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Use of Theory of Conformal Mappings in Harmonic Univalent Mappings with Directional Convexity

OM P. AHUJA
Kent State University, 14111 Claridon Troy Road, Burton, Ohio 44021, U.S.A.
oahuja@kent.edu

Abstract. In this paper we use shear construction to generate certain subclasses of harmonic univalent mappings with directional convexity because it allows us to study such functions by examining their related conformal univalent mappings. In this setting we find growth, distortion, and coefficient bounds for harmonic univalent mappings that are convex in both the directions of real axis and imaginary axis.

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1. Introduction and preliminaries

Let $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk where $\mathbb{C}$ denotes complex plane. A twice continuously differentiable complex-valued function f defined on Δ is said to be harmonic in Δ if f satisfies the partial differential equation

$$\Delta f = 4 f_{z\bar{z}} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

where we use the common notations for its partial derivatives given by

$$f_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad f_{\bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Since Δ is simply connected, a harmonic function f has the canonical representation given by $f = h + \bar{g}$, where h and g are members of the linear space $\mathscr{A}(\Delta)$ of all analytic functions in Δ, and where h and g can be written as a power series representation

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \tag{1.1}$$

respectively. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for a harmonic function f of the form $f = h + \bar{g}$ to be locally univalent

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and sense-preserving in Δ is that $|g'(z)| < |h'(z)|$ for all z in Δ. The analytic dilatation of a harmonic mapping $f = h + \bar{g}$ is defined by $\omega(z) = (g'(z)/h'(z))$. Thus if f is locally univalent and sense-preserving, then $|\omega(z)| < 1$.

The class of all sense-preserving harmonic univalent mappings f of the form $f = h + \bar{g}$, where h and g given by (1.1) is denoted by $\mathscr{S}_H$. An important subclass of $\mathscr{S}_H$ is $\mathscr{S}_H^0$ which consists of all functions$f = h + \bar{g}$ in $\mathscr{S}_H$ with h and g given by (1.1), where $b_1 = 0$. The standard references for the families $\mathscr{S}_H$ and $\mathscr{S}_H^0$ are [3], [4], and [5] and a survey article [1]. The class $\mathscr{S}_H$contains the standard family $\mathscr{S}$ of all analytic normalized univalent functions in Δ. Note that if the co-analytic part, g, is zero, then the function $f = h + \bar{g}$ in $\mathscr{S}_H$ is also in $\mathscr{S}$. Also, note that a function φ in S is known as a conformal univalent mapping in Δ. For standard references for class $\mathscr{S}$ and its subclasses, one may refer to [2], [6], and [7].

Recall that a domain $D \subset \mathbb{C}$is said to be convex in one direction of the line $z = te^{i\theta}$, $t \in \mathbb{R}$ for a given constant $z_0 \in \mathbb{C}$ and given $\theta \in [0, \pi)$if $D \cap \{z_0 + te^{i\theta} : t \in \mathbb{R}\}$ is either connected or empty. Note that two lines $z = te^{i\theta}$ and $z = te^{i(\theta+\pi/2)}$,$t \in \mathbb{R}$ are orthogonal directions for given constant θ. For a given constant θ, define two subclasses of sense-preserving harmonic univalent functions as follows:

$$COD_H(\theta) := \{f \in \mathscr{S}_H : f(\Delta) \text{ is convex in } z = te^{i\theta} \text{ and } z = te^{i(\theta+\pi/2)}, t \in \mathbb{R}\},$$

$$COD_H^0\theta) := \{f \in \mathscr{S}_H^0 : f(\Delta) \text{ is convex in } z = te^{i\theta} \text{ and } z = te^{i(\theta+\pi/2)}, t \in \mathbb{R}\}.$$

Here it suffices to take $\theta \in [0, \pi/2)$ because $\theta \in [\pi/2, \pi)$ produces the same results as $\theta \in [0, \pi/2)$. In particular, we define the classes COD_H and COD_H^0 as follows:

$$COD_H := \{f \in \mathscr{S}_H : f(\Delta) \text{ is convex in the directions of real axis and the imaginary axis}\},$$

$$COD_H^0 := \{f \in \mathscr{S}_H^0 : f(\Delta) \text{ is convex in the directions of real axis and the imaginary axis}\}.$$

