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# On the Simple Sporadic Group He Generated by the $(2,3,t)$ Generators

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**Abstract.** A group $G$ is said to be $(2,3,t)$-generated if it can be generated by two elements $x$ and $y$ such that $o(x)=2$, $o(y)=3$ and $o(xy)=t$. In the present article, we investigate $(2,3,t)$-generations for the Held's sporadic simple group He, where $t$ is a divisor of $|\mathrm{He}|$. Most of the computations were carried out with the aid of computer algebra system $\mathbb{GAP}$ [23].

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## 1. Introduction

A group $G$ is said to be $(2,3)$-generated if it can be generated by an involution $x$ and an element $y$ of order 3. If $o(xy)=t$, we also say $G$ is $(2,3,t)$-generated. The $(2,3)$-generation problem has attracted a wide attention of group theorists. One reason is that $(2,3)$-generated groups are homomorphic images of the modular group $PSL(2,\mathbb{Z})$, which is the free product of two cyclic groups of order two and three. The connection with *Hurwitz groups* and *Riemann surfaces* also plays a role. Recall that a $(2,3,7)$-generated group $G$ which gives rise to compact Riemann surface of genus greater than 2 with automorphism group of maximal order, is called Hurwitz group.

Miller [19] proved that the alternating groups $A_n$, $n \neq 6,7,8$ are $(2,3)$-generated. Macbeath in [18] showed that the projective special linear group $PSL(2,q)$, $q \neq 9$ are $(2,3)$-generated. Woldar [27] determined that all sporadic simple groups, with the exception of $M_{11}$, $M_{22}$, $M_{23}$ and $McL$ are $(2,3)$-generated. In addition to the above, a large number of classical linear groups and Lie groups are $(2,3)$-generated (see [11]). Recently, Liebeck and Shalev in [17] have presented some new probabilistic, non-constructive, methods regarding generations of finite simple groups. In the context of $(2,3)$-generations they show that all finite classical groups are $(2,3)$-generated, with the exception of $PSp(4,2^k)$ and $PSp(4,3^k)$ and finitely many other groups. For the literature concerning the generation of finite simple

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groups by two elements and related ones we encourage the reader to consult [10] in addition to the references cited above.

The Held group He is a sporadic simple group of order $4030387200 = 2^{10}3^3 5^2 7^3 17$. The group He was discovered by Held [15] while attempting to characterize the Mathieu group $M_{24}$ as a simple group with involution centralizer of type $2_+^{1+6}.L_3(2)$. The centralizer of an element $x$ in the $7A$ conjugacy class of the Monster group M is He $\times \langle x \rangle$. The group He has been explicitly constructed as the subgroup of Fischer group $Fi_{24}'$ stabilizing a set of 2058 Fischer's 3-transpositions in $Fi_{24}$.

It is well known that the group He has exactly 11 conjugacy classes of maximal subgroups as determined by Butler [7] and listed in the $\mathbb{ATLAS}$ [9]. Held himself determined much of the local structure of the group as well as the conjugacy classes of its elements. Thompson computed the character table of the group He. The Held group He has exactly two conjugacy classes of involutions denoted by $2A$ and $2B$.

Moori in [21] determined the $(2,3,p)$-generations of the smallest Fischer group $F_{22}$. In [13], Ganief and Moori established $(2,3,t)$-generations of the third Janko group $J_3$. In 2002, Ashrafi [6] computed the $(p,q,r)$-generations for the group He where $p$, $q$ and $r$ are distinct primes. The present paper is devoted to the $(2,3,t)$-generations of He, where $t$ is any divisor of $|\mathrm{He}|$. We will also give the generating triples for He. For more information regarding the study of $(2,3)$-generations and generation of a group by its conjugate involutions as well as computational techniques, the reader is referred to [1–5, 13, 21, 22, 27].

For basic properties of the group He and information on its subgroups the reader is referred to [7, 15]. The $\mathbb{ATLAS}$ of Finite Groups [9] is an important reference and we adopt its notation for subgroups, conjugacy classes, etc. Computations were carried out with the aid of $\mathbb{GAP}$ [23].

