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Uniqueness Results for a Nonlinear Differential Polynomial

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Abstract. In this paper, we study a uniqueness question of meromorphic functions whose nonlinear differential polynomials share a nonzero small function. The results in this paper improve and extend many known results.

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1. Introduction and main results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [5], [6] and [15]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$ as $r \to \infty, r \notin E$.

Let f and g be two nonconstant meromorphic functions, and let a be a finite value. We say that f and g share the value a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM (see [16]). We say that a is a small function of f, if a is a meromorphic function satisfying $T(r,a) = S(r,f)$ (see [15]). We also need the following two definitions.

Definition 1.1. [1, 8] *Let p be a positive integer and $a \in C \cup \{\infty\}$. Then by $N_{p)}(r,1/(f-a))$ we denote the counting function of those zeros of $f-a$ (counted with proper multiplicities) whose multiplicities are not greater than p, by $\overline{N}_{p)}(r,1/(f-a))$ we denote the corresponding reduced counting function (ignoring multiplicities). By $N_{(p}(r,1/\ (f-a))$ we denote the counting function of those zeros of $f-a$ (counted with proper multiplicities) whose multiplicities are not less than p, by $\overline{N}_{(p}(r,1/(f-a))$ we denote the corresponding reduced counting function (ignoring multiplicities).*

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Definition 1.2. *Let a be an any value in the extended complex plane, and let k be an arbitrary nonnegative integer. We define*

$$N_k\left(r,\frac{1}{f-a}\right)=\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}_{(2}\left(r,\frac{1}{f-a}\right)+\cdots+\overline{N}_{(k}\left(r,\frac{1}{f-a}\right).$$

In 1976, Yang asked the following question.

Question 1.1. [17] What can be said about the relationship between two entire functions f and g, if f, g share 0 CM and $f^{(n)}$, $g^{(n)}$ share 1 CM, where n is a nonnegative integer, and $2\delta(0,f)>1$?

In 1990, Yi proved the following theorem, which answered Question 1.1.

Theorem 1.1. [18] *Let f and g be two nonconstant entire functions. Assume that f, g share 0 CM and $f^{(n)}$, $g^{(n)}$ share 1 CM and $2\delta(0,f)>1$, where n is a nonnegative integer. Then either $f^{(n)}g^{(n)}=1$ or $f=g$.*

In 1997, Lahiri posed the following question.

Question 1.2. [7] What can be said about the relationship between two meromorphic functions whose non-linear differential polynomials share 1 CM ?

In 2002, Fang and Fang proved the following result.

Theorem 1.2. [4] *Let f and g be two nonconstant meromorphic functions and $n\geq 13$ be an integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share 1 CM, then $f=g$.*

In 2004, Lin and Yi proved the following result, which corresponds to Theorem B in view of fixed point.

Theorem 1.3. [8] *Let f and g be two nonconstant meromorphic functions and $n\geq 13$ be an integer. If $f^n(f-1)^2f'-z$ and $g^n(g-1)^2g'-z$ share 0 CM, then $f=g$.*

We will deal with Question 1.2. To this end we employ the idea of weighted sharing of values which measures how close a shared value is to being shared IM or to being shared CM. The notion is explained in the following definition.

Definition 1.3. [9] *Let k be a nonnegative integer or infinity. For any $a\in C\cup\{\infty\}$, we denote by $E_k(a,f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m\leq k$, and $k+1$ times if $m>k$. If $E_k(a,f)=E_k(a,g)$, we say that f, g share the value a with weight k.*

Remark 1.1. Definition 1.1 implies that if f, g share a value a with weight k, then z_0 is a zero of $f-a$ with multiplicity $m\,(\leq k)$ if and only if it is a zero of $g-a$ with multiplicity $m\,(\leq k)$, and z_0 is a zero of $f-a$ with multiplicity $m\,(>k)$, if and only if it is a zero of $g-a$ with multiplicity $n\,(>k)$, where m is not necessarily equal to n. Throughout this paper, we write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly, if f, g share (a,k), then f, g share (a,p) for all integer p, $0\leq p<k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞), respectively.

Using the idea of weighted sharing, many mathematicians in the world have got some interesting results on uniqueness questions of meromorphic functions having weighted sharing values by now (see [11] and [12], for example). In this direction, we recall the following result proved by Lahiri and Sahoo in 2008, which improves Theorems 1.1–1.3 and deals with Question 1.2.

Theorem 1.4. [13, Theorem 1.1] *Let f and g be two nonconstant meromorphic functions and $\alpha \not\equiv 0,\infty$ be a small function of f and g, and let n and $k\,(\geq 2)$ be two positive integers such that $f^n(f^k-a)f'-\alpha$ and $g^n(g^k-a)g'-\alpha$ share $(0,m)$, where $a \neq 0$ is a complex number. Then $f=g$ or $f=-g$, provided that one of the following three cases holds:*

(i) $m \geq 2$ *and* $n > \max\{4, k+10-2\Theta(\infty,f)-2\Theta(\infty,g)-\min\{\Theta(\infty,f),\Theta(\infty,g)\}\}$;

(ii) $m=1$ *and* $n > \max\{4, 3k/2+12-3\Theta(\infty,f)-3\Theta(\infty,g)\}$;

(iii) $m=0$ *and* $n > \{4, 4k+22-5\Theta(\infty,f)-5\Theta(\infty,g)-\min\{\Theta(\infty,f),\Theta(\infty,g)\}\}$.

In this direction, Banerjee and Mukherjee proved the following result in 2008.

Theorem 1.5. [2, Theorem 1.1] *Let f and g be two transcendental meromorphic functions, and let $\alpha \not\equiv 0,\infty$ be a small function of f and g. If $f^n(af^2+bf+c)f'-\alpha$ and $g^n(ag^2+bg+c)g'-\alpha$ share $(0,2)$, where $a\neq 0$ is a complex number, b and c are complex numbers satisfying $|b|+|c|\neq 0$. Then one of the following cases will hold:*

(i) *If $b\neq 0$, $c=0$ and $n>\max\{12-2\Theta(\infty,f)-2\Theta(\infty,g)-\min\{\Theta(\infty,f),\Theta(\infty,g)\},\chi\}$, where n is an integer, $\Theta(\infty,f)+\Theta(\infty,g)>0$ and $\chi=4/(\Theta(\infty,f)+\Theta(\infty,g))-2$, then $f=g$.*

(ii) *If $b\neq 0$, $c\neq 0$ and $n>[12-2\Theta(\infty,f)-2\Theta(\infty,g)\}-\min\{\Theta(\infty,f),\Theta(\infty,g)\}]$, the roots of $az^2+bz+c=0$ are distinct and one of f and g is a meromorphic function that is not an entire function and has only multiple poles, then $f=g$.*

(iii) *If $b\neq 0$, $c\neq 0$, $n>[12-2\Theta(\infty,f)-2\Theta(\infty,g)-\min\{\Theta(\infty,f),\Theta(\infty,g)\}]$ and the roots of $az^2+bz+c=0$ coincides, then $f=g$.*

(iv) *If $b=0$, $c\neq 0$ and $n>[12-2\Theta(\infty,f)-2\Theta(\infty,g)-\min\{\Theta(\infty,f),\Theta(\infty,g)\}]$, then either $f=g$ or $f=-g$. Moreover, $f=-g$ does not arise, if n is an even integer.*

We will prove the following result, which improves Theorems 1.4–1.5 and deals with Question 1.2.