Note that $COD_H \equiv COD_H(0)$ and $COD_H^0 \equiv COD_H^0(0)$. Also, note that for every function $F \in \bigcup\{COD_H(\theta) : \theta \in [0, \pi/2)\}$, there exists a function $f \in COD_H$ such that $F(z) = e^{i\theta} f\left(e^{-i\theta} z\right)$ for some $\theta \in [0, \pi/2)$. The classes $COD_H(\theta)$, $COD_H^0(\theta)$, COD_H, and COD_H^0 were introduced in [8]. Finally, we observe that if $f \in \mathscr{S}_H$ and $f(\Delta)$ is convex, then $f \in COD_H(\theta)$ for every θ. This class is justified because the function $L = h + \bar{g}$, where

$$h(z) = \int_0^z \frac{\left(1 - \zeta^3\right)^{-2/3}}{(1 + \varsigma^3)} d\varsigma, \quad g(z) = \int_0^z \frac{\varsigma^3 \left(1 - \varsigma^3\right)^{-2/3}}{(1 + \varsigma^3)} d\varsigma,$$

is univalent and convex in the direction of both real axis and imaginary axis and so L belongs to the family COD_H ; see [8]. Note that the function L is neither convex nor even starlike.

Clunie and Sheil-Small in 1984 [3] discovered a general method, known as 'shear construction' for constructing harmonic univalent mappings with specified properties. This method essentially produces a harmonic univalent mapping onto a convex domain in one direction by "shearing" (or stretching, or translating) a given conformal univalent mapping along parallel lines. The basic shear construction theorem is as follows.

Theorem 1.1. [3] *A harmonic and locally univalent function $f = h + \bar{g}$ is univalent mapping of Δ onto a domain convex in the direction of real axis if and only if $h - g$ is a conformal univalent mapping of Δ onto a domain convex in the direction of real axis.*

Theorem 1.1 has a natural generalization when f is convex in the direction of the line $te^{i\theta}$, $t \in \mathbb{R}$. Thus $e^{-i\theta} f$ and $e^{-i\theta} h - e^{i\theta} g$ are convex in the direction of real axis. It, therefore, follows that the function $h - e^{i2\theta} g$ is convex in the direction of the line $te^{i\theta}$,$t \in \mathbb{R}$. In view of these observations; Theorem A immediately gives the following generalized version of shear construction theorem.

Theorem 1.2. *A harmonic and locally univalent function $f = h + \bar{g}$ is univalent mapping of Δ onto a domain convex in the direction of the line $z = te^{i\theta}$, $0 \leqslant \theta < \pi$, $t \in \mathbb{R}$, if and only if $h - e^{i2\theta} g$ is a conformal univalent mapping of Δ onto a domain convex in the same direction.*

Note that we can use Theorem 1.2 to construct harmonic mappings that are convex in the direction of imaginary axis. Moreover, for $f = h + \bar{g} \in COD_H(\theta)$, where h and g are given by (1.1), Theorem 1.2 has led us to construct the function $T_\theta : \mathscr{A}(\Delta) \to \mathscr{A}(\Delta)$, with suitable normalization, given by

$$T_\theta(z) := \frac{h(z) - e^{i2\theta} g(z)}{1 - e^{i2\theta} b_1}, \qquad 0 \leqslant \theta < \pi. \tag{1.2}$$

Since $f \in \mathscr{S}_H$ is sense-preserving, it follows that $|b_1| < 1$. Thus, in view of Theorem 1.2, it follows that $T_\theta \in \mathscr{S}$. For every θ=0 and θ=π/2, the function T_θ reduces to two special functions

$$\eta(z) := \frac{h(z) - g(z)}{1 - b_1}, \quad \psi(z) := \frac{h(z) + g(z)}{1 + b_1} \tag{1.3}$$

in the family $\mathscr{S}$.

