [9] that included many well-known subclasses of analytic functions with negative coefficients as special cases. In this section, we shall devote attention to the subclass $TS_H^0(\Phi_\sigma, m, \alpha)$ of $S_H^0(\Phi_\sigma, m, \alpha)$ consisting of multivalent harmonic functions $f = h + \bar{g}$ of the form

$$h(z) = z^m - \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1}, \quad g(z) = \sigma \sum_{n=2}^{\infty} b_{n+m-1} z^{n+m-1}, \tag{3.1}$$

where $a_{n+m-1} \geq 0$, $b_{n+m-1} \geq 0$. Let $\Phi_\sigma(z) = \phi_1(z) + \sigma\overline{\phi_2(z)}$ where ϕ_1 and ϕ_2 are of the form

$$\phi_1(z) = z^m + \sum_{n=2}^{\infty} A_{n+m-1} z^{n+m-1}, \quad \phi_2(z) = z^m + \sum_{n=2}^{\infty} B_{n+m-1} z^{n+m-1}, \tag{3.2}$$

with $A_{n+m-1} \geq 0, B_{n+m-1} \geq 0$. The subclass $TS_H^0(\Phi_\sigma, m, \alpha)$ includes as special cases the subclasses $TS_H^0(m, \alpha)$ and $TK_H^0(m, \alpha)$.

Theorem 3.1. *For f of the form (3.1), $f \in TS_H^0(\Phi_\sigma, m, \alpha)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{n+m(1-\alpha)-1}{m(1-\alpha)} a_{n+m-1} A_{n+m-1} + \sum_{n=2}^{\infty} \frac{n+m(1+\alpha)-1}{m(1-\alpha)} b_{n+m-1} B_{n+m-1} \leq 1. \tag{3.3}$$

Proof. If f belongs to $TS_H^0(\Phi_\sigma, m, \alpha)$, then (1.3) is equivalent to

$$\text{Re } \psi(z) > 0$$

for $z \in U$, where $\psi = F/G$ and

$$F(z) = m(1-\alpha)z^m - \sum_{n=2}^{\infty} (n+m(1-\alpha)-1) a_n A_{n+m-1} z^{n+m-1}$$
$$- \sum_{n=2}^{\infty} (n+m(1+\alpha)-1) b_{n+m-1} B_{n+m-1} \bar{z}^{n+m-1}$$

$$G(z) = z^m - \sum_{n=2}^{\infty} a_{n+m-1} \phi_{n+m-1} z^{n+m-1} + \sum_{n=2}^{\infty} b_{n+m-1} \phi_{n+m-1} \bar{z}^{n+m-1}.$$

Letting $z \to 1^-$ through real values yields condition (3.3). Conversely, for h and g given by (3.1),

$$\left| (h*\phi_1) * \left[\frac{z^m + \frac{x+1-2m+2m\alpha}{2m-2m\alpha} z^{m+1}}{(1-z)^2} \right] - \sigma\overline{(g*\phi_2)} * \left[\frac{\frac{x+\alpha}{1-\alpha}\bar{z}^m - \frac{(2m-1)x-1+2m\alpha}{2m-2m\alpha}\bar{z}^{m+1}}{(1-\bar{z})^2} \right] \right|$$
$$> |z| \Bigg[1 - \sum_{n=2}^{\infty} \frac{n+m(1-\alpha)-1}{m(1-\alpha)} |a_{n+m-1}||A_{n+m-1}|$$
$$- \sum_{n=2}^{\infty} \frac{n+m(1+\alpha)-1}{m(1-\alpha)} |b_{n+m-1}||B_{n+m-1}| \Bigg]$$

which is non-negative by hypothesis, thus proving sufficiency of condition (3.3). ∎

Corollary 3.1. *Let Φ_σ be of the form (3.2) with $A_{n+m-1} \geq B_{m+1}, B_{n+m-1} \geq B_{m+1}$, $(n \geq 2)$. If $f \in TS_H^0(\Phi_\sigma, m, \alpha)$, then for $|z| = r < 1$,*

$$r^m - \frac{m(1-\alpha)}{(1+m(1-\alpha))B_{m+1}} r^{m+1} \leq |f(z)| \leq r^m + \frac{m(1-\alpha)}{(1+m(1-\alpha))B_{m+1}} r^{m+1}.$$

The result is sharp with equality for $f(z) = z^m - \frac{m(1-\alpha)}{(1+m(1-\alpha))B_{m+1}} z^{m+1}$.