Theorem 1.6. *Let f and g be two nonconstant meromorphic functions. If $f^n(f^{k_1}+af^{k_2}+b)f'-\alpha$ and $g^n(g^{k_1}+ag^{k_2}+b)g'-\alpha$ share $(0,m)$, where n,k_1,k_2 are three positive integers such that $n>k_1+2$ and $k_1>k_2$, a and b are complex numbers such that $|a|+|b|\neq 0$, m is a nonnegative integer, $\alpha(\not\equiv 0,\infty)$ be a small function of f and g, and if the algebraic equation $\omega^{k_1}+a\omega^{k_2}+b=0$ has no multiple roots, then*

$$\frac{f^{n+k_1+1}}{n+k_1+1}+\frac{af^{n+k_2+1}}{n+k_2+1}+\frac{bf^{n+1}}{n+1}=\frac{g^{n+k_1+1}}{n+k_1+1}+\frac{ag^{n+k_2+1}}{n+k_2+1}+\frac{bg^{n+1}}{n+1}, \tag{1.1}$$

provided that one of the following three cases holds.

(i) $m\geq 2$ *and* $n>\max\{\chi_1,\chi_2\}$ *where*

$$\chi_1=k_1+10-\delta(0,f)-2\Theta(0,f)-2\Theta(\infty,f)-\delta(0,g)-2\Theta(0,g)-3\Theta(\infty,g) \tag{1.2}$$

and

$$\chi_2=k_1+10-\delta(0,g)-2\Theta(0,g)-2\Theta(\infty,g)-\delta(0,f)-2\Theta(0,f)-3\Theta(\infty,f). \tag{1.3}$$

(ii) $m=1$ *and* $n>\max\{\chi_3,\chi_4\}-1$, *where*

$$\chi_3=12+\frac{3k_1}{2}-4\Theta(0,f)-3\Theta(\infty,f)-3\Theta(0,g)-3\Theta(\infty,g) \tag{1.4}$$

and

$$\chi_4=12+\frac{3k_1}{2}-4\Theta(0,g)-3\Theta(\infty,g)-3\Theta(0,f)-3\Theta(\infty,f). \tag{1.5}$$

(iii) *$m = 0$ and $n > \max\{\chi_5, \chi_6\}$, where*

$$\chi_5 = 21 + 4k_1 - 6\Theta(0, f) - 6\Theta(\infty, f) - 5\Theta(0, g) - 5\Theta(\infty, g) \tag{1.6}$$

and

$$\chi_6 = 21 + 4k_1 - 6\Theta(0, g) - 6\Theta(\infty, g) - 5\Theta(0, f) - 5\Theta(\infty, f). \tag{1.7}$$

By Theorem 1.6 we get the following uniqueness theorem.

Theorem 1.7. *Let f and g be two nonconstant meromorphic functions such that $f^n(f^{k_1} + af^{k_2} + b)f' - \alpha$ and $g^n(g^{k_1} + ag^{k_2} + b)g' - \alpha$ share $(0, m)$, where $\alpha \not\equiv 0, \infty$ is a small function of f and g, and n, k_1, k_2 are positive integers such that $n > k_1 + 2$ and $k_1 > k_2$, m is a nonnegative integer, a and b are two complex numbers satisfying $|a| + |b| \neq 0$, such that the algebraic equation $\omega^{k_1} + a\omega^{k_2} + b = 0$ has no multiple roots. Suppose that one of* (i)-(iii) *in Theorem 1.6 holds. Then*

(i) *If $a = 0$ and $\Theta(\infty, f) > 2/(n+2)$, then $f = tg$, where t is a constant satisfying $t^{k_1} = 1$.*

(ii) *If $b = 0$ and $\Theta(\infty, f) > 2/(n + k_2 + 1)$, then $f = tg$, where t is a constant satisfying $t^{k_1 - k_2} = 1$.*

(iii) *If $a \neq 0$ and $b \neq 0$, and if g is a meromorphic function that has only multiple poles and*

$$\overline{N}(r, |f/g = \eta_j, g \neq \infty) = S(r, f/g) \tag{1.8}$$

for $l + 1 \leq j \leq n + k_1 + 1$, where $\overline{N}(r, |f/g = \eta_j, g \neq \infty)$ denotes the reduced counting function of those η_j-points of f/g which are not poles of g, and η_j $(l + 1 \leq j \leq n + k_1 + 1)$ are $n + k_1 - l + 1$ distinct roots of $\omega^{n+k_1+1} = 1$ that are not the common roots of $\omega^{n+k_1+1} = 1$ and $\omega^{n+k_2+1} = 1$, then $f = tg$, where t is a constant satisfying $t^{k_1} = t^{k_2} = 1$.

Remark 1.2. Let f and g be two nonconstant meromorphic functions, and let

$$F = \frac{f^{n+3}}{n+3} - \frac{2f^{n+2}}{n+2} + \frac{f^{n+1}}{n+1}, \quad G = \frac{g^{n+3}}{n+3} - \frac{2g^{n+2}}{n+2} + \frac{g^{n+1}}{n+1}.$$

In Theorem 1.2, the condition that $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share 1 CM is equivalent to the condition that F' and G' share 1 CM, in Theorem 1.3, the condition that $f^n(f-1)^2 f' - z$ and $g^n(g-1)^2 g' - z$ share 0 CM is equivalent to the condition that $F' - z$ and $G' - z$ share 0 CM. Thus Theorems 1.2 and 1.3 improve Theorem 1.1 for $n = 1$. Similarly we can see that Theorems 1.4 improves Theorem 1.3, Theorem 1.5 improves Theorem 1.4, and Theorem 1.7 improves Theorem 1.5. Thus the differential polynomials appeared in Theorems 1.2–1.5 and Theorems 1.6 and 1.7 are important.