We define the following subclasses of the class $COD_H(\theta)$generated by T_θ :

$$\begin{aligned}
\mathscr{S}^*_\alpha COD_H(\theta) &:= \{f \in COD_H(\theta) : T_\theta \in \mathscr{S}^*(\alpha)\},\\
\mathscr{K}_\alpha COD_H(\theta) &:= \{f \in COD_H(\theta) : T_\theta \in \mathscr{K}(\alpha)\},\\
\mathscr{S}^*_\alpha \hat{C}_H &:= \bigcup_{\theta \in [0,\pi/2)} \mathscr{S}^*_\alpha COD_H(\theta),\\
\mathscr{K}_\alpha \hat{C}_H &:= \bigcup_{\theta \in [0,\pi/2)} \mathscr{K}_\alpha COD_H(\theta).
\end{aligned}$$

where $0 \leqslant \theta < \pi/2$ and $0 \leqslant \alpha \leqslant 1$. In particular, we define

$$\mathscr{S}^*_\alpha COD_H := \mathscr{S}^*_\alpha COD_H(0), \quad \mathscr{K}_\alpha COD_H := \mathscr{K}_\alpha COD_H(0).$$

We next recall the following two well-known subclasses of $\mathscr{S}$:

$$\mathscr{S}^*(\alpha) := \left\{\varphi \in S : \mathrm{Re}\, \frac{z\varphi'(z)}{\varphi(z)} \geqslant \alpha,\ z \in \Delta,\ 0 \leqslant \alpha \leqslant 1\right\},$$

$$\mathscr{K}(\alpha) := \left\{\varphi \in S : \mathrm{Re}(1 + \frac{z\varphi''(z)}{\varphi'(z)}) \geqslant \alpha,\ z \in \Delta,\ 0 \leqslant \alpha \leqslant 1\right\}.$$

A function f in $\mathscr{S}^*(\alpha)$ or $\mathscr{K}(\alpha)$ is called starlike of order α or convex of order α, respectively. Note that $\mathscr{S}^* \equiv \mathscr{S}^*(0)$ and $\mathscr{K} \equiv \mathscr{K}(0)$.

In the present paper, we apply some classical results of the growth, distortion, and coefficient estimates of the families $\mathscr{S}^*(\alpha)$ and $\mathscr{K}(\alpha)$ to the classes $\mathscr{S}^*_\alpha \hat{C}_H$, $\mathscr{K}_\alpha \hat{C}_H$, $\mathscr{S}^*_\alpha COD_H(\theta)$, and $\mathscr{K}_\alpha COD_H(\theta)$. In particular, we also address the case when $b_1 = 0$ for the corresponding subclasses $\mathscr{S}^*_\alpha \hat{C}^0_H$, $\mathscr{S}^*_\alpha COD^0_H(\theta)$, $\mathscr{K}_\alpha \hat{C}^0_H$, and $\mathscr{K}_\alpha COD^0_H(\theta)$.

2. Lemmas

The following observation offers a simple relationship between the classes $\mathscr{S}^*_\alpha \hat{C}_H$ and$\mathscr{S}^*_\alpha COD_H$.

Lemma 2.1. *For every function F in $\mathscr{S}^*_\alpha \hat{C}_H$, there exits a function f in $\mathscr{S}^*_\alpha COD_H$ such that $F(z) = e^{i\theta} f(e^{-i\theta} z)$ for any constant $\theta \in [0, \pi/2)$.*

A similar observation for the relationship between the classes $\mathscr{K}_\alpha \hat{C}_H$ and $\mathscr{K}_\alpha COD_H$ is stated in the following.

Lemma 2.2. *For every function F in $\mathscr{K}_\alpha \hat{C}_H$, there exists a function f in $K_\alpha COD_H$ such that $F(z) = e^{i\theta} f(e^{-i\theta} z)$ for any constant $\theta \in [0, \pi/2)$.*

In next six lemmas, we recall some well-known results of the classical theory of conformal mappings.

