BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **35***(3) (2012), 765–774*

Meromorphic Function Sharing Two Sets with its Difference Operator

BAOQIN CHEN AND ZONGXUAN CHEN
School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, P. R. China
chenbaoqin_chbq@126.com, chzx@vip.sina.com

Abstract. We investigate the relationship between a meromorphic function $f(z)$ and its difference operator when they share two sets, and give some interesting results.

2010 Mathematics Subject Classification: Primary 30D35; Secondary 39B32

Keywords and phrases: Uniqueness theory, difference operator, shared sets.

1. Introduction and main results

Throughout this paper, a meromorphic function always means meromorphic in the whole complex plane. We assume that the reader is familiar with the basic notations of the Nevanlinna theory of meromorphic functions such as $T(r,f)$, $m(r,f)$, $N(r,f)$ and $\overline{N}(r,f)$ for a meromorphic function $f(z)$ (see, e.g., [5, 10, 15]). In particular, for $a \in \mathbb{C} \cup \{\infty\} := \hat{\mathbb{C}}$, we denote $N_2(r, 1/(f-a))$ the counting function of zeros of $f-a$ such that simple zeros are counted once and multiple zeros twice. The notation $S(r,f)$ is defined to be any quantity satisfying $S(r,f) = o(T(r,f))$, as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. Then a meromorphic function $a(z)$ is called to be a small function of $f(z)$ provided that $T(r,a) = S(r,f)$. Let $S(f)$ be the set of all small functions of $f(z)$. Let f and g be two meromorphic functions and $a \in \hat{\mathbb{C}}$. We say that f and g share a CM, resp. IM, if $f-a$ and $g-a$ share the same zeros counting multiplicities, resp. ignoring multiplicities.

We also need the following definitions in this paper.

Definition 1.1. [7] *Let k be a nonnegative integer or infinity. For $a \in \hat{\mathbb{C}}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$.*

Remark 1.1. From Definition 1.1, we can see that each element in $E_1(a;f)$ is counted exactly once in $N_2(r, 1/(f-a))$.

Definition 1.2. [7] *Let k be a nonnegative integer or infinity. If for $a \in \hat{\mathbb{C}}$, $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.*

Communicated by V. Ravichandran.
Received: July 9, 2011; *Revised:* October 7, 2011.

From these two definitions, one can find that if f, g share the value a with weight k, then f, g share the value a with weight p for all integer p, $0 \le p < k$. We also note that f, g share a value a CM or IM if and only if f, g share the value a with weight ∞ or 0 respectively.

Definition 1.3. [7] *For $S \subset \hat{\mathbb{C}}$, we define $E_f(S,k)$ as $E_f(S,k) = \bigcup_{a \in S} E_k(a;f)$, where k is a nonnegative integer or infinity.*

In this paper, we assume that $S_1 = \{1, \omega, \ldots, \omega^{n-1}\}$ and $S_2 = \{\infty\}$, where $\omega^n = 1$ and n is a positive integer. The investigation on the uniqueness of meromorphic functions sharing sets is an important subfield of the uniqueness theory. Yi [16], Li and Yang [13], Yi and Yang [18] had proved several results on the uniqueness problems of two meromorphic functions when they share two sets around 1995. Then in 2006, Lahiri and Banerjee [8] considered these problems with the idea of weighted sharing of sets. One can refer to [1,3,8,9,12–14, 16,18] for these results which are related to this paper.

In what follows, c always means a non-zero constant. For a meromorphic function $f(z)$, we denote its shift and difference operator by $f(z+c)$ and $\Delta_c f := f(z+c) - f(z)$, respectively. Recently, numbers of papers (including [2,4,6,11,19]) have focused on value distribution in difference analogues of meromorphic functions. Many papers (including [6,19]) mainly deal with some uniqueness questions for a meromorphic function that shares values or common sets with its shift or its difference operator. We recall the following two results proved by Zhang [19], in which the relation between $f(z)$ and its shift $f(z+c)$ when they share two sets, is discussed.

Theorem 1.1. [19] *Let $c \in \mathbb{C}$. Suppose that $f(z)$ is a nonconstant meromorphic function of finite order such that $E_{f(z)}(S_j,\infty) = E_{f(z+c)}(S_j,\infty)$ $(j = 1,2)$. If $n \ge 4$, then $f(z) \equiv t f(z+c)$, where $t^n = 1$.*

Theorem 1.2. [19] *Let $m \ge 2$, $n \ge 2m+4$ with n and $n-m$ having no common factors. Let a and b be two non-zero constants such that the equation $\omega^n + a\omega^{n-m} + b = 0$ has no multiple roots. Let $S = \{\omega | \omega^n + a\omega^{n-m} + b = 0\}$. Suppose that $f(z)$ is a nonconstant meromorphic function of finite order. Then $E_{f(z)}(S,\infty) = E_{f(z+c)}(S,\infty)$ and $E_{f(z)}(\{\infty\},\infty) = E_{f(z+c)}(\{\infty\},\infty)$ imply $f(z) \equiv f(z+c)$.*

An interesting question is what can be said if we replace $f(z+c)$ with $\Delta_c f$. Regarding this question, we prove the following results.