Remark 1.3. The following result can be found in [18]: If and only if

$$\frac{b^{k_1 - k_2}}{a^{k_1}} \neq \frac{(-1)^{k_1} k_2^{k_2} (k_1 - k_2)^{k_1 - k_2}}{k_1^{k_1}},$$

the algebraic equation $\omega^{k_1} + a\omega^{k_2} + b = 0$ has k_1 distinct simple roots and no multiple roots, where k_1 and k_2 are two positive integer such that $k_1 > k_2$, a and b are two nonzero complex numbers.

2. Some lemmas

Lemma 2.1. [14] *Let f be a nonconstant meromorphic function, and let*

$$F = \sum_{k=0}^{p} a_k f^k \Big/ \sum_{j=0}^{q} b_j f^j$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_p \neq 0$ and $b_q \neq 0$. Then $T(r,F) = d\,T(r,f) + O(1)$, where $d = \max\{p,q\}$.

Lemma 2.2. *Let f and g be two nonconstant meromorphic functions and $\alpha \not\equiv 0, \infty$ be a small function of f and g. If n, k_1, k_2 are three positive integers such that $n \geq 4$ and $k_1 > k_2$, and if the algebraic equation $\omega^{k_1} + a\omega^{k_2} + b = 0$ has no multiple roots, where a and b are two complex numbers such that $|a| + |b| \neq 0$, then $f^n(f^{k_1} + af^{k_2} + b)f'g^n(g^{k_1} + ag^{k_2} + b)g' \not\equiv \alpha^2$.*

Remark 2.1. Lemma 2.2 improves Lemma 2.8 in [2]. For convenience we will give the proof of Lemma 2.2, the tools and methods of the proof are the same as those in the proof of [2, Lemma 2.8] originally.

Proof. Suppose that

$$f^n(f^{k_1} + af^{k_2} + b)f'g^n(g^{k_1} + ag^{k_2} + b)g' = \alpha^2. \tag{2.1}$$

Let $z_0 \notin S_0$ be a zero of f with multiplicity p, where S_0 is a set defined as

$$S_0 = \{z : \alpha(z) = 0\} \cup \{z : \alpha(z) = \infty\}. \tag{2.2}$$

Then it follows from (2.1) that z_0 is a pole of g with multiplicity q, say, such that

$$np + p - 1 = nq + k_1 q + q + 1, \tag{2.3}$$

By rewriting (2.3) as

$$k_1 q + 2 = (n+1)(p-q), \tag{2.4}$$

we get $q \geq (n-1)/k_1$. This together with (2.4) implies

$$p \geq \frac{1}{n+1} \cdot \left(\frac{(n+k_1+1)(n-1)}{k_1} + 2 \right) = \frac{n+k_1-1}{k_1}. \tag{2.5}$$

Let $z_1 \notin S_0$ be a zero of $f^{k_1} + af^{k_2} + b$ with multiplicity p_1. Then it follows from (2.1) that z_1 is a pole of g with multiplicity q_1, say. This together with (2.1) implies that $2p_1 - 1 = (n+k_1+1)q_1 + 1 \geq n + k_1 + 2$. Thus

$$p_1 \geq \frac{n+k_1+3}{2}. \tag{2.6}$$

Let $z_2 \notin S_0$ be a pole of f. Then it follows from (2.1) that z_2 is either a zero of g' or a zero of $g^n(g^{k_1}+ag^{k_2}+b)$. Thus from (2.5)–(2.6) and Lemma 2.1 we get

$$\begin{aligned}\overline{N}(r,f) &\le \overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g^{k_1}+ag^{k_2}+b}\right)+\overline{N}_0\left(r,\frac{1}{g'}\right)+S(r,f)+S(r,g)\\ &\le \frac{k_1}{n+k_1-1}N\left(r,\frac{1}{g}\right)+\frac{2}{n+k_1+3}N\left(r,\frac{1}{g^{k_1}+ag^{k_2}+b}\right)\\ &\quad+\overline{N}_0\left(r,\frac{1}{g'}\right)+S(r,f)+S(r,g)\\ &\le \left(\frac{k_1}{n+k_1-1}+\frac{2k_1}{n+k_1+3}\right)T(r,g)+\overline{N}_0\left(r,\frac{1}{g'}\right)+S(r,f)+S(r,g),\end{aligned}$$

where $\overline{N}_0(r,1/g')$ denotes the reduced counting function of those zeros of g' which are not zeros of $g(g^{k_1}+ag^{k_2}+b)$.

Let

$$f^{k_1}+af^{k_2}+b=(f-\mu_1)(f-\mu_2)\cdots(f-\mu_{k_1}), \tag{2.7}$$

where $\mu_1, \mu_2, \cdots, \mu_{k_1-1}, \mu_{k_1}$ are k_1 distinct roots of the algebraic equation

$$\omega^{k_1}+a\omega^{k_2}+b=0.$$

We discuss the following two cases.

Case 1. Suppose that $b\neq 0$. Then $\mu_j\neq 0$ for $1\le j\le k_1$. By (2.5)–(2.7) and the second fundamental theorem we get

$$\begin{aligned}k_1T(r,f) &\le \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\sum_{j=1}^{k_1}\overline{N}\left(r,\frac{1}{f-\mu_j}\right)-\overline{N}_0\left(r,\frac{1}{f'}\right)+S(r,f)\\ &\le \left(\frac{k_1}{n+k_1-1}+\frac{2k_1}{n+k_1+3}\right)\{T(r,f)+T(r,g)\}\\ &\quad+\overline{N}_0\left(r,\frac{1}{g'}\right)-\overline{N}_0\left(r,\frac{1}{f'}\right)+S(r,f)+S(r,g).\end{aligned}$$

Similarly

$$\begin{aligned}k_1T(r,g) &\le \left(\frac{k_1}{n+k_1-1}+\frac{2k_1}{n+k_1+3}\right)\{T(r,f)+T(r,g)\}\\ &\quad+\overline{N}_0\left(r,\frac{1}{f'}\right)-\overline{N}_0\left(r,\frac{1}{g'}\right)+S(r,f)+S(r,g).\end{aligned}$$

From the above inequality we get

$$\begin{aligned}k_1\{T(r,f)+T(r,g)\} &\le \left(\frac{2k_1}{n+k_1-1}+\frac{4k_1}{n+k_1+3}\right)\{T(r,f)+T(r,g)\}\\ &\quad+S(r,f)+S(r,g).\end{aligned}$$

From (2.8) we get

$$k_1\le \frac{2k_1}{n+k_1-1}+\frac{4k_1}{n+k_1+3},$$

and so

$$\frac{2}{n+k_1-1}+\frac{4}{n+k_1+3}\geq 1, \tag{2.8}$$

which contradicts the assumptions $n\geq 4$ and $k_1\geq 2$.