## 2. Preliminary results

Throughout this paper our notation is standard and taken mainly from [3, 5, 6, 13, 21, 25]. In particular, for a finite group $G$ with $C_1, C_2, \ldots, C_k$ conjugacy classes of its elements and $g_k$ a fixed representative of $C_k$, we denote $\Delta(G) = \Delta_G(C_1, C_2, \ldots, C_k)$ the number of distinct tuples $(g_1, g_2, \ldots, g_{k-1})$ with $g_i \in C_i$ such that $g_1 g_2 \ldots g_{k-1} = g_k$. It is well known that $\Delta_G(C_1, C_2, \ldots, C_k)$ is structure constant for the conjugacy classes $C_1, C_2, \ldots, C_k$ and can easily be computed from the character table of $G$ (see [16, p. 45]) by the following formula

$$\Delta_G(C_1, C_2, \ldots, C_k) = \frac{|C_1||C_2|\ldots|C_{k-1}|}{|G|} \times \sum_{i=1}^{m} \frac{\chi_i(g_1)\chi_i(g_2)\ldots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{[\chi_i(1_G)]^{k-2}}$$

where $\chi_1, \chi_2, \ldots, \chi_m$ are the irreducible complex characters of $G$. Further let $\Delta^*(G) = \Delta_G^*(C_1, C_2, \ldots, C_k)$ denote the number of distinct tuples $(g_1, g_2, \ldots, g_{k-1})$ with $g_i \in C_i$ and $g_1 g_2 \ldots g_{k-1} = g_k$ such that $G = \langle g_1, g_2, \ldots, g_{k-1} \rangle$. If $\Delta_G^*(C_1, C_2, \ldots, C_k) > 0$, then we say that $G$ is $(C_1, C_2, \ldots, C_k)$-generated. If $H$ **is** any subgroup of $G$ containing the fixed element $g_k \in C_k$, then $\Sigma_H(C_1, C_2, \cdots, C_{k-1}, C_k)$ denotes the number of distinct tuples $(g_1, g_2, \cdots, g_{k-1}) \in C_1 \times C_2 \times \cdots \times C_{k-1}$ such that $g_1 g_2 \cdots g_{k-1} = g_k$ and $\langle g_1, g_2, \cdots, g_{k-1} \rangle \leq H$ where $\Sigma_H(C_1, C_2, \cdots, C_k)$ is obtained by summing the structure constants $\Delta_H(c_1, c_2, \cdots, c_k)$ of $H$ over all $H$-conjugacy classes $c_1, c_2, \cdots, c_{k-1}$ satisfying $c_i \subseteq H \cap C_i$ for $1 \leq i \leq k-1$.

A general conjugacy class of elements of order $n$ in $G$ is denoted by $nX$. For example $2A$ represents the first conjugacy class of involutions in a group $G$. In most instances it will be clear from the context to which conjugacy classes we are referring. Thus we shall often suppress the conjugacy classes, using $\Delta(G)$ and $\Delta^*(G)$ as abbreviated notation for $\Delta_G(lX,mY,nZ)$ and $\Delta^*_G(lX,mY,nZ)$, respectively.

The following results in certain situations are very effective at establishing non-generations.

**Theorem 2.1.** (Scott's Theorem, [8] and [24]) *Let $x_1,x_2,\ldots,x_m$ be elements generating a group $G$ with $x_1x_2\cdots x_n = 1_G$, and $V$ be an irreducible module for $G$ of dimension $n \geq 2$. Let $C_V(x_i)$ denote the fixed point space of $\langle x_i\rangle$ on $V$, and let $d_i$ be the codimension of $C_V(x_i)$ in $V$. Then $d_1+d_2+\cdots+d_m \geq 2n$.*

**Lemma 2.1.** (Woldar [27]) *Let $G$ be a finite centerless group and suppose $lX$, $mY$, $nZ$ are $G$-conjugacy classes for which $\Delta^*(G) = \Delta^*_G(lX,mY,nZ) < |C_G(z)|, z \in nZ$. Then $\Delta^*(G) = 0$ and therefore $G$ is not $(lX,mY,nZ)$-generated.*

The following result will be crucial in determining generating triples.