Lemma 2.3. [7, 9]. *Let $z \in \Delta$, $|z| = r$ and suppose $\varphi \in \mathscr{K}(\alpha)$. If $0 \leqslant \alpha \leqslant 1$, then*

$$\frac{1}{(1+r)^{2(1-\alpha)}} \leqslant |\varphi'(z)| \leqslant \frac{1}{(1-r)^{2(1-\alpha)}}.$$

If $\alpha \neq 1/2$, then

$$\frac{(1+r)^{2\alpha-1} - 1}{2\alpha-1} \leqslant |\varphi(z)| \leqslant \frac{1-(1-r)^{2\alpha-1}}{2\alpha-1}.$$

If $\alpha = 1/2$, then

$$\ln(1+r) \leqslant |\varphi(z)| \leqslant -\ln(1-r).$$

All these inequalities are sharp for

$$\varphi(z) = \begin{cases} \frac{1-(1-z)^{2\alpha-1}}{2\alpha-1} & \text{if } \alpha \neq \frac{1}{2} \\ -\ln(1-z) & \text{if } \alpha = \frac{1}{2}. \end{cases} \tag{2.1}$$

Lemma 2.4. [7, 9]. *Let $z \in \Delta$ and $|z| = r$. If $\varphi \in \mathscr{S}^*(\alpha)$with $0 \leqslant \alpha \leqslant 1$, then*

$$\frac{r}{(1+r)^{2(1-\alpha)}} \leqslant |\varphi(z)| \leqslant \frac{r}{(1-r)^{2(1-\alpha)}}.$$

These bounds are sharp for

$$k_0(z) = \frac{z}{(1-z)^{2(1-\alpha)}}. \tag{2.2}$$

Lemma 2.5. [7, 9]. *If $\varphi(z) = z + \sum_{n=2}^{\infty} A_n z^n \in \mathscr{S}^*(\alpha)$, $0 \leqslant \alpha \leqslant 1$, then*

$$|A_n| \leqslant \frac{1}{(n-1)!} \prod_{k=2}^{n} (k - 2\alpha)$$

for $n = 2, 3, \ldots$. These bounds are sharp for the function (2.2).

Lemma 2.6. [7, 9]. *If $\varphi(z) = z + \sum_{n=2}^{\infty} A_n z^n \in \mathscr{K}(\alpha)$, $0 \leqslant \alpha \leqslant 1$, then*

$$|A_n| \leqslant \frac{1}{n!} \prod_{k=2}^{n} (k - 2\alpha)$$

for $n = 2, 3, \ldots$. These bounds are sharp for the function (2.1).

3. Distortion and growth theorems

Theorem 3.1. *Let $z \in \Delta$, $|z| = r$ and suppose a function $f = h + \bar{g}$ with h and g given by* (1.1) *is in $\mathscr{K}_\alpha \hat{C}_H$. If $0 \leqslant \alpha \leqslant 1$, then*

$$\frac{1+|b_1|^2}{(1+r)^{4(1-\alpha)}} \leqslant |h'(z)|^2 + |g'(z)|^2 \leqslant \frac{1+|b_1|^2}{(1-r)^{4(1-\alpha)}}. \tag{3.1}$$

If $\alpha \neq 1/2$, then

$$\frac{((1+r)^{2\alpha-1}-1)^2(1+|b_1|^2)}{(2\alpha-1)^2} \leqslant |h(z)|^2 + |g(z)|^2 \leqslant \frac{(1-(1-r)^{2\alpha-1})^2(1+|b_1|^2)}{(2\alpha-1)^2}. \tag{3.2}$$

If $\alpha = 1/2$, then

$$(\ln(1+r))^2(1+|b_1|^2) \leqslant |h(z)|^2 + |g(z)|^2 \leqslant (\ln(1-r))^2(1+|b_1|^2). \tag{3.3}$$

All of these inequalities are sharp.