Case 2. Suppose that $b=0$. Then $|a|\neq 0$. From the condition that the algebraic equation $\omega^{k_1}+a\omega^{k_2}+b=0$ has no multiple roots we get $k_2=1$. Without loss of generality, let $\mu_j\neq 0$ for $1\leq j\leq k_1-1$ and $\mu_{k_1}=0$. Also (2.1) can be rewritten as

$$f^{n+1}(f^{k_1-1}+a)f'g^{n+1}(g^{k_1-1}+a)g'=\alpha^2. \tag{2.9}$$

Proceeding as in Case 1, we can get (2.8) from (2.9), which is impossible. Lemma 2.2 is thus completely proved. ∎

Lemma 2.3. [10] *Let f and g be two nonconstant meromorphic functions sharing $(1,2)$. If $f\not\equiv g$ and $fg\not\equiv 1$, then*

$$T(r)\leq N_2\left(r,\frac{1}{f}\right)+N_2\left(r,\frac{1}{g}\right)+N_2(r,f)+N_2(r,g)+S(r),$$

where $T(r)=\max\{T(r,f),T(r,g)\}$, $S(r)=o\{T(r)\}$, as $r\longrightarrow\infty$, possibly outside a set of finite linear measure.

Lemma 2.4. [16, Lemma 1.10] *Let f_1 and f_2 be two nonconstant meromorphic functions such that $f_1+f_2=1$. Then*

$$T(r,f_1)\leq \overline{N}\left(r,\frac{1}{f_1}\right)+\overline{N}\left(r,\frac{1}{f_2}\right)+\overline{N}(r,f_1)+S(r,f_1).$$

Lemma 2.5. *Let*

$$F=\frac{f^{n+k_1+1}}{n+k_1+1}+\frac{af^{n+k_2+1}}{n+k_2+1}+\frac{bf^{n+1}}{n+1} \tag{2.10}$$

and

$$G=\frac{g^{n+k_1+1}}{n+k_1+1}+\frac{ag^{n+k_2+1}}{n+k_2+1}+\frac{bg^{n+1}}{n+1}, \tag{2.11}$$

where k_1, k_2, n are positive integers such that $n>k_1+2$ and $k_1>k_2$, a and b are complex numbers such that $|a|+|b|\neq 0$. If $F'=G'$, then $F=G$.

Remark 2.2. Lemma 2.5 improves [2, Lemma 2.9]. For convenience we will give the proof of Lemma 2.5, the tools and methods of the proof are the same as those in the proof of [2, Lemma 2.9] originally.

Proof. By (2.11), (2.12) and the condition $F'=G'$ we know that f and g share ∞ CM and

$$F-G=c, \tag{2.12}$$

where c is some constant. From (2.11), (2.12) and Lemma 2.1 we get

$$T(r,F)=(n+k_1+1)T(r,f)+O(1)=T(r,G)+O(1)=(n+k_1+1)T(r,g)+O(1). \tag{2.13}$$

Let

$$F_1=\frac{f^{k_1}}{n+k_1+1}+\frac{af^{k_2}}{n+k_2+1}+\frac{b}{n+1} \tag{2.14}$$

and

$$G_1=\frac{g^{k_1}}{n+k_1+1}+\frac{ag^{k_2}}{n+k_2+1}+\frac{b}{n+1}. \tag{2.15}$$

If $c\neq 0$, from (2.13)-(2.16) and Lemma 2.4 we get

$$\begin{aligned}(n+k_1+1)T(r,f)&\le \overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+S(r,f_1)\\&\le \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{F_1}\right)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{G_1}\right)+\overline{N}(r,f)+S(r,f)\\&\le (k_1+2)T(r,f)+(k_1+1)T(r,g)+S(r,f)\\&=(2k_1+3)T(r,f)+S(r,f),\end{aligned}$$

which implies $n+k_1+1\le 2k_1+3$, and so $n\le k_1+2$, this contradicts the condition $n>k_1+2$. Thus $c=0$. This together with (2.12) reveals the conclusion of Lemma 2.5. ∎

Lemma 2.6. *Let*

$$F_0=\frac{F'}{\alpha}\quad \textit{and}\quad G_0=\frac{G'}{\alpha}, \tag{2.16}$$

where

$$F=\frac{f^{n+k_1+1}}{n+k_1+1}+\frac{af^{n+k_2+1}}{n+k_2+1}+\frac{bf^{n+1}}{n+1} \tag{2.17}$$

and

$$G=\frac{g^{n+k_1+1}}{n+k_1+1}+\frac{ag^{n+k_2+1}}{n+k_2+1}+\frac{bg^{n+1}}{n+1}, \tag{2.18}$$

in which f and g are nonconstant meromorphic functions, $\alpha\not\equiv 0,\infty$ is a small function of f and g, and k_1, k_2, n are positive integers. Then

$$T(r,F)\le N\left(r,\frac{1}{F}\right)+T(r,F_0)-N\left(r,\frac{1}{F_0}\right)+S(r,F) \tag{2.19}$$

and

$$T(r,G)\le N\left(r,\frac{1}{G}\right)+T(r,G_0)-N\left(r,\frac{1}{G_0}\right)+S(r,G). \tag{2.20}$$

Proof. By (2.16)–(2.19) and the first fundamental theorem, we get

$$\begin{aligned}T(r,F)&=T\left(r,\frac{1}{F}\right)+O(1)\\&=N\left(r,\frac{1}{F}\right)+m\left(r,\frac{1}{F}\right)+O(1)\\&\le N\left(r,\frac{1}{F}\right)+m\left(r,\frac{F_0}{F}\right)+m\left(r,\frac{1}{F_0}\right)+O(1)\\&\le N\left(r,\frac{1}{F}\right)+T\left(r,F_0\right)-N\left(r,\frac{1}{F_0}\right)+S(r,F),\end{aligned}$$

which implies (2.19). Similarly we get (2.20). This proves Lemma 2.6. ∎

Lemma 2.7. [3] *Let f and g be two nonconstant meromorphic functions sharing $(1,m)$, where m is a nonnegative integer, and let*

$$H=\left(\frac{f''}{f'}-\frac{2f'}{f-1}\right)-\left(\frac{g''}{g'}-\frac{2g'}{g-1}\right).$$