Proof. We have

$$\begin{aligned}
&[1+m(1-\alpha)]B_{m+1} \left[\sum_{n=2}^{\infty} (a_{n+m-1} + b_{n+m-1}) \right] \\
&\qquad \leq \sum_{n=2}^{\infty} [(n+m(1-\alpha)-1)a_{n+m-1}A_{n+m-1} + (n+m(1+\alpha)-1)b_{n+m-1}B_{n+m-1}] \\
&\qquad \leq m(1-\alpha).
\end{aligned}$$

Thus,

$$\begin{aligned}
|f(z)| &= \left| z^m - \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1} + \sigma \sum_{n=2}^{\infty} b_{n+m-1} \overline{z}^{n+m-1} \right| \\
&\leq r^m + r^{m+1} \left[\sum_{n=2}^{\infty} (a_{n+m-1} + b_{n+m-1}) \right] \\
&\leq r^m + \frac{m(1-\alpha)}{[1+m(1-\alpha)]B_{m+1}} r^{m+1}.
\end{aligned}$$

The sharp lower bound is obtained in a similar manner. ∎

We now determine its extreme points.

Theorem 3.2. *Let*

$$h_m(z) := z^m, h_{n+m-1}(z) := z^m - \frac{m(1-\alpha)}{[n+m(1-\alpha)-1]A_{n+m-1}} z^{n+m-1},$$

and

$$g_{n+m-1}(z) := z^m + \frac{m(1-\alpha)}{\sigma[n+m(1+\alpha)-1]B_{n+m-1}} \overline{z}^{n+m-1}, \quad (n = 2, 3, \ldots).$$

A function $f \in TS_H^0(\Phi_\sigma, m, \alpha)$ if and only if f can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} (\lambda_{n+m-1} h_{n+m-1}(z) + \gamma_{n+m-1} g_{n+m-1}(z)),$$

where $\lambda_{n+m-1} \geq 0, \gamma_{n+m-1} \geq 0, \lambda_m = 1 - \sum_{n=2}^{\infty} (\lambda_{n+m-1} + \gamma_{n+m-1})$ and $\gamma_m = 0$. In particular, the extreme points of $TS_H^0(\Phi_\sigma, m, \alpha)$ are $\{h_{n+m-1}\}$ and $\{g_{n+m-1}\}$.

Proof. Let

$$\begin{aligned}
f(z) &= \sum_{n=1}^{\infty} (\lambda_{n+m-1} h_{n+m-1}(z) + \gamma_{n+m-1} g_{n+m-1}(z)) \\
&= z^m - \sum_{n=2}^{\infty} \lambda_{n+m-1} \frac{m(1-\alpha)}{(n+m(1-\alpha)-1)A_{n+m-1}} z^{n+m-1}
\end{aligned}$$

$$+\sigma\sum_{n=1}^{\infty}\gamma_{n+m-1}\frac{m(1-\alpha)}{(n+m(1+\alpha)-1)B_{n+m-1}}\overline{z}^{n+m-1}.$$

Since

$$\begin{aligned}\sum_{n=2}^{\infty}&\frac{n+m(1-\alpha)-1}{m(1-\alpha)}\lambda_{n+m-1}\frac{m(1-\alpha)}{(n+m(1-\alpha)-1)A_{n+m-1}}A_{n+m-1}\\&+\sum_{n=2}^{\infty}\frac{n+m(1+\alpha)-1}{m(1-\alpha)}\gamma_{n+m-1}\frac{m(1-\alpha)}{(n+m(1+\alpha)-1)B_{n+m-1}}B_{n+m-1}\\&=\sum_{n=2}^{\infty}(\lambda_{n+m-1}+\gamma_{n+m-1})=1-\lambda_m\leq 1,\end{aligned}$$

it follows from Theorem 3.1 that $f\in TS_H^0(\Phi_\sigma,m,\alpha)$.

Conversely, if $f\in TS_H^0(\Phi_\sigma,m,\alpha)$, then

$$a_{n+m-1}\leq\frac{m(1-\alpha)}{(n+m(1-\alpha)-1)A_{n+m-1}},\quad b_{n+m-1}\leq\frac{m(1-\alpha)}{(n+m(1+\alpha)-1)B_{n+m-1}}.$$

Set

$$\lambda_{n+m-1}=\frac{(n+m(1-\alpha)-1)}{m(1-\alpha)}a_{n+m-1}A_{n+m-1},$$
$$\gamma_{n+m-1}=\frac{(n+m(1+\alpha)-1)}{m(1-\alpha)}b_{n+m-1}B_{n+m-1}$$

and

$$\lambda_m=1-\sum_{n=2}^{\infty}(\lambda_{n+m-1}+\gamma_{n+m-1}),\quad \gamma_m=0.$$