**Theorem 2.2.** (Moori [14]) *Let $G$ be a finite group and $H$ a subgroup of $G$ containing a fixed element $x$ such that $gcd(o(x),[N_G(H){:}H]) = 1$. Then the number $h$ of conjugates of $H$ containing $x$ is $\chi_H(x)$, where $\chi_H$ is the permutation character of $G$ with action on the conjugates of $H$. In particular,*

$$h = \sum_{i=1}^{m} \frac{|C_G(x)|}{|C_{N_G(H)}(x_i)|},$$

*where $x_1,\ldots,x_m$ are representatives of the $N_G(H)$-conjugacy classes that fuse to the $G$-class $[x]_G$.*

In the course of our work we will frequently apply Scott's Theorem to the complex irreducible 51-dimensional module $V$ of He. For the reader's convenience, we record in Table I below all values $d_{nX}$ for relevant conjugacy classes $nX$ of the Held group He.

Table 1. The codimensions $d_{nX} = \dim(V/C_V(nX))$

| $d_{2A}$ | $d_{2B}$ | $d_{3A}$ | $d_{3B}$ | $d_{8A}$ | $d_{10A}$ | $d_{12A}$ | $d_{12B}$ | $d_{14A}$ | $d_{14B}$ | $d_{15A}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 20 | 24 | 30 | 34 | 44 | 44 | 44 | 46 | 47 | 47 | 46 |

## 3. $(2,3,t)$-Generations of He

The Held group He has exactly two conjugacy classes of elements of order 2 and 3 each. The group He acts on a set $\Omega$ of 2058 points. The point stabilizer of this action is isomorphic to $S_4(4){:}2$ and the orbits of point stabilizer on $\Omega$ are of lengths 1, 136, 136, 425 and 1360. The permutation character of He on $\Omega$ (the set of conjugates of $S_4(4){:}2$) is given by $\chi_{S_4(4):2} = \underline{1a} + \underline{51ab} + \underline{680a} + \underline{1275a}$.

In this section we investigate the $(2,3,t)$-generations of the Held's group He where $t$ is a divisor of $|\text{He}|$. It is a well known fact that if $G$ is $(2,3,t)$-generated simple group, then $1/2+1/3+1/t < 1$. It follows that in $(2,3,t)$-generations of the Held group He, we only need to consider $t \in \{7,8,10,12,14,15,17,21,28\}$.

**Lemma 3.1.** *The Held's group* He *is $(2X,3Y,7Z)$-generated if and only if $X = Y = B$ and $Z \in \{D,E\}$.*

*Proof.* This has been proved in [27] and [6]. ∎

**Lemma 3.2.** *The sporadic group* He *is* $(2X,3Y,8A)$*-generated if and only if the ordered pair* $(X,Y)=(B,B)$.

*Proof.* For the triple $(2B,3A,8A)$, non-generation follows immediately since the structure constant $\Delta(2B,3A,8A)=0$.

If the group He is $(pX,qY,rZ)$-generated, then $d_{pX}+d_{qY}+d_{rZ}\geq 102$ by Scott's theorem. It is clear from Table I that the triples $(2A,3A,8A)$ and $(2A,3B,8A)$ violate Scott's theorem and are therefore not generating triples for He.