Theorem 1.3. *Suppose that $f(z)$ is a nonconstant meromorphic function of finite order such that $E_{f(z)}(S_1,2) = E_{\Delta_c f}(S_1,2)$ and $E_{f(z)}(S_2,\infty) = E_{\Delta_c f}(S_2,\infty)$. If $n \ge 7$, then $\Delta_c f \equiv t f(z)$, where $t^n = 1$ and $t \ne -1$.*

The following corollary follows directly from Theorem 1.3 and it can be seen as a counterpart result to Theorem 1.1.

Corollary 1.1. *Suppose that $f(z)$ is a nonconstant meromorphic function of finite order such that $E_{f(z)}(S_1,\infty) = E_{\Delta_c f}(S_1,\infty)$ and $E_{f(z)}(S_2,\infty) = E_{\Delta_c f}(S_2,\infty)$. If $n \ge 7$, then $\Delta_c f \equiv t f(z)$, where $t^n = 1$ and $t \ne -1$.*

Using a similar method as in the proof of Theorem 1.3, we get the following corollary.

Corollary 1.2. *Under the assumptions of Corollary 1.1, if $f(z)$ is a nonconstant entire function of finite order and $n \ge 5$, then the conclusion of Corollary 1.1 still holds.*

Now another interesting question is whether the conditions for the shared sets S_1 or S_2 in Theorem 1.3 can be replaced by other conditions or not. Considering this question, we prove the following results.

Theorem 1.4. *Suppose that $f(z)$ is a nonconstant meromorphic function of finite order satisfying $E_{f(z)}(S_1,0) = E_{\Delta_c f}(S_1,0)$ and $E_{f(z)}(S_2,\infty) = E_{\Delta_c f}(S_2,\infty)$. If there exists a constant α $(0 < \alpha \le 2)$ such that*

$$\overline{N}(r,f(z)) + \overline{N}\left(r,\frac{1}{f(z)}\right) < \alpha T(r,f(z)), \tag{1.1}$$

and if $n \ge 15\alpha/2+4$, then $\Delta_c f \equiv tf(z)$, where $t^n = 1$ and $t \ne -1$.

Theorem 1.5. *Suppose that $f(z)$ is a nonconstant meromorphic function of finite order satisfying $E_{f(z)}(S_1,2) = E_{\Delta_c f}(S_1,2)$ and $E_{f(z)}(S_2,0) = E_{\Delta_c f}(S_2,0)$. If*

$$\limsup_{r\to\infty} \frac{N\left(r,\frac{1}{f(z)}\right)}{T(r,f(z))} < 1, \tag{1.2}$$

and if $n \ge 7$, then $\Delta_c f \equiv tf(z)$, where $t^n = 1$ and $t \ne -1$.

The following Theorem 1.6 is to reduce the lower bound of n in Theorem 1.3. Its proof is similar to the proof of Theorem 1.3 and hence omitted.

Theorem 1.6. *Suppose that $f(z)$ is a nonconstant meromorphic function of finite order satisfying $E_{f(z)}(S_1,2) = E_{\Delta_c f}(S_1,2)$ and $E_{f(z)}(S_2,\infty) = E_{\Delta_c f}(S_2,\infty)$. If*

$$\overline{N}(r,f(z)) + \overline{N}\left(r,\frac{1}{f(z)}\right) = S(r,f),$$

and if $n \ge 3$, then $\Delta_c f \equiv tf(z)$, where $t^n = 1$ and $t \ne -1$.

Now we get the following theorem corresponding to Theorem 1.2.

Theorem 1.7. *Let $m \ge 2$, $n \ge 2m+4$ with n and $n-m$ having no common factors. Let a and b be two non-zero constants such that the equation $\omega^n + a\omega^{n-m} + b = 0$ has no multiple roots. Let $S = \{\omega | \omega^n + a\omega^{n-m} + b = 0\}$. Suppose that $f(z)$ is a nonconstant meromorphic function of finite order satisfying $E_{f(z)}(S,\infty) = E_{\Delta_c f}(S,\infty)$ and $E_{f(z)}(\{\infty\},\infty) = E_{\Delta_c f}(\{\infty\},\infty)$. If*

$$N\left(r,\frac{1}{\Delta_c f}\right) = T(r,f(z)) + S(r,f), \tag{1.3}$$

then $\Delta_c f \equiv f(z)$.