Suppose that $H\not\equiv 0$. If $m=1$, then

$$T(r,f)\le N_2\left(r,\frac{1}{f}\right)+N_2\left(r,\frac{1}{g}\right)+N_2(r,f)+N_2(r,g)$$
$$+\frac{1}{2}\overline{N}\left(r,\frac{1}{f}\right)+\frac{1}{2}\overline{N}(r,f)+S(r,f)+S(r,g).$$

If $m=0$, then

$$T(r,f)\le N_2\left(r,\frac{1}{f}\right)+N_2\left(r,\frac{1}{g}\right)+N_2(r,f)+N_2(r,g)+2\overline{N}\left(r,\frac{1}{f}\right)$$
$$+\overline{N}\left(r,\frac{1}{g}\right)+2\overline{N}(r,f)+\overline{N}(r,g)+S(r,f)+S(r,g).$$

Lemma 2.8. [20] *Let $s>0$ and t are relatively prime integers, and let c be a complex number such that $c^s=1$. Then there exists one and only one common zero of ω^s-1 and ω^t-c.*

3. Proof of theorems

Proof of Theorem 1.6. Let F_0 and G_0 be defined as in (2.16), where F and G are defined as in (2.17) and (2.18) respectively. We discuss the following three cases.

Case 1. Suppose that $m\ge 2$. Then from (2.16)–(2.18), Lemma 2.2, Lemma 2.3 and the condition that $F'-\alpha$ and $G'-\alpha$ share $(0,m)$ we know that either $F'=G'$ or

$$T_0(r)\le N_2\left(r,\frac{1}{F_0}\right)+N_2\left(r,\frac{1}{G_0}\right)+N_2(r,F_0)+N_2(r,G_0)+S(r,F_0)+S(r,G_0), \tag{3.1}$$

where

$$T_0(r)=\max\{T(r,F_0),T(r,G_0)\}. \tag{3.2}$$

If $F'=G'$, from Lemma 2.5 we get the conclusion of Theorem 1.6. Next we suppose that $F'\not\equiv G'$. Then (3.1) holds. From (2.16), (2.17) and Lemma 2.6 we get (2.19) and (2.20), where

$$N\left(r,\frac{1}{F}\right)-N\left(r,\frac{1}{F_0}\right)$$
$$=N\left(r,\frac{1}{f}\right)+N\left(r,1/\{\frac{f^{k_1}}{n+k_1+1}+\frac{af^{k_2}}{n+k_2+1}+\frac{b}{n+1}\}\right)$$
$$-N\left(r,\frac{1}{f^{k_1}+af^{k_2}+b}\right)-N\left(r,\frac{1}{f'}\right)+S(r,f)$$
$$\le k_1T(r,f)+N\left(r,\frac{1}{f}\right)-N\left(r,\frac{1}{f^{k_1}+af^{k_2}+b}\right)-N\left(r,\frac{1}{f'}\right)+S(r,f)$$

From (2.16)–(2.18), (3.1) and [15, Theorem 1.24] we get

$$\begin{aligned}T(r,F_0) &\leq 2\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{k_1}+af^{k_2}+b}\right)+N\left(r,\frac{1}{f'}\right)+2\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{g}\right)\\&\quad+N\left(r,\frac{1}{g^{k_1}+ag^{k_2}+b}\right)+N\left(r,\frac{1}{g'}\right)+2\overline{N}(r,g)+S(r,f)+S(r,g)\\&\leq 2\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{k_1}+af^{k_2}+b}\right)+N\left(r,\frac{1}{f'}\right)+2\overline{N}(r,f)\\&\quad+2\overline{N}\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g}\right)+3\overline{N}(r,g)+k_1T(r,g)+S(r,f)+S(r,g).\end{aligned}$$

Thus from (2.19) and Lemma 2.1 we get

$$\begin{aligned}nT(r,f) &\leq (4-\delta(0,f)-2\Theta(0,f)-2\Theta(\infty,f)+\varepsilon)T(r,f)\\&\quad+(k_1+6-\delta(0,g)-2\Theta(0,g)-3\Theta(\infty,g)+\varepsilon)T(r,g)+S(r,f)+S(r,g)\\&\leq (k_1+10-\delta(0,f)-2\Theta(0,f)-2\Theta(\infty,f)-\delta(0,g)-2\Theta(0,g))T(r)\\&\quad+(2\varepsilon-3\Theta(\infty,g))T(r)+S(r),\end{aligned}$$

where and in what follows, $T(r)=\max\{T(r,f),T(r,g)\}$ and $S(r)=o\{T(r)\}$, possibly outside a set of finite linear measure, ε is an arbitrary positive number. Similarly, from (2.20) and Lemma 2.1 we have

$$\begin{aligned}nT(r,g) &\leq (k_1+10-\delta(0,g)-2\Theta(0,g)-2\Theta(\infty,g)-\delta(0,f)-2\Theta(0,f))T(r)\\&\quad+(2\varepsilon-3\Theta(\infty,f))T(r)+S(r).\end{aligned}$$

From the above inequalities we get

$$n\leq (k_1+10-\delta(0,f)-2\Theta(0,f)-2\Theta(\infty,f)-\delta(0,g)-2\Theta(0,g)-3\Theta(\infty,g))$$

and

$$n\leq k_1+10-\delta(0,g)-2\Theta(0,g)-2\Theta(\infty,g)-\delta(0,f)-2\Theta(0,f)-3\Theta(\infty,f),$$

which contradicts the condition $n>\max\{\chi_1,\chi_2\}$, where χ_1 and χ_2 are defined as in (1.2) and (1.3) respectively.