Finally, we consider the triple $(2B,3B,8A)$. Amongst the maximal subgroups of He with order divisible by $2\times3\times8$, the only maximal subgroups having non-empty intersection with **each** conjugacy class in this triple are isomorphic to $H_1\cong 2^2{\cdot}L_3(4).S_3$, $H_2\cong 2^6{:}3{\cdot}S_6$ (two non-conjugate copies), $H_3\cong 2_+^{1+6}.L_3(2)$ and $H_4\cong 7^2{:}2L_2(7)$. We see that $\Delta(\mathrm{He})=272$, $\Sigma(H_2)=24$, $\Sigma(H_3)=16$ and $\Sigma(H_1)=0=\Sigma(H_4)$. Now since $\Sigma(H_1)=0=\Sigma(H_4)$, it follows that $H_1$ and $H_4$ are not $(2B,3B,8A)$-generated. A fixed element of order 8 in He is contained in a unique copy of each $H_2$ and $H_3$. Therefore, $\Delta^*(\mathrm{He})\geq 272-2(24)-1(16)>0$ and the $(2B,3B,8A)$-generation of He follows. This completes the proof. ∎

**Lemma 3.3.** *The group* He *is* $(2X,3Y,10A)$*-generated, where* $X,Y\in\{A,B\}$*, if and only if* $X=Y=B$.

*Proof.* The structure constant $\Delta_{\mathrm{He}}(2A,3B,10A)=0$, proving the non-generation of He by this triple. The triples $(2A,3A,10A)$ and $(2B,3A,10A)$ violate Scott's theorem, resulting in the non-generation of He by these triples.

Finally, we calculate $\Delta_{\mathrm{He}}(2B,3B,10A)=460$. The maximal subgroups of He that may contain $(2B,3B,10A)$-generated subgroups are isomorphic to $2^2{\cdot}L_3(4).S_3$, $2^6{:}3{\cdot}S_6$ (two non-conjugate classes), $3{\cdot}S_7$ and $5^2{:}4A_4$. Our calculations show that $\Sigma(2^2{\cdot}L_3(4).S_3)=0=\Sigma(5^2{:}4A_4)$, $\Sigma(2^6{:}3{\cdot}S_6)=20$ and $\Sigma(3{\cdot}S_7)=30$. Further a fixed element of order 10 is contained in a unique conjugate copy of each $2^6{:}3{\cdot}S_6$ and in $\chi_{3\cdot S_7}(3A)=2$ (see [1]) conjugate copies of $3{\cdot}S_7$. Hence $2^6{:}3{\cdot}S_6$ and $3{\cdot}S_7$ with their conjugates contribute at most $2\times20$ and $2\times30$, respectively, to $\Delta_{\mathrm{He}}(2B,3B,10A)$ and we obtain $\Delta^*_{\mathrm{He}}(2B,3B,10A)\geq 360$. Therefore, He is $(2B,3B,10A)$-generated and the result follows. ∎

**Lemma 3.4.** *The group* He *is* $(2X,3Y,12Z)$*-generated for* $X,Y,Z\in\{A,B\}$*, if and only if* $X=Y=B$.

*Proof.* Scott's theorem eliminates the cases $(2A,3A,12A),(2A,3B,12A),(2A,3B,12B)$, $(2B,3A,12A),(2B,3A,12B)$.

Next, we consider the triples $(2B,3B,12A)$ and $(2B,3B,12B)$. We calculate the structure constants $\Delta_{\mathrm{He}}(2B,3B,12A)=372$ and $\Delta_{\mathrm{He}}(2B,3B,12B)=248$. Observe that the only maximal subgroups of He which meet each of the classes in these triples are $2^2{\cdot}L_3(4).S_3$, $2^6{:}3{\cdot}S_6$ (two non-conjugate classes), $3{\cdot}S_7$ and $S_4\times L_3(2)$. We calculate that

$$\begin{aligned}\Delta^*_{\mathrm{He}}(2B,3B,12A)&\geq\Delta(\mathrm{He})-\Sigma(2^2{\cdot}L_3(4).S_3)-4\Sigma(2^6{:}3{\cdot}S_6)-\Sigma(3{\cdot}S_7)-\Sigma(S_4\times L_3(2))\\&=372-4(12)-24>0,\\ \Delta^*_{\mathrm{He}}(2B,3B,12B)&\geq\Delta(\mathrm{He})-\Sigma(2^2{\cdot}L_3(4).S_3)-2\Sigma(2^6{:}3{\cdot}S_6)-\Sigma(3{\cdot}S_7)-\Sigma(S_4\times L_3(2))\\&=348-2(4)-8>0.\end{aligned}$$