Example. Let $f(z) = e^z$, $n \ge 3$ be a given integer and c be a constant satisfying $e^c = 1 + e^{2\pi i/n}$. Then we see that $\Delta_c f(z) = e^{2\pi i/n} f(z)$ and hence

$$\Pi_{k=0}^{n-1}(\Delta_c f(z) - e^{\frac{2k\pi i}{n}}) = \Pi_{k=0}^{n-1}\left(f(z) - e^{\frac{2k\pi i}{n}}\right).$$

This infers that $E_{f(z)}(S_1,\infty) = E_{\Delta_c f}(S_1,\infty)$ and $E_{f(z)}(S_2,\infty) = E_{\Delta_c f}(S_2,\infty)$. This example satisfies Theorems 1.3–1.6.

But we still wonder whether the lower bound of n in our results is sharp or not.

2. Proof of Theorem 1.3

We recall the following result which takes a key role when concerning questions about a meromorphic function $f(z)$ and its difference operator $\Delta_c f$.

Lemma 2.1. [4] *Let $c \in \mathbb{C}$, $n \in \mathbb{N}$, and let $f(z)$ be a meromorphic function of finite order. Then for any small periodic function $a(z) \in S(f)$ with period c,*

$$m\left(r, \frac{\Delta_c^n f}{f(z)-a(z)}\right) = S(r,f),$$

where the exceptional set associated with $S(r,f)$ is of at most finite logarithmic measure.

The proof of Theorem 1.3 is based on a result in [1], which can be read as follows:

Lemma 2.2. [1] *Let F and G be two nonconstant meromorphic functions defined in $\mathbb{C}$. If $E_2(1;F) = E_2(1;G)$ and $E_k(\infty;F) = E_k(\infty;G)$, where $0 \le k \le \infty$, then one of the following cases occurs:*

(i) $T(r,F)+T(r,G) \le 2\{N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)$
$+\overline{N}_*(r,\infty;F,G)\}+S(r,F)+S(r,G)$;

(ii) $F \equiv G$;

(iii) $FG \equiv 1$,

where $\overline{N}_*(r,\infty;F,G)$ denotes the reduced counting function of those poles of F whose multiplicities differ from the multiplicities of the corresponding poles of G.

Proof of Theorem 1.3. Denote $F = (\Delta_c f)^n$ and $G = f(z)^n$. By the condition that $E_{f(z)}(S_1,2) = E_{\Delta_c f}(S_1,2)$, we see that F and G share 1 with weight 2, that is, $E_2(1;F) = E_2(1;G)$. Since $f(z)$ and $\Delta_c f$ share ∞ CM, we have $N(r,\Delta_c f) = N(r,f(z))$ and also $\overline{N}(r,\Delta_c f) = \overline{N}(r,f(z))$. Moreover, we deduce that F and G share ∞ with weight k, for any $0 \le k \le \infty$, and

$$\overline{N}_*(r,\infty;F,G) = 0. \tag{2.1}$$

Furthermore, we note that

$$\begin{aligned} N_2\left(r,\frac{1}{F}\right) &= 2\overline{N}\left(r,\frac{1}{\Delta_c f}\right),\ \overline{N}(r,F) = \overline{N}(r,\Delta_c f), \\ N_2\left(r,\frac{1}{G}\right) &= 2\overline{N}\left(r,\frac{1}{f(z)}\right),\ \overline{N}(r,G) = \overline{N}(r,f(z)). \end{aligned} \tag{2.2}$$

Combining with (2.1) and (2.2) gives

$$\begin{aligned} &N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}_*(r,\infty;F,G) \\ =&2\overline{N}\left(r,\frac{1}{\Delta_c f}\right)+2\overline{N}\left(r,\frac{1}{f(z)}\right)+\overline{N}(r,\Delta_c f)+\overline{N}(r,f(z)) \\ \le&2T\left(r,\frac{1}{\Delta_c f}\right)+2T\left(r,\frac{1}{f(z)}\right)+T(r,\Delta_c f)+T(r,f(z))+S(r,\Delta_c f)+S(r,f) \\ \le&3\{T(r,\Delta_c f)+T(r,f(z))\}+S(r,\Delta_c f)+S(r,f). \end{aligned} \tag{2.3}$$

On the other hand, we have

$$T(r,F)+T(r,G) = n\{T(r,\Delta_c f)+T(r,f(z))\}. \tag{2.4}$$

Suppose that the Case (i) in Lemma 2.2 holds. Then by (2.3) and (2.4), we deduce that

$$(n-6)\{T(r,\Delta_c f)+T(r,f(z))\}\le S(r,\Delta_c f)+S(r,f),$$

which is a contradiction with $n\ge 7$. Hence by Lemma 2.2, we have $F\equiv G$ or $FG\equiv 1$.