Then it is easily seen that $\sum_{n=1}^{\infty}(\lambda_{n+m-1}h_{n+m-1}(z)+\gamma_{n+m-1}g_{n+m-1}(z))=f(z)$. ∎

4. The Dziok-Srivastava linear operator

Let us denote by $S(m)$ the class of all analytic functions f in U of the form

$$f(z)=z^m+\sum_{n=2}^{\infty}a_{n+m-1}z^{n+m-1}.$$

For $\alpha_j\in\mathbb{C}\quad(j=1,2,\ldots,p)$ and $\beta_n\in\mathbb{C}\setminus\{0,-1,-2,\ldots\}$ $(n=1,2,\ldots q)$, the *generalized hypergeometric function* ${}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z)$ in U is defined by the infinite series

$${}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z):=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}\quad(p\leq q+1;p,q\in\mathbb{N}_0:=\{0,1,2,\ldots\}),$$

where $(a)_n$ is the Pochhammer symbol given by

$$(a)_n:=\frac{\Gamma(a+n)}{\Gamma(a)}=\begin{cases}1, & (n=0);\\ a(a+1)(a+2)\cdots(a+n-1), & (n\in\mathbb{N}).\end{cases}$$

It is known [32, p. 43] that the ${}_pF_q$ series is absolutely convergent in $\mathbb{C}$ if $p\leq q$, and in U if $p=q+1$. Furthermore, if

$$\operatorname{Re}\left(\sum_{j=1}^{q}\beta_j-\sum_{j=1}^{p}\alpha_j\right)>0,$$

then the ${}_pF_q$ series is absolutely convergent for $|z| = 1$. Corresponding to the function ${}_pF_q(\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;z)$, the Dziok-Srivastava operator [18]

$$H^{(p,q)}(\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q): S(m) \to S(m)$$

is defined by the Hadamard product

$$\begin{aligned} H^{(p,q)}(\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q)f(z) &:= z^m \ {}_pF_q(\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;z) * f(z) \\ &= z^m + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1}\dots(\alpha_p)_{n-1}}{(\beta_1)_{n-1}\cdots(\beta_q)_{n-1}} \frac{a_{n+m-1}z^{n+m-1}}{(n-1)!}. \end{aligned}$$

For convenience, we write

$$z\ {}_pF_q[\alpha;\beta;z] := z\ {}_pF_q(\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;z),$$

$$H^{p,q}[\alpha;\beta]f(z) := H^{(p,q)}(\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q)f(z).$$

Corresponding to $f = h + \overline{g}$ given by (1.2), we define an operator $\mathscr{L}$ on f given by

$$\mathscr{L}[f] = \Phi_\sigma * f = (\phi_1 + \sigma\overline{\phi_2}) * (h + \overline{g}), \tag{4.1}$$

where

$$\begin{aligned} \phi_1(z) &= z^m \ {}_pF_q[\lambda;\beta;z] = z^m + \textstyle\sum_{n=2}^{\infty} A_{n+m-1}z^{n+m-1}, \\ \phi_2(z) &= z^m \ {}_lF_m[c;d;z] = z^m + \textstyle\sum_{n=2}^{\infty} B_{n+m-1}z^{n+m-1}, \end{aligned} \tag{4.2}$$

and

$$A_{n+m-1} = \frac{(\lambda_1)_{n-1}\cdots(\lambda_p)_{n-1}}{(\beta_1)_{n-1}\cdots(\beta_q)_{n-1}} \frac{1}{(n-1)!}, \qquad B_{n+m-1} = \frac{(c_1)_{n-1}\cdots(c_l)_{n-1}}{(d_1)_{n-1}\cdots(d_m)_{n-1}} \frac{1}{(n-1)!}. \tag{4.3}$$

Here we are certainly assuming that none of the denominator parameters can be zero or a negative integer. A similar operator to $\mathscr{L}$ defined by (4.1) was studied by Ahuja *et al.* [8]. Harmonic starlikeness and convexity of integral operators related to hypergeometric series was also recently investigated in [1]. It is interesting to note that the functions ϕ_1 and ϕ_2 considered in [1] are taken to be integral operators associated with the Gauss hypergeometric functions.