Hence the group He is $(2B,3B,12Z)$-generated for all $Z \in \{A,B\}$

Finally, the non-generation of He by the triple $(2A,3A,12B)$ is immediate since $\Delta_{He}(2A, 3A,12B)=0$. Hence the lemma follows. ∎

**Lemma 3.5.** *The group* He *is* $(2X,3Y,14Z)$*-generated, where* $X,Y \in \{A,B\}$ *and* $Z \in \{A,B, C,D\}$*, if and only if* $Y=B$.

*Proof.* Let $14R$ denote the class $14A$ or $14B$ and $14S$ denote the class $14C$ or $14D$. The non-generation of He by the triple $(2A,3A,14S)$ is clear as the structure constant $\Delta_{\rm He}(2A,3A,14S) = 0$. Further, an application of Scott's theorem shows that He is not $(2A,3A,14R)$-generated. Next consider the triple $(2B,3A,14S)$. For this triple we obtain that $\Delta_{\rm He}(2B,3A,14R) = 14 < 56 = |C_{\rm He}(14R)|$. So, by Lemma 2.2, $\Delta^*_{\rm He}(2B,3A,14R)=0$ and the group He is not $(2B,3A,14R)$-generated.

For the triple $(2B,3A,14S)$, the only maximal subgroup $M$ of He, up to isomorphism, that may admit $(2B,3A,14S)$-generation is isomorphic to $7^{1+2}_{+}{:}(S_3 \times 3)$. Further, an element of order 14 is contained in a unique conjugate class of $M$. We calculate

$$\Delta^*_{\rm He}(2B,3A,14S) \le \Delta_{\rm He}(2B,3A,14S) - \Sigma_M(2B,3A,14S) = 0$$

and so $\Delta^*({\rm He})=0$, proving that $(2B,3A,14S)$ is not a generating triple of He.

Next we show that the group He is $(2X,3B,14R)$-generated. The only maximal subgroups of He that may contain $(2X,3B,14R)$-generated proper subgroups are isomorphic to $H_1 \cong 2^2.L_3(4).S_3$, $H_2 \cong S_4 \times L_3(2)$ and $H_3 \cong 7{:}3 \times L_3(2)$. Our calculations give $\Delta_{\rm He}(2A,3B, 14R)=56$, $\Delta_{\rm He}(2B,3B,14R)=504$ and $\Sigma(H_1)=\Sigma(H_2)=\Sigma(H_3)=0$. So, the group He is $(2X,3B,14R)$-generated as $\Delta^*_{\rm He}(2X,3B,14R)=\Delta_{\rm He}(2X,3B,14R)>0$.

Finally, consider the case $(2X,3B,14S)$. In this case we have $\Delta_{\rm He}(2A,3B,14S)=49$ and $\Delta_{\rm He}(2B,3B,14S)=392$. The proper subgroups of He that may admit $(2X,3B,14S)$-generation of He are contained in the maximal subgroups isomorphic to $M_1 \cong 2^{1+6}_{+}.L_3(2)$, $M_2 \cong 7^2{:}2L_2(7)$ and $M_3 \cong 7^{1+2}{:}(S_3 \times 3)$. We calculate $\Sigma_{M_1}(2A,3B,14S)=7$, $\Sigma_{M_1}(2B,3B, 14S)=14$, $\Sigma_{M_2}(2B,3B,14S)=0=\Sigma_{M_3}(2B,3B,14S)$. Therefore,

$$\Delta^*_{\rm He}(2A,3B,14S) \ge \Delta_{\rm He}(2A,3B,14S) - \Sigma_{M_1}(2A,3B,14S) = 49-7>0,$$
$$\Delta^*_{\rm He}(2B,3B,14S) \ge \Delta_{\rm He}(2B,3B,14S) - \Sigma_{M_1}(2B,3B,14S) = 392-14>0.$$