If $F\equiv G$, that is, $(\Delta_c f)^n = f(z)^n$. Then there exists a constant $t\in\mathbb{C}$ such that $\Delta_c f\equiv t f(z)$, where $t^n=1$. As $f(z)$ is a nonconstant meromorphic function, $t\ne -1$.

If $FG\equiv 1$, that is

$$(\Delta_c f)^n \equiv \frac{1}{f(z)^n}. \tag{2.5}$$

Since $f(z)$ and $\Delta_c f$ share ∞ CM, we see that

$$N\left(r,\frac{\Delta_c f}{f(z)}\right)\le N\left(r,\frac{1}{f(z)}\right)\le T(r,f(z))+S(r,f). \tag{2.6}$$

From (2.5), (2.6) and Lemma 2.1, we obtain that

$$\begin{aligned}2nT(r,f(z)) &= T\left(r,\frac{1}{f(z)^{2n}}\right)+O(1)=T\left(r,\frac{1}{f(z)^n}\cdot(\Delta_c f)^n\right)+O(1)\\ &= nm\left(r,\frac{\Delta_c f}{f(z)}\right)+nN\left(r,\frac{\Delta_c f}{f(z)}\right)+O(1)\le nT(r,f(z))+S(r,f).\end{aligned}$$

Thus, $T(r,f(z))=S(r,f)$, which is impossible. Theorem 1.3 is thus proved. ∎

3. Proof of Theorem 1.4

Lemma 3.1. [17] *Let $f(z)$ and $g(z)$ be two meromorphic functions. If $f(z)$ and $g(z)$ share 1 IM, and if*

$$\limsup_{r\to\infty}\frac{N^*(r,f)+N^*(r,g)+N^*\left(r,\frac{1}{f}\right)+N^*\left(r,\frac{1}{g}\right)}{T(r,f)+T(r,g)}<1,$$

where $N^*(r,f)=2N_2(r,f)+3\overline{N}(r,f)$, then $f\equiv g$ or $fg\equiv 1$.

Proof of Theorem 1.4. By the condition of Theorem 1.4, we still have $N(r,\Delta_c f)=N(r,f(z))$ and also $\overline{N}(r,\Delta_c f)=\overline{N}(r,f(z))$. Then by Lemma 2.1, we have

$$\begin{aligned}T(r,\Delta_c f)&=m(r,\Delta_c f)+N(r,\Delta_c f)\\ &\le m\left(r,\frac{\Delta_c f}{f(z)}\right)+m(r,f(z))+N(r,f(z))\le T(r,f(z))+S(r,f).\end{aligned} \tag{3.1}$$

It is immediate to see that $S(r,\Delta_c f)=o(T(r,f(z)))$.

Denote $F=(\Delta_c f)^n$ and $G=f(z)^n$. Since $E_{f(z)}(S_1,0)=E_{\Delta_c f}(S_1,0)$, it follows that F and G share 1 IM. Applying the second main theorem and by (1.1), we have

$$\begin{aligned}nT(r,f(z))=T(r,G)&\le\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)+S(r,G)\\ &\le\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{F-1}\right)+S(r,G)\\ &\le\overline{N}(r,f(z))+\overline{N}\left(r,\frac{1}{f(z)}\right)+T\left(r,\frac{1}{F-1}\right)+S(r,f)\\ &<\alpha T(r,f(z))+nT(r,\Delta_c f)+S(r,f).\end{aligned} \tag{3.2}$$

It follows from (3.2) that

$$T(r,f(z)) < \frac{n}{n-\alpha}T(r,\Delta_c f)+S(r,f). \tag{3.3}$$

Then from (3.3) and the condition $n \geq 15\alpha/2+4$, we deduce that

$$\begin{aligned} T(r,F)+T(r,G) &= n\{T(r,\Delta_c f)+T(r,f(z))\} > n(1+\frac{n-\alpha}{n})T(r,f(z))+S(r,f) \\ &= (2n-\alpha)T(r,f(z))+S(r,f) \geq (14\alpha+8)T(r,f(z))+S(r,f). \end{aligned} \tag{3.4}$$

On the other hand, we note that

$$\begin{aligned} &N^*(r,G) = 2N_2(r,G)+3\overline{N}(r,G) \leq 7\overline{N}(r,f(z)), \\ &N^*\left(r,\frac{1}{G}\right) = 2N_2\left(r,\frac{1}{G}\right)+3\overline{N}\left(r,\frac{1}{G}\right) \leq 7\overline{N}\left(r,\frac{1}{f(z)}\right). \end{aligned} \tag{3.5}$$

Then by (1.1) and (3.5), we get

$$N^*(r,G)+N^*\left(r,\frac{1}{G}\right) \leq 7\left\{\overline{N}(r,f(z))+\overline{N}\left(r,\frac{1}{f(z)}\right)\right\} < 7\alpha T(r,f(z)). \tag{3.6}$$