Proof. We first assume that $f \in \mathscr{K}_\alpha COD_H(\theta)$ and therefore $f \in COD_H(\theta)$ and T_θ given by (1.2) is in $\mathscr{K}(\alpha)$ for each $\theta (0 \leqslant \theta < \pi/2)$. Using Lemma 2.3, we obtain

$$\frac{1}{(1+r)^{2(1-\alpha)}} \leqslant \left| \frac{h'(z) - e^{i2\theta} g'(z)}{1 - b_1 e^{i2\theta}} \right| \leqslant \frac{1}{(1-r)^{2(1-\alpha)}}.$$

Choosing $\theta = 0$ and $\theta \to (\pi/2)^-$, we have $f \in \mathscr{K}_\alpha COD_H$ and

$$\frac{|1-b_1|^2}{(1+r)^{4(1-\alpha)}} \leqslant |h'(z) - g'(z)|^2 \leqslant \frac{|1-b_1|^2}{(1-r)^{4(1-\alpha)}},$$

and

$$\frac{|1+b_1|^2}{(1+r)^{4(1-\alpha)}} \leqslant |h'(z) + g'(z)|^2 \leqslant \frac{|1+b_1|^2}{(1-r)^{4(1-\alpha)}}.$$

Adding corresponding parts of the last two inequalities, we obtain (3.1) provided that$f \in \mathscr{K}_\alpha COD_H$. Because of Lemma 2.2, it follows that (3.1) also holds for any f in $\mathscr{K}_\alpha \hat{C}_H$. Using Lemma 2.3 and Lemma 2.2 and following the proof of (3.1), it is routine to prove the inequalities (3.2) and (3.3). In view of (1.2) and (2.1), it is a routine to verify that the inequalities (3.1) to (3.3) are sharp for the functions $f = h + \bar{g}$ in $\mathscr{K}_\alpha \hat{C}_H$, where

$$h(z) = \begin{cases} \frac{1-(1-z)^{2\alpha-1}}{2\alpha-1} & \text{if } \alpha \neq \frac{1}{2}, \\ -\ln(1-z) & \text{if } \alpha = \frac{1}{2}, \end{cases} \tag{3.4}$$

$$g(z) = \begin{cases} \frac{b_1(1-(1-z)^{2\alpha-1})}{2\alpha-1}, & \text{if } \alpha \neq \frac{1}{2}, \\ -b_1 \ln(1-z) & \text{if } \alpha = \frac{1}{2}. \end{cases} \tag{3.5}$$

∎

Corollary 3.1. *Let $z \in \Delta$, $|z| = r$. If a function $f = h + \bar{g}$ with h and g given by* (1.1) *is in $\mathscr{K}_\alpha \hat{C}_H$, then*

$$\frac{\sqrt{2}}{2(1+r)^{2(1-\alpha)}} < |h'(z)| < \frac{\sqrt{2}}{(1-r)^{2(1-\alpha)}}, \tag{3.6}$$

$$0 \leqslant |g'(z)| < \frac{1}{(1-r)^{2(1-\alpha)}}. \tag{3.7}$$

Proof. From compound inequality (3.1) and using $0 \leqslant |b_1| < 1$, we have

$$\frac{1}{(1+r)^{4(1-\alpha)}} \leqslant |h'(z)|^2 + |g'(z)|^2 < \frac{2}{(1-r)^{4(1-\alpha)}}. \tag{3.8}$$

Since f is sense-preserving, we have $0 \leqslant |g'(z)| < |h'(z)|$ for all z in Δ. Therefore,

$$0 \leqslant 2|g'(z)|^2 < |h'(z)|^2 + |g'(z)|^2 < \frac{2}{(1-r)^{4(1-\alpha)}}.$$