Case 2. Suppose that $m=1$. Let

$$H_0=\left(\frac{F_0''}{F_0'}-\frac{2F_0'}{F_0-1}\right)-\left(\frac{G_0''}{G_0'}-\frac{2G_0'}{G_0-1}\right). \tag{3.3}$$

Suppose that $H_0 \not\equiv 0$. Then from Lemma 2.7 and the condition that $F' - \alpha$ and $G' - \alpha$ share $(0,1)$ we get

$$\begin{aligned}
T(r,F_0) \le\ & N_2\left(r,\frac{1}{F_0}\right) + N_2\left(r,\frac{1}{G_0}\right) + N_2(r,F_0) + N_2(r,G_0) \\
& + \frac{1}{2}\overline{N}\left(r,\frac{1}{F_0}\right) + \frac{1}{2}\overline{N}(r,F_0) + S(r,F_0) + S(r,G_0) \\
\le\ & 2\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^{k_1}+af^{k_2}+b}\right) + N\left(r,\frac{f}{f'}\right) + 2\overline{N}\left(r,\frac{1}{g}\right) \\
& + N\left(r,\frac{1}{g^{k_1}+ag^{k_2}+b}\right) + N\left(r,\frac{g}{g'}\right) + 2\overline{N}(r,f) + 2\overline{N}(r,g) \\
& + \frac{1}{2}\left\{\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{k_1}+af^{k_2}+b}\right) + N\left(r,\frac{f}{f'}\right)\right\} \\
& + \frac{1}{2}\overline{N}(r,f) + S(r,f) + S(r,g).
\end{aligned}$$

Proceeding as in Case 1, we get

$$\begin{aligned}
N\left(r,\frac{1}{F}\right) - N\left(r,\frac{1}{F_0}\right) \le\ & k_1T(r,f) + N\left(r,\frac{1}{f}\right) - N\left(r,\frac{1}{f^{k_1}+af^{k_2}+b}\right) \\
& - N\left(r,\frac{1}{f'}\right) + S(r,f) \\
=\ & k_1T(r,f) + \overline{N}\left(r,\frac{1}{f}\right) - N\left(r,\frac{1}{f^{k_1}+af^{k_2}+b}\right) \\
& - N\left(r,\frac{f}{f'}\right) + S(r,f).
\end{aligned}$$

From Lemma 2.6 we get (2.19). From (2.19), Lemma 2.1 and the above two inequalities we get

$$\begin{aligned}
(n+k_1+1)T(r,f) \le\ & k_1T(r,f) + 3\overline{N}\left(r,\frac{1}{f}\right) + 2\overline{N}\left(r,\frac{1}{g}\right) \\
& + N\left(r,\frac{1}{g^{k_1}+ag^{k_2}+b}\right) + N\left(r,\frac{g}{g'}\right) + 2\overline{N}(r,f) + 2\overline{N}(r,g) \\
& + \frac{1}{2}\left(\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{k_1}+af^{k_2}+b}\right) + N\left(r,\frac{f}{f'}\right)\right) \\
& + \frac{1}{2}\overline{N}(r,f) + S(r,f) \\
\le\ & k_1T(r,f) + 3\overline{N}\left(r,\frac{1}{f}\right) + 2\overline{N}\left(r,\frac{1}{g}\right) + k_1T(r,g) + \overline{N}(r,g) \\
& + \overline{N}\left(r,\frac{1}{g}\right) + 2\overline{N}(r,f) + 2\overline{N}(r,g) + \frac{1}{2}\overline{N}\left(r,\frac{1}{f}\right) + \frac{k_1}{2}T(r,f) \\
& + \frac{1}{2}\left(\overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right)\right) + \frac{1}{2}\overline{N}(r,f) + S(r,f) + S(r,g),
\end{aligned}$$

i.e.,

$$
\begin{aligned}
(n+1)T(r,f) &\le 4\overline{N}\left(r,\frac{1}{f}\right) + 3\overline{N}(r,f) + 3\overline{N}\left(r,\frac{1}{g}\right) + 3\overline{N}(r,g) \\
&\quad + k_1 T(r,g) + \frac{k_1}{2}T(r,f) + S(r,f) + S(r,g) \\
&\le \left(4(1-\Theta(0,f)) + 3(1-\Theta(\infty,f)) + \frac{k_1}{2} + \varepsilon\right)T(r,f) \\
&\quad + \{3(1-\Theta(0,g)) + 3(1-\Theta(\infty,g) + k_1 + \varepsilon)\}T(r,g) + S(r,f) + S(r,g) \\
&\le \{4(1-\Theta(0,f)) + 3(1-\Theta(\infty,f)) + 3(1-\Theta(0,g))\}T(r) \\
&\quad + 3(1-\Theta(\infty,g))T(r) + \left(\frac{3k_1}{2} + 2\varepsilon\right)T(r) + S(r).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
(n+1)T(r,g) &\le \{4(1-\Theta(0,g)) + 3(1-\Theta(\infty,g)) + 3(1-\Theta(0,f))\}T(r) \\
&\quad + 3(1-\Theta(\infty,f))T(r) + \left(\frac{3k_1}{2} + 2\varepsilon\right)T(r) + S(r).
\end{aligned}
$$

From the above two inequalities we get

$$
n+1 \le 4(1-\Theta(0,f)) + 3(1-\Theta(\infty,f)) + 3(1-\Theta(0,g)) + 3(1-\Theta(\infty,g)) + \frac{3k_1}{2} \tag{3.4}
$$

and

$$
n+1 \le 4(1-\Theta(0,g)) + 3(1-\Theta(\infty,g)) + 3(1-\Theta(0,f)) + 3(1-\Theta(\infty,f)) + \frac{3k_1}{2}. \tag{3.5}
$$

Let χ_3 and χ_4 be defined as in (1.4) and (1.5) respectively. From (3.4), (3.5) and the condition $n+1 > \max\{\chi_3, \chi_4\}$ we get a contradiction. Thus $H_0 = 0$, this together with (3.3) gives

$$
\frac{1}{F_0 - 1} = \frac{A}{G_0 - 1} + B, \tag{3.6}
$$

where A and B are constants and $A \neq 0$. From (3.6) we see that F and G share 1 CM, which implies that

$$
N\left(r, \frac{1}{F' - \alpha}\right) - N_E(r,\alpha) = S(r,f) \tag{3.7}
$$

and

$$
N\left(r, \frac{1}{G' - \alpha}\right) - N_E(r,\alpha) = S(r,g), \tag{3.8}
$$

where $N_E(r,\alpha)$ denotes the counting function of those common zeros of $F_0 - \alpha$ and $G_0 - \alpha$, each such common zero has the same multiplicities related to $F_0 - \alpha$ and $G_0 - \alpha$, and is counted according to its multiplicity. Next in the same manner as in Case 1 we can get from (3.7) and (3.8) that $F' = G'$, this together with Lemma 2.5 gives the conclusion of Theorem 1.6.