Hence, the group He is $(2X,3B,14S)$-generated and the result follows. ∎

**Lemma 3.6.** *The group* He *is* $(2X,3Y,15A)$*-generated, where* $X,Y \in \{A,B\}$*, if and only if* $X=Y=B$.

*Proof.* An application of Scott's theorem eliminates the cases $(2A,3A,15A),(2A,3B,15A)$ and $(2B,3A,15A)$. Next we consider the triple $(2B,3B,15A)$. From the list of maximal subgroups of He, we observe that, up to isomorphism, $H_1 \cong 2^2.L_3(4).S_3$, $H_2 \cong 2^6{:}3{\cdot}S_6$ (two non-conjugate classes), and $H_3 \cong 3{\cdot}S_7$ are the only maximal subgroups that admit $(2B,3B,15A)$-generated subgroups. We calculate $\Delta_{\rm He}(2B,3B,15A)=390$, $\Sigma_{H_1}(2B,3B,15A) = 45$, $\Sigma_{H_2}(2B,3B,15A)=15$ and $\Sigma_{H_3}(2B,3B,15A)=0$. We compute

$$\Delta^*_{\rm He}(2B,3B,15A) \ge \Delta_{\rm He}(2B,3B,15A) - 2\Sigma_{H_1}(2B,3B,15A) - 4\Sigma_{H_2}(2B,3B,15A)$$
$$- \Sigma_{H_3}(2B,3B,15A) = 390-2(45)-4(15)>0.$$

This shows that $(2B,3B,15A)$ is a generating pair of He, proving the lemma. ∎

**Lemma 3.7.** *The group* He *is* $(2X,3Y,17Z)$*-generated if and only if* $(X,Y) \notin \{(A,A),(B,A)\}$.

*Proof.* See [6]. ∎

**Lemma 3.8.** *The group* He *is* $(2X,3Y,21Z)$*-generated for all* $X,Y \in \{A,B\}$ *and* $Z \in \{A,B,C,D\}$*, except when* $(X,Y,Z) = (A,A,Z),(B,A,A),(B,A,B),(A,B,C)$ *or* $(A,B,D)$.

*Proof.* Let $21U$ denote the class $21A$ or $21B$ and $21V$ denote the class $21C$ or $21D$. The non-generation of He by the triples $(2A,3A,21U)$, $(2B,3A,21U)$ and $(2A,3A,21V)$ is immediate as the values of structure constant for each of these triple is zero.

Next consider the triple $(2B,3A,21V)$. We compute $\Delta_{\mathrm{He}}(2B,3A,21V) = 21$ and observe that the only maximal subgroups of He having non-empty intersection with the conjugacy classes in this triple are isomorphic to $2^2{\cdot}L_3(4).S_3$, $7_+^{1+2}{:}(S_3\times 3)$ and $S_4\times L_3(2)$. However, we obtain

$$\Sigma(2^2{\cdot}L_3(4).S_3) = \Sigma(7_+^{1+2}{:}(S_3\times 3)) = \Sigma(S_4\times L_3(2)) = 0.$$

Therefore,

$$\Delta^*_{\mathrm{He}}(2B,3A,21V) = \Delta_{\mathrm{He}}(2B,3A,21V) = 21,$$

proving that $(2B,3A,21V)$ is a generating triple for He.

Now we show that He is $(2B,3B,21U)$-generated. We observe from the fusion maps into He that if $M$ is a maximal subgroup with non-empty intersection with the classes in this triple, then $M$ is isomorphic to either $3{\cdot}S_7$ or $7_+^{1+2}{:}(S_3\times 3)$. However, we obtain that $\Sigma_M(2B,3B,21U) = 0$ for both the these maximal subgroups and hence

$$\Delta^*_{\mathrm{He}}(2B,3B,21U) = \Delta_{\mathrm{He}}(2B,3B,21U) = 378,$$

proving the generation of He by the triple $(2B,3B,21U)$.