Since $f(z)$ and $\Delta_c f$ share ∞ CM, and by (1.1), we have

$$N^*(r,F) \leq 7\overline{N}(r,\Delta_c f) = 7\overline{N}(r,f(z)) < 7\alpha T(r,f(z)).$$

Similarly as to (3.6), we get

$$\begin{aligned} N^*(r,F)+N^*\left(r,\frac{1}{F}\right) &\leq 7\left\{\overline{N}(r,\Delta_c f)+\overline{N}\left(r,\frac{1}{\Delta_c f}\right)\right\} \\ &< 7\{\alpha T(r,f(z))+T(r,\Delta_c f)\}+S(r,\Delta_c f) \leq 7(\alpha+1)T(r,f(z))+S(r,f). \end{aligned} \tag{3.7}$$

Combining with (3.6) and (3.7), yields

$$N^*(r,F)+N^*(r,G)+N^*\left(r,\frac{1}{F}\right)+N^*\left(r,\frac{1}{G}\right) < (14\alpha+7)T(r,f(z))+S(r,f). \tag{3.8}$$

So by (3.4) and (3.8), we see that

$$\limsup_{r\to\infty} \frac{N^*(r,F)+N^*(r,G)+N^*\left(r,\frac{1}{F}\right)+N^*\left(r,\frac{1}{G}\right)}{T(r,F)+T(r,G)} \leq \frac{14\alpha+7}{14\alpha+8} < 1.$$

Therefore, by Lemma 3.1, $F \equiv G$ or $FG \equiv 1$.

Then using the same method as in the proof of Theorem 1.3 to discuss the two cases, we can also get the conclusion of Theorem 1.4. ∎

4. Proof of Theorem 1.5

Lemma 4.1. [2] *Let $f(z)$ be a meromorphic function of finite order ρ and c be a non-zero complex constant. Then, for each $\varepsilon > 0$, we have*

$$T(r,f(z+c)) = T(r,f(z))+O(r^{\rho-1+\varepsilon})+O(\log r).$$

Proof of Theorem 1.5. By (1.2), for any $\varepsilon > 0$ is small and $R > 0$ is large, we get that

$$\overline{N}\left(r,\frac{1}{f(z)}\right) \leq N\left(r,\frac{1}{f(z)}\right) \leq (1-\varepsilon)T(r,f(z)), \qquad r \geq R. \tag{4.1}$$

Denote $F=(\Delta_c f)^n$ and $G=f(z)^n$. Then (2.4) still holds. By the condition that $E_{f(z)}(S_1,2)=E_{\Delta_c f}(S_1,2)$, we also have $E_2(1;F)=E_2(1;G)$. Since $f(z)$ and $\Delta_c f$ share ∞ IM, we have $\overline{N}(r,\Delta_c f)=\overline{N}(r,f(z))$. Moreover, we deduce that $E_{n-1}(\infty;F)=E_{n-1}(\infty;G)$, and

$$\begin{aligned}\overline{N}_*(r,\infty;F,G)&=\overline{N}_*(r,\infty;G,F)\le \min\{\overline{N}(r,f(z)),\overline{N}(r,\Delta_c f)\}\\&\le\frac{1}{2}\{\overline{N}(r,f(z))+\overline{N}(r,\Delta_c f)\}\le\frac{1}{2}\{T(r,f(z))+T(r,\Delta_c f)\}+S(r,f)+S(r,\Delta_c f).\end{aligned}\tag{4.2}$$

Obviously, by Lemma 4.1, we see that

$$T(r,\Delta_c f)\le T(r,f(z+c))+T(r,f(z))+O(1)\le 2T(r,f(z))+S(r,f),$$

which implies that $S(r,\Delta_c f)=o(T(r,f(z)))$. Then by (4.1) and (4.2), we deduce that

$$\begin{aligned}&N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}_*(r,\infty;F,G)\\&=2\overline{N}\left(r,\frac{1}{\Delta_c f}\right)+2\overline{N}\left(r,\frac{1}{f(z)}\right)+\overline{N}(r,\Delta_c f)+\overline{N}(r,f(z))\\&\quad+\frac{1}{2}\{T(r,f(z))+T(r,\Delta_c f)\}+S(r,f)+S(r,\Delta_c f)\\&\le 2T\left(r,\frac{1}{\Delta_c f}\right)+2\overline{N}\left(r,\frac{1}{f(z)}\right)+\frac{3}{2}\{T(r,f(z))+T(r,\Delta_c f)\}+S(r,f)\\&\le\frac{7}{2}T(r,\Delta_c f)+\frac{3}{2}T(r,f(z))+2(1-\varepsilon)T(r,f(z))+S(r,f)\\&=\frac{7}{2}\{T(r,\Delta_c f)+T(r,f(z))\}-2\varepsilon T(r,f(z))+S(r,f),\qquad r\ge R.\end{aligned}\tag{4.3}$$

Now suppose that the case (i) in Lemma 2.2 holds. Hence, from (2.4) and (4.3), we have

$$(n-7)\{T(r,\Delta_c f)+T(r,f(z))\}+4\varepsilon T(r,f(z))\le S(r,f),$$

which contradicts with $n\ge 7$ and $\varepsilon>0$. Therefore, by Lemma 2.2, we have $F\equiv G$ or $FG\equiv 1$.