Case 3. Suppose that $m = 0$ and set (3.3). Suppose that $H_0 \not\equiv 0$. Then by Lemma 2.7 and in the same manner as in Case 2 we get

$$
\begin{aligned}
(n+1)T(r,f) &\le 2k_1 T(r,f) + 2k_1 T(r,g) + 6\overline{N}\left(r,\frac{1}{f}\right) + 6\overline{N}(r,f) \\
&\quad + 5\overline{N}\left(r,\frac{1}{g}\right) + 5\overline{N}(r,g) + S(r,f) + S(r,g) \\
&\le \{2k_1 + 6(1-\Theta(0,f) + 6(1-\Theta(\infty,f)) + \varepsilon)\}T(r,f) \\
&\quad + \{2k_1 + 5(1-\Theta(0,g) + 5(1-\Theta(\infty,g)) + \varepsilon\}T(r,g) + S(r,f) + S(r,g) \\
&\le \{22+4k_1 - 6\Theta(0,f) - 6\Theta(\infty,f) - 5\Theta(0,g) - 5\Theta(\infty,g) + 2\varepsilon\}T(r) + S(r),
\end{aligned}
$$

where ε is an arbitrary positive number. Similarly

$$(n+1)T(r,g) \le \{22+4k_1 - 6\Theta(0,g) - 6\Theta(\infty,g) - 5\Theta(0,f) - 5\Theta(\infty,f) + 2\varepsilon\}T(r) + S(r).$$

From the above inequalities we have

$$n \le 21 + 4k_1 - 6\Theta(0,f) - 6\Theta(\infty,f) - 5\Theta(0,g) - 5\Theta(\infty,g) \tag{3.9}$$

and

$$n \le 21 + 4k_1 - 6\Theta(0,g) - 6\Theta(\infty,g) - 5\Theta(0,f) - 5\Theta(\infty,f). \tag{3.10}$$

Let χ_5 and χ_6 be defined as in (1.6) and (1.7). From (3.9), (3.10) and the condition $n > \max\{\chi_5, \chi_6\}$ we get a contradiction. Thus $H_0 = 0$. Next from the end of Case 2 we get the conclusion of Theorem 1.6. Theorem 1.6 is thus completely proved. ∎

Proof of Theorem 1.7. By Theorem 1.6 we get (1.1). We discuss the following three cases.

Case 1. Suppose that $a = 0$ and $b \ne 0$. Then (1.1) is rewritten as

$$f^{n+1}\left(\frac{f^{k_1}}{n+k_1+1} + \frac{b}{n+1}\right) = g^{n+1}\left(\frac{g^{k_1}}{n+k_1+1} + \frac{b}{n+1}\right). \tag{3.11}$$

Let

$$h = \frac{g}{f}, \tag{3.12}$$

and let u_1, u_2, $\cdots$, u_{k_1-1}, u_{k_1} be k_1 distinct roots of the algebraic equation

$$\frac{\omega^{k_1}}{n+k_1+1} + \frac{b}{n+1} = 0. \tag{3.13}$$

From (3.11)–(3.12) we get

$$g^{k_1}(1-h^{n+k_1+1}) = \frac{b(n+k_1+1)}{n+1}\cdot(h^{n+1}-1)h^{k_1}. \tag{3.14}$$

If h is a constant such that $h^{n+k_1+1} \ne 1$, from (3.14) we get

$$g^{k_1} = \frac{b(n+k_1+1)(h^{n+1}-1)h^{k_1}}{(n+1)(1-h^{n+k_1+1})}. \tag{3.15}$$

From (3.15) we know that g is a constant, which is impossible. Thus $h^{n+k_1+1} = 1$. This together with (3.14) and $h \not\equiv 0, \infty$ implies $h^{n+1} - 1 = 0$. Thus $h^{k_1} = 1$, which and (3.12) reveals the conclusion (i) of Theorem 1.7.

Suppose that h is not a constant. By rewriting (3.14) as (3.15) we know that every zero of $h-u^k$ $(1 \le k \le n+1)$ is of multiplicity $\ge k_1 \ge 2$, this together with the second fundamental theorem gives

$$\begin{aligned}(n-1)T(r,h) &\le \sum_{k=1}^{n+1} \overline{N}\left(r, \frac{1}{h-u^k}\right) + S(r,h)\\ &\le \frac{1}{2}\sum_{k=1}^{n+1} N\left(r, \frac{1}{h-u^k}\right) + S(r,h)\\ &\le \frac{n+1}{2}T(r,h) + S(r,h),\end{aligned}$$

i.e.,

$$\frac{n-3}{2}T(r,h) = S(r,h),$$

which and the condition $n > 4$ implies $T(r,h) = S(r,h)$, and so h is a constant, this is impossible.

Case 2. Suppose that $a \ne 0$ and $b = 0$. Then (1.1) is rewritten as

$$f^{n+k_2+1}\left(\frac{f^{k_1-k_2}}{n+k_1+1} + \frac{a}{n+k_2+1}\right) = g^{n+k_2+1}\left(\frac{g^{k_1-k_2}}{n+k_1+1} + \frac{a}{n+k_2+1}\right). \tag{3.16}$$

Next we set (3.12). Then $h \not\equiv 0, \infty$. From (3.12) and (3.16) we get

$$(n+k_2+1)(1-h^{n+k_1+1})g^{k_1-k_2} = a(n+k_1+1)(h^{n+k_2+1}-1)h^{k_1-k_2}. \tag{3.17}$$

Suppose that h is a constant. If $h^{n+k_1+1} \ne 1$, by rewriting (3.17) as

$$g^{k_1-k_2} = \frac{a(n+k_1+1)(h^{n+k_2+1}-1)h^{k_1-k_2}}{(n+k_2+1)(1-h^{n+k_1+1})}, \tag{3.18}$$

we know that g is a constant, which is impossible. Thus $h^{n+k_1+1} = 1$, and so it follows from (3.17) that $h^{n+k_2+1} = 1$. Thus $h^{k_1-k_2} = 1$. This together with (3.12) reveals the conclusion (ii) of Theorem 1.7.