For the triples $(2A,3B,21U)$ and $(2B,3B,21V)$, the maximal subgroups that may contain $(2A,3B,21U)$-, and $(2B,3B,21V)$-generated proper subgroups are isomorphic to $2^2{\cdot}L_3(4).S_3$, $3{\cdot}S_7$, $S_4{\times}L_3(2)$ and $7{:}3\times L_3(2)$. We obtain

$$\Delta^*_{\mathrm{He}}(2A,3B,21U) \geq \Delta(\mathrm{He}) - \Sigma(3{\cdot}S_7) - 2\Sigma(7{:}3\times L_3(2))56 - 14 - 2(7) > 0,$$
$$\Delta^*_{\mathrm{He}}(2B,3B,21V) \geq \Delta(\mathrm{He}) - \Sigma(2^2{\cdot}L_3(4).S_3) - \Sigma(S_4{\times}L_3(2)) = 427 - 63 - 7 > 0.$$

Therefore, the triples $(2A,3B,21U)$ and $(2B,3B,21V)$ are the generating triples for He, proving the lemma.

Finally we consider the triple $(2A,3B,21V)$. Here we have

$$\Delta_{\mathrm{He}}(2A,3B,21V) = 42.$$

We prove that He is not $(2A,3B,21V)$-generated by constructing the $(2A,3B,21V)$-generated subgroup of the group He explicitly. We use the "standard generators" of the group He given by Wilson in [25] and also available in [26]. The group He has a 51-dimensional irreducible representation over $GF(2)$. Using this representation we generate $\mathrm{He} = \langle a,b\rangle$, where $a$ and $b$ are $51\times 51$ matrices over $GF(2)$ with orders 2 and 7 respectively such that $ab$ has order 17. Using $\mathbb{GAP}$, we see that $a\in 2A$, $b\in 7A$. We produce $c = (a^2b^5a^3)^4$, $d = (b^{-15}ab^3)^4$, $e = (ab^{-7}(ab)^{-19}a^{-5}b^{-10}c^{-a-c}d^{-19})^{-28}$, $z = ae$ such that $c,d,e\in 3B$ and $z\in 21V$. Let $H = \langle a,e\rangle$ then $H < \mathrm{He}$ with $|H| = 241920$. We compute that $\Sigma_H(2A,3B,21V) = 21$ and $z$ is contained in exactly two conjugate copies of $H$. Thus the total contribution from $H$ to the distinct ordered pairs $(\alpha,\beta)$ with $\alpha\in 2A$, $\beta\in 3B$ such that $\alpha\beta = z$ is equal to $2\times 21$ and consequently, we obtain that $\Delta^*_{\mathrm{He}}(2A,3B,21V) = 0$. Hence the group He is not $(2A,3B,21V)$-generated. This completes the lemma. ∎

**Lemma 3.9.** *The group* He *is* $(2X,3Y,28Z)$*-generated, where* $X,Y,Z \in \{A,B\}$ *except when* $(2X,3Y,28Z) = (2A,3A,28Z)$.

*Proof.* The non-generation of He by the triple $(2A,3A,28Z)$ is immediate since

$$\Delta_{\mathrm{He}}(2A,3A,28Z) = 0.$$

The maximal subgroups of He having elements of order 28 are isomorphic to $H_1 \cong 2^2{\cdot}L_3(4)$ $.S_3$, $H_2 \cong S_4{\times}L_3(2)$, $H_3 \cong 7{:}3{\times}L_3(2)$. We now consider following three cases.

**Case** $(2B,3A,28Z)$**:** The maximal subgroups that contain possible $(2B,3A,28Z)$-generated subgroups are isomorphic to $H_1$ and $H_2$. However, $\Sigma_{H_1}(2B,3A,28Z) = 0 = \Sigma_{H_2}(2B,3A,28Z)$ and hence $\Delta^*_{\mathrm{He}}(2B,3A,28Z) = \Delta_{\mathrm{He}}(2B,3A,28Z) = 28$. Therefore, the group He is $(2B,3A,$ $28Z)$-generated.