Theorem 4.1. *Let $f = h + \overline{g} \in S_H^0(m)$ be of the form* (1.2)*, where the coefficients a_{n+m-1} and b_{n+m-1} satisfy*

$$|a_{n+m-1}| \le \frac{m(1-\alpha)}{n+m(1-\alpha)-1}, \quad \text{and} \quad |b_{n+m-1}| \le \frac{m(1-\alpha)}{n+m(1+\alpha)-1}, \ (n \ge 2). \tag{4.4}$$

Let ϕ_1 and ϕ_2 of the form (4.2) *satisfy*

$$\sum_{j=1}^{q} \beta_j > \sum_{j=1}^{p} |\lambda_j|, \quad \sum_{j=1}^{m} d_j > \sum_{j=1}^{l} |c_j|,$$

where $\beta_j > 0$ $(j = 1,\dots,q)$ and $d_j > 0$ $(j = 1,\dots,m)$. If

$${}_pF_q[|\lambda|;\beta;1] + {}_lF_m[|c|;d;1] \le 3 \tag{4.5}$$

holds, then $\mathscr{L}[f] \in S_H(m,\alpha)$.

Proof. In view of Theorem 2.2, it suffices to show that $S \leq m(1-\alpha)$, where

(4.6)
$$S := \sum_{n=2}^{\infty}(n+m(1-\alpha)-1)|a_{n+m-1}||A_{n+m-1}| + \sum_{n=2}^{\infty}(n+m(1+\alpha)-1)|b_{n+m-1}||B_{n+m-1}|,$$

where A_{n+m-1} and B_{n+m-1} are given by (4.3). Thus

$$\begin{aligned} S &\leq m(1-\alpha)\sum_{n=2}^{\infty}[|A_{n+m-1}|+|B_{n+m-1}|] \\ &\leq m(1-\alpha)\left\{\sum_{n=2}^{\infty}\frac{(|\lambda_1|)_{n-1}\cdots(|\lambda_p|)_{n-1}}{(\beta_1)_{n-1}\cdots(\beta_q)_{n-1}}\frac{1}{(n-1)!}+\sum_{n=2}^{\infty}\frac{(|c_1|)_{n-1}\cdots(|c_l|)_{n-1}}{(d_1)_{n-1}\cdots(d_m)_{n-1}}\frac{1}{(n-1)!}\right\} \\ &= m(1-\alpha)\left\{{}_pF_q[|\lambda|;\beta;1]-1+{}_lF_m[|c|;d;1]-1\right\} \\ &\leq m(1-\alpha), \end{aligned}$$

provided (4.5) holds. ∎

Note that the hypergeometric condition (4.5) is independent of α.

Example 4.1. Let $l=2=p$, $m=1=q$, $\beta>1+|\lambda|$, and $d>1+|c|$ in Theorem 4.1. The Gauss summation formula [32, p. 30] gives

$${}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad Re(c-a-b)>0.$$

Using the property that $\Gamma(z+1)=z\Gamma(z)$ and the Gauss summation formula, the condition (4.5) reduces to

$$\frac{\beta-1}{\beta-|\lambda|-1}+\frac{d-1}{d-|c|-1}\leq 3.$$

Acknowledgement. The authors thank the referees for their useful comments. The authors would like to thank Prof. Dato' Rosihan M. Ali for his valuable guidance.

References

[1] O. P. Ahuja, Harmonic starlikeness and convexity of integral operators generated by hypergeometric series, *Integral Transforms Spec. Funct.* **20** (2009), no. 7–8, 629–641.

[2] O. P. Ahuja, Use of theory of conformal mappings in harmonic univalent mappings with directional convexity, *Bull. Malays. Math. Sci. Soc. (2)* **35** (2012), no. 3, 775–784.

[3] O. P. Ahuja and H. Ö. Güney, Multivalent harmonic mappings convoluted with a multivalent analytic function, *Aust. J. Math. Anal. Appl.* **5** (2008), no. 2, Art. 3, 7 pp.

[4] O. P. Ahuja, H. Ö. Güney and F. Müge Sakar, Certain classes of harmonic multivalent functions based on Hadamard product, *J. Inequal. Appl.* **2009**, Art. ID 759251, 12 pp.

[5] O. P. Ahuja and J. M. Jahangiri, Multivalent harmonic starlike functions, *Ann. Univ. Mariae Curie-Skłodowska Sect. A* **55** (2001), 1–13.

[6] O. P. Ahuja and J. M. Jahangiri, Multivalent harmonic convex functions, *Math. Sci. Res. J.* **11** (2007), no. 9, 537–543.

[7] O. P. Ahuja, J. M. Jahangiri and H. Silverman, Convolutions for special classes of harmonic univalent functions, *Appl. Math. Lett.* **16** (2003), no. 6, 905–909.

[8] O. P. Ahuja, S. B. Joshi and A. Swaminathan, Multivalent harmonic convolution operators associated with generalized hypergeometric functions, preprint.