This proves (3.7). In order to prove (3.6), we note that

$$|h'(z)|^2 \leqslant |h'(z)|^2 + |g'(z)|^2 < \frac{2}{(1-r)^{4(1-\alpha)}}$$

and

$$2|h'(z)|^2 > |h'(z)|^2 + |g'(z)|^2 \geqslant \frac{1}{(1+r)^{4(1-\alpha)}}$$

by (3.8). The desired compound inequality (3.6) immediately follows. ∎

Corollary 3.2. *Let $z \in \Delta$, $|z| = r$. If a function $f = h + \bar{g}$ with h and g given by* (1.1) *is in $\mathscr{K}_\alpha \hat{C}_H^0$, then*

$$\frac{1}{\sqrt{1+r^2}(1+r)^{2(1-\alpha)}} \leqslant |h'(z)| \leqslant \frac{1}{(1-r)^{2(1-\alpha)}}, \tag{3.9}$$

$$0 \leqslant |g'(z)| < \frac{r}{(1-r)^{2(1-\alpha)}}. \tag{3.10}$$

Proof. Since $b_1 = 0$, it follows from (3.1) that

$$\frac{1}{(1+r)^{4(1-\alpha)}} \leqslant |h'(z)|^2 + |g'(z)|^2 \leqslant \frac{1}{(1-r)^{4(1-\alpha)}}. \tag{3.11}$$

Again, since the analytic dilatation $\omega = g'/h'$ of the function $f = h + \bar{g}$ in $\mathscr{S}_H^0$ satisfies the condition $|\omega| < 1$, therefore, Schwartz lemma yields

$$|g'(z)| \leqslant |z||h'(z)|, \qquad z \in \Delta. \tag{3.12}$$

In view of the inequalities (3.11) and (3.12), we obtain

$$(1+r^2)|h'(z)|^2 \geqslant |h'(z)|^2 + |g'(z)|^2 \geqslant \frac{1}{(1+r)^{4(1-\alpha)}},$$

and

$$|h'(z)|^2 \leqslant |h'(z)|^2 + |g'(z)|^2 \leqslant \frac{1}{(1-r)^{4(1-\alpha)}}.$$

These two inequalities together prove (3.9). On the other hand, (3.10) follows from (3.12) and the right side inequality of (3.9). ∎

Theorem 3.2. *Let $z \in \Delta$, $|z| = r$ and suppose a function $f = h + \bar{g}$ with h and g given by* (1.1) *is in $\mathscr{S}_\alpha^* \hat{C}_H$. Then*

$$\frac{r^2(1+|b_1|^2)}{(1+r)^{4(1-\alpha)}} \leqslant |h(z)|^2 + |g(z)|^2 \leqslant \frac{r^2(1+|b_1|^2)}{(1-r)^{4(1-\alpha)}}, \tag{3.13}$$

$$0 \leqslant |h(z)| < \frac{\sqrt{2}r}{(1-r)^{2(1-\alpha)}}, \tag{3.14}$$

$$0 \leqslant |g(z)| < \frac{\sqrt{2}r}{(1-r)^{2(1-\alpha)}}, \tag{3.15}$$

$$0 \leqslant |f(z)| < \frac{2r}{(1-r)^{2(1-\alpha)}}. \tag{3.16}$$

The bounds in (3.13) *are sharp for the functions* $f = h + \bar{g}$ *in* $\mathscr{S}^*_\alpha \hat{C}_H$*, where* h *and* g *are given by*

$$h(z) = \frac{z}{(1-z)^{2(1-\alpha)}}, \tag{3.17}$$

$$g(z) = \frac{b_1 z}{(1-z)^{2(1-\alpha)}}. \tag{3.18}$$

Proof. Let $f \in \mathscr{S}^*_\alpha COD_H(\theta)$ so that $f \in COD_H(\theta)$ and $T_\theta \in \mathscr{S}^*(\alpha)$for each $\theta \in [0, \pi/2)$. Applying Lemma 2.4 we obtain

$$\frac{r}{(1+r)^{2(1-\alpha)}} \leqslant \left| \frac{h(z) - e^{i2\theta} g(z)}{1 - b_1 e^{i2\theta}} \right| \leqslant \frac{r}{(1-r)^{2(1-\alpha)}}. \tag{3.19}$$