**Case** $(2A,3B,28Z)$**:** We calculate that $\Delta_{\mathrm{He}}(2A,3B,28Z) = 56$. Any maximal subgroup with non-empty intersection with the classes $2A$, $3B$ and $28Z$ is isomorphic to $H_1$, $H_2$ or $H_3$. Our calculations gives $\Sigma_{H_2}(2A,3B,28Z) = 0 = \Sigma_{H_3}(2A,3B,28Z)$ and $\Sigma_{H_1}(2A,3B,28Z) = 28$. Thus $\Delta^*_{\mathrm{He}}(2A,3B,28Z) \geq \Delta_{\mathrm{He}}(2A,3B,28Z) - \Sigma_{H_1}(2A,3B,28Z) > 0$, proving the generation of He by this triple.

**Case** $(2B,3B,28Z)$**:** The $(2B,3B,28Z)$-generated proper subgroups of He are contained in the maximal subgroups isomorphic to $H_1$ and $H_2$. A simple computation reveals that $\Delta_{\mathrm{He}}(2B,3B,28Z) = 308$, $\Sigma_{H_1}(2B,\ 3B,28Z) = 0$ and $\Sigma_{H_2}(2B,3B,28Z) = 28$. We obtain $\Delta^*_{He}(2B,3B,28Z) \geq 280$, proving generation of He by this triple. ∎

We now state our main results of this section.

**Theorem 3.1.** *The Held group* He *is not* $(2A,3A,tX)$*-generated for any integer* $t$*. Moreover, the group He is* $(2B,3B,tX)$*-generated for* $t \geq 7$.

*Proof.* This follows from Lemmas $3.1-3.9$. ∎

**Theorem 3.2.** *The Held group* He *is* $(2A,3B,tX)$*-generated if and only if* $tX \in \{14,17,21AB,$ $28\}$. *Moreover, the group* He *is* $(2B,3A,tX)$*-generated if and only if* $tX \in \{21CD,28\}$.

*Proof.* The proof follows from the Lemmas $3.1-3.9$. ∎

## 4. On the ranks of Held group He

We know that two involutions cannot possibly generate a finite simple group as they generate a dihedral group. So we would like to investigate the generation by three involutions. It is well known that sporadic simple groups are generated by three involutions. For example Moori [22] proved that the Fischer group $F_{22}$ can be generated by three conjugate involutions. The work of Liebeck and Shalev shows that all but finitely many simple classical groups can be generated by three involutions (see [17]). However, the problem of finding simple classical groups which can be generated by three conjugate involutions is still very much open.

The problem of generating a group by a set of involutions of minimal size is closely related to the $(2,3,t)$-generation of the group. Let $G$ be a finite group. We denote the rank of $G$ in a conjugacy class $X$ by $\mathrm{rank}(G:X)$, the minimum number of elements of $X$ generating $G$. In this section we determine $\mathrm{rank}(G{:}X)$ where $X$ is an involution in the group

He. Of course, $\text{rank}(G:X) \leq 2$ implies $G$ is cyclic or dihedral. The following result is quite useful in this context.

**Lemma 4.1.** [3, 12] *Let $G$ be a $(2X,3Y,tZ)$-generated finite simple group. Then $G$ is $(2X,2X,2X,(tZ)^3)$-generated.*

*Proof.* Let $x \in 2X$, $y \in 3Y$ with $G = \langle x,y \rangle$ such that $z = xy \in tZ$. Then $\langle x, x^y, x^{y^2} \rangle$ is a non-trivial normal subgroup of $G$, whence $G = \langle x, x^y, x^{y^2} \rangle$. Also, $xx^yx^{y^2} = (xy)^3 = z^3$, proving the result. ∎

**Corollary 4.1.** $\text{rank}(\text{He} : 2A) = 3 = \text{rank}(\text{He} : 2B)$.

*Proof.* The first equality follows via application of Lemma 4.1 to the $(2A,3B,21A)$-generation of He established in Lemma 3.8. The second equality follows by applying Lemma 4.1 to the $(2B,3B,7D)$-generation cited in Lemma 3.1. ∎

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