[9] R. M. Ali, M. H. Khan, V. Ravichandran and K.G. Subramanian, A class of multivalent functions with negative coefficients defined by convolution, *Bull. Korean Math. Soc.* **43** (2006), no. 1, 179–188.

[10] R. M. Ali, B. A. Stephen and K. G. Subramanian, Subclasses of harmonic mappings defined by convolution, *Appl. Math. Lett.* **23** (2010), no. 10, 1243–1247.
[11] R. M. Ali, B. A. Stephen, K. G. Subramanian and S. K. Lee, Convolution of harmonic mappings on the exterior unit disk and the generalized hypergeometric functions, *Bull. Belg. Math. Soc. Simon Stevin* **18** (2011), no. 2, 239–251.
[12] K. Al-Shaqsi and M. Darus, On harmonic functions defined by derivative operator, *J. Inequal. Appl.* **2008**, Art. ID 263413, 10 pp.
[13] W. G. Atshan, S. R. Kulkarni and R. K. Raina, A class of multivalent harmonic functions involving a generalized Ruscheweyh type operator, *Mat. Vesnik* **60** (2008), no. 3, 207–213.
[14] Sh. Chen, S. Ponnusamy and X. Wang, Coefficient estimates and Landau-Bloch's constant for planar harmonic mappings, *Bull. Malays. Math. Sci. Soc. (2)* **34** (2011), no. 2, 255–265.
[15] M. Chuaqui, P. Duren and B. Osgood, Curvature properties of planar harmonic mappings, *Comput. Methods Funct. Theory* **4** (2004), no. 1, 127–142.
[16] J. Clunie and T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **9** (1984), 3–25.
[17] P. Duren, *Harmonic Mappings in the Plane*, Cambridge Tracts in Mathematics, 156, Cambridge Univ. Press, Cambridge, 2004.
[18] J. Dziok and H. M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, *Integral Transforms Spec. Funct.* **14** (2003), no. 1, 7–18.
[19] J. M. Jahangiri, Coefficient bounds and univalence criteria for harmonic functions with negative coefficients, *Ann. Univ. Mariae Curie-Skłodowska Sect. A* **52** (1998), no. 2, 57–66.
[20] J. M. Jahangiri, Harmonic functions starlike in the unit disk, *J. Math. Anal. Appl.* **235** (1999), no. 2, 470–477.
[21] J. M. Jahangiri, G. Murugusundaramoorthy and K. Vijaya, Salagean-type harmonic univalent functions, *Southwest J. Pure Appl. Math.* **2002**, no. 2, 77–82.
[22] S. B. Joshi and M. Darus, Unified treatment for harmonic univalent functions, *Tamsui Oxf. J. Math. Sci.* **24** (2008), no. 3, 225–232.
[23] P. Li, S. Ponnusamy and X. Wang, Some properties of planar *p*-harmonic and log-*p*-harmonic mappings, *Bull. Malays. Math. Sci. Soc. (2)*, accepted.
[24] G. Murugusundaramoorthy, A class of Ruscheweyh-type harmonic univalent functions with varying arguments, *Southwest J. Pure Appl. Math.* **2003**, no. 2, 90–95.
[25] R. Omar and S. A. Halim, Multivalent harmonic functions defined by Dziok-Srivastava operator, *Bull. Malays. Math. Sci. Soc. (2)* **35** (2012), no. 3, 601–610.
[26] T. Rosy, B. Adolf Stephen, K.G. Subramanian and J. M. Jahangiri, Goodman-Rønning-type harmonic univalent functions, *Kyungpook Math. J.* **41** (2001), no. 1, 45–54.
[27] T. Rosy, B. Adolf Stephen, K.G. Subramanian and J. M. Jahangiri, Goodman-type harmonic convex functions, *J. Nat. Geom.* **21** (2002), no. 1–2, 39–50.
[28] P. Sharma and N. Khan, Harmonic multivalent functions involving a linear operator, *Int. J. Math. Anal. (Ruse)* **3** (2009), no. 5–8, 295–308.
[29] T. Sheil-Small, Constants for planar harmonic mappings, *J. London Math. Soc. (2)* **42** (1990), no. 2, 237–248.
[30] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.* **51** (1975), 109–116.
[31] H. Silverman, Harmonic univalent functions with negative coefficients, *J. Math. Anal. Appl.* **220** (1998), no. 1, 283–289.
[32] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Ellis Horwood Series: Mathematics and its Applications, Horwood, Chichester, 1984.
[33] S. Yalçin, M. Öztürk and M. Yamankaradeniz, On the subclass of Salagean-type harmonic univalent functions, *JIPAM. J. Inequal. Pure Appl. Math.* **8** (2007), no. 2, Article 54, 9 pp.