Suppose that h is not a constant. If $k_1 - k_2 = 1$, then (3.18) is rewritten as

$$g = \frac{a(n+k_2+2)h(h^{n+k_2+1}-1)}{(n+k_2+1)(1-h^{n+k_2+2})}. \tag{3.19}$$

From Lemma 2.8, we know that $\omega = 1$ is the only common root of $\omega^{n+k_2+1} = 1$ and $\omega^{n+k_2+2} = 1$. This together with (3.19) and Lemma 2.1 implies

$$T(r,g) = (n+k_2+1)T(r,h) + O(1) \tag{3.20}$$

and

$$\overline{N}(r,g) = \sum_{k=1}^{n+k_2+1} \overline{N}(r, \frac{1}{h-\upsilon^k}), \tag{3.21}$$

where $\upsilon = \exp((2\pi i)/(n+k_2+2))$. From (3.19)–(3.21) and the second fundamental theorem we get

$$\begin{aligned}\frac{n+k_2-1}{n+k_2+1}T(r,g)+O(1) &= (n+k_2-1)T(r,h)\\ &\le \sum_{k=1}^{n+k_2+1}\overline{N}\left(r,\frac{1}{h-\upsilon^k}\right)+S(r,h)\\ &\le \overline{N}(r,g)+S(r,g)\\ &\le \{1-\Theta(\infty,g)+\varepsilon\}T(r,g)+S(r,g),\end{aligned}$$

which implies $\Theta(\infty,g) \le 2/(n+k_2+1)$, from this and the condition $\Theta(\infty,g) > 2/(n+k_2+1)$ we get a contradiction.

If $k_1-k_2 \ge 2$, by rewriting (3.17) as (3.18) and from the fact that there exists at most k_1-k_2 distinct common roots of $\omega^{n+k_1+1}=1$ and $\omega^{n+k_2+1}=1$ such that they are the roots of $\omega^{k_1-k_2}=1$, we know that every zero of $h-\upsilon_k$ $(1\le k\le n+k_2+1)$ is of multiplicity ≥ 2, where $\upsilon_1, \upsilon_2, \cdots, \upsilon_{n+k_2}, \upsilon_{n+k_2+1}$ are $n+k_2+1$ distinct simple roots of $\omega^{n+k_2+1}=1$, but they are not the roots of $\omega^{k_1-k_2}=1$. Thus from (3.18) and the second fundamental theorem we get

$$\begin{aligned}(n+k_2-1)T(r,h) &\le \sum_{k=1}^{n+k_2+1}\overline{N}\left(r,\frac{1}{h-\upsilon_k}\right)+S(r,h)\\ &\le \frac{1}{2}\sum_{k=1}^{n+k_2+1}N\left(r,\frac{1}{h-\upsilon_k}\right)+S(r,h)\\ &\le \frac{n+k_2+1}{2}T(r,h)+S(r,h),\end{aligned}$$

i.e.,

$$(n+k_2-3)T(r,h)=S(r,h). \tag{3.22}$$

From (3.22) and the condition $n>4$ we know that h is a constant, which contradicts the above supposition.

Case 3. Suppose that $a \ne 0$ and $b \ne 0$. By rewriting (1.1) we get

$$Af^{n+k_1+1}+Bf^{n+k_2+1}+Cf^{n+1}=Ag^{n+k_1+1}+Bg^{n+k_2+1}+Cg^{n+1}, \tag{3.23}$$

where

$$A=\frac{1}{n+k_1+1},\quad B=\frac{a}{n+k_2+1},\quad C=\frac{b}{n+1}. \tag{3.24}$$

Let

$$\eta=\frac{f}{g}. \tag{3.25}$$

Then $\eta \not\equiv 0,\infty$. From (3.23) and (3.25) we get

$$Ag^{k_1}(\eta^{n+k_1+1}-1)+Bg^{k_2}(\eta^{n+k_2+1}-1)+C(\eta^{n+1}-1)=0. \tag{3.26}$$

Suppose that η is not a constant. By rewriting (3.26) we get

$$Ag^{k_1}=-Bg^{k_2}\cdot\frac{\eta^{n+k_2+1}-1}{\eta^{n+k_1+1}-1}-\frac{C(\eta^{n+1}-1)}{\eta^{n+k_1+1}-1}. \tag{3.27}$$

Next we denote by $\{1,\eta_1,\eta_2,\cdots,\eta_l\}\subseteq\{\eta:\eta^{k_1-k_2}=1\}$ the set of the distinct common roots of $\eta^{n+k_2+1}=1$ and $\eta^{n+k_1+1}=1$, where l is some positive integer satisfying $1\le l\le k_1-k_2$, and let $\eta_{l+1},\eta_{l+2},\cdots,\eta_{n+k_1},\eta_{n+k_1+1}\notin\{\omega:\omega^{n+k_2+1}=1\}$ be $n+k_1-l+1$ distinct roots of $\omega^{n+k_1+1}=1$. Let $z_1\notin\{z:\eta^{n+k_1+1}(z)=1\}$ be a pole of g of multiplicity $p\ge 2$. Then it follows from (3.27) that $k_1p\le k_2p$, i.e, $k_1\le k_2$, which contradicts the condition $k_1>k_2$. This together with (1.8), (3.27) and the condition that every pole of g is of multiplicity ≥ 2 gives

$$\overline{N}\left(r,\frac{1}{h-\eta_j}\right)=\overline{N}_{(2}\left(r,\frac{1}{h-\eta_j}\right)+S(r,h) \tag{3.28}$$

for $l+1\le j\le n+k_1+1$. Thus from (3.28) we have $\Theta(\eta_k,\eta)\ge 1/2$ for $l+1\le k\le n+k_1+1$. This together with the fact $\sum_{k=l+1}^{n+k_1+1}\Theta(\eta_k,\eta)\le 2$ implies $(n+k_1-l+1)/2\le 2$, i.e, $n+k_1\le l+3$. Thus from $l\le k_1-k_2$ we get $n\le 3-k_2$, which contradicts the condition $n>4$.

Suppose that η is a constant. Then $\eta\ne 0,\infty$. If $\eta^{n+k_1+1}-1\ne 0$, by rewriting (3.26) as (3.27) we know that g is a constant, which is impossible. If $\eta^{n+k_1+1}-1=0$ and $\eta^{n+k_2+1}-1\ne 0$, then from (3.26) we get

$$g^{k_2}=-\frac{C(\eta^{n+1}-1)}{B(\eta^{n+k_2+1}-1)},$$

which implies that g is a constant, this is impossible. If $\eta^{n+k_1+1}-1=0$ and $\eta^{n+k_2+1}-1=0$, from (3.26) we get $\eta^{n+1}=1$. Thus $\eta^{k_1}=\eta^{k_2}=1$. This together with (3.25) reveals the conclusion (iii) of Theorem 1.7. Theorem 1.7 is thus completely proved. ∎

4. Concluding remarks

Regarding the above Theorem 1.6, we pose the following question.

Question 4.1. Whether the conditions with $\chi_1,\chi_2,\cdots,\chi_6$ in Theorem 1.6 are sharp or not?

Regarding the above Theorem 1.7, we make the following conjecture.

Conjecture 4.1. *If we remove the condition (1.8), then the case (iii) in Theorem 1.7 will hold.*

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