If $F\equiv G$, then our conclusion holds.

If $FG\equiv 1$, that is $(\Delta_c f)^n\equiv 1/(f(z)^n)$, which implies $T(r,\Delta_c f)=T(r,f(z))+O(1)$. Then by the above formula and (1.2), we see that

$$\begin{aligned}N\left(r,\frac{\Delta_c f}{f(z)}\right)&\le N\left(r,\frac{1}{f(z)}\right)+N(r,\Delta_c f)\\&\le(1-\varepsilon)T(r,f(z))+T(r,\Delta_c f)+S(r,f)+S(r,\Delta_c f)\\&\le(2-\varepsilon)T(r,f(z))+S(r,f),\qquad r\ge R.\end{aligned}\tag{4.4}$$

From (4.4) and Lemma 2.1, we obtain that

$$\begin{aligned}2nT(r,f(z))&=T\left(r,\frac{1}{f(z)^{2n}}\right)+O(1)=T\left(r,\frac{1}{f(z)^n}\cdot(\Delta_c f)^n\right)+O(1)\\&=nm\left(r,\frac{\Delta_c f}{f(z)}\right)+nN\left(r,\frac{\Delta_c f}{f(z)}\right)+O(1)\le(2n-\varepsilon n)T(r,f(z))+S(r,f).\end{aligned}$$

which is impossible for $\varepsilon>0$. Thus, Theorem 1.5 is proved. ∎

5. Proof of Theorem 1.7

The following lemma is a difference analogue of the second main theorem of Nevanlinna theory, which is given by Halburd and Korhonen in [4].

Lemma 5.1. [4] *Let $c \in \mathbb{C}$, and let $f(z)$ be a meromorphic function of finite order such that $\Delta_c f \not\equiv 0$. Let $q \geq 2$, and let $a_1(z), \ldots, a_q(z)$ be distinct meromorphic periodic functions with period c such that $a_k \in S(f)$ for all $k = 1, \ldots, q$. Then*

$$m(r,f) + \sum_{k=1}^{q} m\left(r, \frac{1}{f-a_k}\right) \leq 2T(r,f) - N_{pair}(r,f) + S(r,f), \tag{5.1}$$

where

$$N_{pair}(r,f) := 2N(r,f) - N(r,\Delta_c f) + N\left(r, \frac{1}{\Delta_c f}\right),$$

and the exceptional set associated with $S(r,f)$ is of at most finite logarithmic measure.

Proof of Theorem 1.7. It is evident that $\Delta_c f \not\equiv 0$. By the condition of Theorem 1.7, we still have $N(r,\Delta_c f) = N(r, f(z))$ and (3.1). Since $E_{f(z)}(S,\infty) = E_{\Delta_c f}(S,\infty)$, where $S = \{\omega | \omega^n + a\omega^{n-m} + b = 0\}$ and the equation $\omega^n + a\omega^{n-m} + b = 0$ has no multiple roots, we see that $(\Delta_c f)^n + a(\Delta_c f)^{n-m} + b$ and $f(z)^n + af(z)^{n-m} + b$ share 0 CM. Thus, from $E_{f(z)}(\{\infty\},\infty) = E_{\Delta_c f}(\{\infty\},\infty)$, we have

$$\frac{(\Delta_c f)^n + a(\Delta_c f)^{n-m} + b}{f(z)^n + af(z)^{n-m} + b} = e^{p(z)}, \tag{5.2}$$

where $p(z)$ is a polynomial.

Now suppose that $e^{p(z)} \equiv 1$. Then $(\Delta_c f)^n + a(\Delta_c f)^{n-m} \equiv f(z)^n + af(z)^{n-m}$. By denoting $h(z) = (\Delta_c f)/(f(z))$, we get

$$f(z)^m (h(z)^n - 1) = -a(h(z)^{n-m} - 1). \tag{5.3}$$

If $h(z)$ is not a constant, we rewrite (5.3) as

$$\begin{aligned} &f(z)^m (h(z)-1)(h(z)-\mu)\cdots(h(z)-\mu^{n-1}) \\ &= -a(h(z)-1)(h(z)-\nu)\cdots(h(z)-\nu^{n-m-1}), \end{aligned} \tag{5.4}$$

where $\mu = \cos(2\pi/n) + i\sin(2\pi/n)$ and $\nu = \cos(2\pi/(n-m)) + i\sin(2\pi/(n-m))$.