Letting $\theta = 0$ and $\theta \to (\pi/2)^-$, squaring and adding respective sides of the inequalities so obtained, we find that (3.13) holds for $f \in \mathscr{S}^*_\alpha COD_H$. In view of Lemma 2.1, it follows that (3.13) also holds true for$f \in \mathscr{S}^*_\alpha \hat{C}_H$. For proving (3.14), first note that (3.13) yields

$$\frac{r^2}{(1+r)^{4(1-\alpha)}} \leqslant |h(z)|^2 + |g(z)|^2 < \frac{2r^2}{(1-r)^{4(1-\alpha)}},$$

because $0 \leqslant |b_1| < 1$. Since $|g(z)| \geqslant 0$, the above inequality gives

$$|h(z)|^2 \leqslant |h(z)|^2 + |g(z)|^2 < \frac{2r^2}{(1-r)^{4(1-\alpha)}} \tag{3.20}$$

and therefore (3.14) follows. The proof of (3.15) is similar. Also, (3.16) is proven from

$$|f(z)| \leqslant \sqrt{2(|h(z)|^2 + |g(z)|^2)} < \frac{2r}{(1-r)^{2(1-\alpha)}},$$

by using (3.20). Finally, (3.17 and (3.8) follow by using (1.2) in (2.2). ∎

4. Coefficient bounds

Theorem 4.1. *If a function*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n} \tag{4.1}$$

is in $\mathscr{K}_\alpha \hat{C}_H$*, then*

$$|a_n|^2 + |b_n|^2 \leqslant \frac{1 + |b_1|^2}{(n!)^2} \prod_{k=2}^{n} (k - 2\alpha)^2 \tag{4.2}$$

for $n = 2, 3, \ldots$. *The bounds in* (3.13) *are sharp for the functions* $f = h + \bar{g}$ *in* $\mathscr{K}_\alpha \hat{C}_H$*, where* h *and* g *are given by* (3.4) *and* (3.5).

Proof. Suppose $f \in \mathscr{K}_\alpha COD_H(\theta)$ so that $f \in COD_H(\theta)$ and the function

$$T_\theta(z) = \frac{h(z) - e^{i2\theta} g(z)}{1 - e^{i2\theta} b_1} = z + \sum_{n=2}^{\infty} \frac{a_n - e^{i2\theta} b_n}{1 - e^{i2\theta} b_1} z^n \tag{4.3}$$

is in $\mathscr{K}(\alpha)$ for each $\theta \in [0, \pi/2)$. Applying Lemma 2.6, we obtain

$$\left| \frac{a_n - e^{i2\theta} b_n}{1 - e^{i2\theta} b_1} \right| \leqslant \frac{1}{n!} \prod_{k=2}^{n} (k - 2\alpha)$$

for $n = 2, 3, \ldots$. Setting $\theta = 0$ and $\theta \to (\pi/2)^-$, we have

$$|a_n - b_n|^2 \leqslant \frac{|1 - b_1|^2}{(n!)^2} \prod_{k=2}^{n} (k - 2\alpha)^2,$$

$$|a_n + b_n|^2 \leqslant \frac{|1 + b_1|^2}{(n!)^2} \prod_{k=2}^{n} (k - 2\alpha)^2.$$

Adding respective sides in the last two compound inequalities and simplifying, it follows that (4.1) holds for $f \in \mathscr{K}_\alpha COD_H$. In view of Lemma 2.2, (4.2) holds true for $f \in \mathscr{K}_\alpha \hat{C}_H$. ∎

Corollary 4.1. *If a function f given by* (4.1) *is in $\mathscr{K}_\alpha \hat{C}_H$, then*

$$|a_n| < \frac{\sqrt{2}}{n!} \prod_{k=2}^{n} (k - 2\alpha), \tag{4.4}$$

$$|b_n| < \frac{\sqrt{2}}{n!} \prod_{k=2}^{n} (k - 2\alpha) \tag{4.5}$$

for $n = 2, 3, \ldots$.