Since n and $n-m$ have no common factors, $\mu, \ldots, \mu^{n-1}, \nu, \ldots, \nu^{n-m-1}$ are different. Suppose z_0 is a μ^j–point of $h(z)$ of multiplicity $s_j > 0$, where $1 \leq j \leq n-1$. Then we note that $-a(h(z_0)-1)(h(z_0)-\nu)\cdots(h(z_0)-\nu^{n-m-1})$ is a constant. By (5.4), we know that z_0 is a pole of $f(z)^m$. Obviously, $s_j \geq m$.

It follows that, for $1 \leq j \leq n-1$,

$$m\overline{N}\left(r, \frac{1}{h(z)-\mu^j}\right) \leq N\left(r, \frac{1}{h(z)-\mu^j}\right) \leq T(r,h(z)) + S(r,h). \tag{5.5}$$

Then by (5.5), we have

(5.6)

$$2 \geq \sum_{j=1}^{n-1} \Theta(\mu^j, h(z)) = \sum_{j=1}^{n-1} \left(1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}\left(r, \frac{1}{h(z)-\mu^j}\right)}{T(r,h(z))}\right) \geq \sum_{j=1}^{n-1}\left(1 - \frac{1}{m}\right) = (n-1)\left(1 - \frac{1}{m}\right),$$

which is impossible with $m \geq 2$ and $n \geq 2m+4$. Therefore, $h(z)$ is a constant. Since $f(z)$ is a nonconstant meromorphic function, we deduce from (5.3) that $h(z) \equiv 1$. Hence $\Delta_c f \equiv f(z)$.

Suppose that $e^{p(z)} \not\equiv 1$. Noting that $S = \{\omega | \omega^n + a\omega^{n-m} + b = 0\}$ and the equation $\omega^n + a\omega^{n-m} + b = 0$ has no multiple roots, we assume that $\omega_1, \ldots, \omega_n$ are all different roots of the equation $\omega^n + a\omega^{n-m} + b = 0$. From (5.2) and Lemma 2.1, we get

$$\begin{aligned} T(r, e^{p(z)}) &= m(r, e^{p(z)}) = m\left(r, \frac{(\Delta_c f)^n + a(\Delta_c f)^{n-m} + b}{f(z)^n + af(z)^{n-m} + b}\right) \\ &= m\left(r, \frac{(\Delta_c f - \omega_1)\cdots(\Delta_c f - \omega_n)}{(f(z) - \omega_1)\cdots(f(z) - \omega_n)}\right) \\ &\leq \sum_{i=1}^{n} m\left(r, \frac{\Delta_c f}{f(z) - \omega_i}\right) + \sum_{i=1}^{n} m\left(r, \frac{1}{f(z) - \omega_i}\right) + O(1) \\ &= \sum_{i=1}^{n} m\left(r, \frac{1}{f(z) - \omega_i}\right) + S(r, f). \end{aligned} \tag{5.7}$$

It follows from Lemma 5.1 that

$$\begin{aligned} \sum_{i=1}^{n} m\left(r, \frac{1}{f(z) - \omega_i}\right) &\leq 2T(r, f(z)) - m(r, f(z)) - 2N(r, f(z)) + N(r, \Delta_c f) \\ &\quad - N\left(r, \frac{1}{\Delta_c f}\right) + S(r, f) = T(r, f(z)) - N\left(r, \frac{1}{\Delta_c f}\right) + S(r, f). \end{aligned} \tag{5.8}$$

Combining (1.3), (5.7) and (5.8) gives

$$T(r, e^{p(z)}) \leq T(r, f(z)) - N\left(r, \frac{1}{\Delta_c f}\right) + S(r, f) = S(r, f). \tag{5.9}$$

Rewriting (5.2), we get

$$(\Delta_c f)^{n-m}[(\Delta_c f)^m + a] = [f(z)^n + af(z)^{n-m} + b - be^{-p(z)}]e^{p(z)}. \tag{5.10}$$

Set $g(z) = f(z)^n + af(z)^{n-m}$. By the standard Valiron-Mohon'ko theorem (see [10]) and $m > 0$, we have

$$T(r, g(z)) = nT(r, f(z)) + S(r, f). \tag{5.11}$$

Obviously, $S(r, g) = S(r, f)$.

Applying the second main theorem for three small target functions, by (3.1) and (5.10), we get

$$\begin{aligned} T(r, g(z)) &\leq \overline{N}(r, g(z)) + \overline{N}\left(r, \frac{1}{g(z)}\right) + \overline{N}\left(r, \frac{1}{g(z) + b - be^{-p(z)}}\right) + S(r, g) \\ &\leq \overline{N}(r, f(z)) + \overline{N}\left(r, \frac{1}{f(z)^{n-m}[f(z)^m + a]}\right) + \overline{N}\left(r, \frac{1}{(\Delta_c f)^{n-m}}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{(\Delta_c f)^m + a}\right) + S(r, f) \\ &\leq \overline{N}(r, f(z)) + \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f(z)^m + a}\right) + \overline{N}\left(r, \frac{1}{\Delta_c f}\right) \end{aligned} \tag{5.12}$$

$$\begin{aligned}
&+T\left(r,\frac{1}{(\Delta_c f)^m+a}\right)+S(r,f)\\
\leq &T(r,f(z))+T\left(r,\frac{1}{f(z)}\right)+T\left(r,\frac{1}{f(z)^m+a}\right)+T\left(r,\frac{1}{\Delta_c f}\right)\\
&+mT(r,\Delta_c f)+S(r,f)\\
\leq &(m+2)T(r,f(z))+(m+1)T(r,\Delta_c f)+S(r,f)\\
\leq &(2m+3)T(r,f(z))+S(r,f).
\end{aligned}$$

By (5.11) and (5.12), we get

$$(n-2m-3)T(r,f(z))\leq S(r,f),$$

which contradicts with $n\geq 2m+4$. This completes our proof of Theorem 1.7. ∎

Acknowledgement. This work was supported by the National Natural Science Foundation of China (No: 11171119). The authors would like to thank the referees for their valuable suggestions which have greatly improved this paper.

References

[1] A. Banerjee, Meromorphic functions sharing two sets, *Czechoslovak Math. J.* **57(132)** (2007), no. 4, 1199–1214.

[2] Y.-M. Chiang and S.-J. Feng, On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane, *Ramanujan J.* **16** (2008), no. 1, 105–129.

[3] J. Dou, X.-G. Qi and L.-Z. Yang, Entire functions that share fixed-points, *Bull. Malays. Math. Sci. Soc. (2)* **34** (2011), no. 2, 355–367.

[4] R. G. Halburd and R. J. Korhonen, Nevanlinna theory for the difference operator, *Ann. Acad. Sci. Fenn. Math.* **31** (2006), no. 2, 463–478.

[5] W. K. Hayman, *Meromorphic Functions*, Oxford Mathematical Monographs Clarendon Press, Oxford, 1964.

[6] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo and J. Zhang, Value sharing results for shifts of meromorphic functions, and sufficient conditions for periodicity, *J. Math. Anal. Appl.* **355** (2009), no. 1, 352–363.

[7] I. Lahiri, Weighted sharing and uniqueness of meromorphic functions, *Nagoya Math. J.* **161** (2001), 193–206.

[8] I. Lahiri and A. Banerjee, Weighted sharing of two sets, *Kyungpook Math. J.* **46** (2006), no. 1, 79–87.

[9] I. Lahiri and P. Sahoo, Uniqueness of meromorphic functions sharing three weighted values, *Bull. Malays. Math. Sci. Soc. (2)* **31** (2008), no. 1, 67–75.

[10] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, de Gruyter Studies in Mathematics, 15, de Gruyter, Berlin, 1993.

[11] I. Laine and C.-C. Yang, Clunie theorems for difference and q-difference polynomials, *J. Lond. Math. Soc. (2)* **76** (2007), no. 3, 556–566.

[12] X.-M. Li and H.-X. Yi, On uniqueness theorems of meromorphic functions concerning weighted sharing of three values, *Bull. Malays. Math. Sci. Soc. (2)* **33** (2010), no. 1, 1–16.

[13] P. Li and C.-C. Yang, Some further results on the unique range sets of meromorphic functions, *Kodai Math. J.* **18** (1995), no. 3, 437–450.

[14] J. Xia and Y. Xu, Normality criterion concerning sharing functions II, *Bull. Malays. Math. Sci. Soc. (2)* **33** (2010), no. 3, 479–486.

[15] C.-C. Yang and H.-X. Yi, *Uniqueness Theory of Meromorphic Functions*, Mathematics and its Applications, 557, Kluwer Acad. Publ., Dordrecht, 2003.

[16] H.-X. Yi, Unicity theorems for entire functions, *Kodai Math. J.* **17** (1994), no. 1, 133–141.

[17] H.-X. Yi, Meromorphic functions that share one or two values. II, *Kodai Math. J.* **22** (1999), no. 2, 264–272.

[18] H.-X. Yi and L.-Z. Yang, Meromorphic functions that share two sets, *Kodai Math. J.* **20** (1997), no. 2, 127–134.

[19] J. Zhang, Value distribution and shared sets of differences of meromorphic functions, *J. Math. Anal. Appl.* **367** (2010), no. 2, 401–408.