Proof. By Theorem 4.1, we have

$$\begin{aligned} |a_n| &\leqslant \sqrt{\frac{1 + |b_1|^2}{(n!)^2} \prod_{k=2}^{n} (k - 2\alpha)^2 - |b_n|^2} \\ &\leqslant \frac{\sqrt{1 + |b_1|^2}}{n!} \prod_{k=2}^{n} (k - 2\alpha) < \frac{\sqrt{2}}{n!} \prod_{k=2}^{n} (k - 2\alpha). \end{aligned} \tag{4.6}$$

This proves (4.4). The proof of (4.5) is similar to that of (4.4) because

$$|b_n| \leqslant \sqrt{\frac{1 + |b_1|^2}{(n!)^2} \prod_{k=2}^{n} (k - 2\alpha)^2 - |a_n|^2} \leqslant \frac{\sqrt{1 + |b_1|^2}}{n!} \prod_{k=2}^{n} (k - 2\alpha). \tag{4.7}$$

∎

Remark 4.1. If a function f given by (4.1) is in $\mathscr{K} \hat{C}_H$, then $|a_n| \leqslant \sqrt{2}$ and $|b_n| \leqslant \sqrt{2}$ for $n = 2, 3, \ldots$.

Letting $b_1 = 0$ in (4.6) and (4.7), we obtain

Corollary 4.2. *If a function f given by* (4.1) *is in $\mathscr{K}_\alpha \hat{C}_H^0$, then*

$$|a_n| \leqslant \frac{1}{n!} \prod_{k=2}^{n} (k - 2\alpha), \quad |b_n| \leqslant \frac{1}{n!} \prod_{k=2}^{n} (k - 2\alpha)$$

for $n = 2, 3, \ldots$.

Theorem 4.2. *If a function f of the form* (4.1) *is in $\mathscr{S}^*_\alpha\hat{C}_H$, then*

$$|a_n|^2+|b_n|^2 \leqslant \frac{1+|b_1|^2}{((n-1)!)^2}\prod_{k=2}^{n}(k-2\alpha)^2 \tag{4.8}$$

for $n=2,3,\ldots$. This inequality is sharp for the functions $f=h+\bar{g}$ in $\mathscr{S}^*_\alpha\hat{C}_H$, where h and g are given by (3.17) *and* (3.18).

Proof. Suppose $f\in\mathscr{S}^*_\alpha COD_H(\theta)$so that $f\in COD_H(\theta)$ and T_θ given by (4.3) is in $\mathscr{S}^*(\alpha)$ for each $\theta\in[0,\pi/2)$. As an application of Lemma 2.5 we obtain

$$\left|\frac{a_n-e^{i2\theta}b_n}{1-e^{i2\theta}b_1}\right| \leqslant \frac{1}{(n-1)!}\prod_{k=2}^{n}(k-2\alpha)$$

for every $n=2,3,\ldots$ and for all $\theta\in[0,\pi/2)$. Suppose $\theta=0$ and $\theta\to(\pi/2)^-$. Then it follows that $f\in\mathscr{S}^*_\alpha COD_H$ and

$$|a_n-b_n|^2 \leqslant \frac{|1-b_1|^2}{((n-1)!)^2}\prod_{k=2}^{n}(k-2\alpha)^2,\quad |a_n+b_n|^2 \leqslant \frac{|1+b_1|^2}{((n-1)!)^2}\prod_{k=2}^{n}(k-2\alpha)^2.$$

Adding respective sides in these two compound inequalities and simplifying, it follows that (4.8) holds for $f\in\mathscr{S}^*_\alpha COD_H$. Now Lemma 2.1 concludes the theorem for $f\in\mathscr{S}^*_H\hat{C}_H$. ∎

Corollary 4.3. *If a function f given by* (4.1) *is in $\mathscr{S}^*_\alpha\hat{C}_H$, then*

$$|a_n| < \frac{\sqrt{2}}{(n-1)!}\prod_{k=2}^{n}(k-2\alpha),\qquad |b_n| < \tfrac{\sqrt{2}}{(n-1)!}\prod_{k=2}^{n}(k-2\alpha)$$

for $n=2,3,\ldots$.

Proof. Since $|b_1|<1$, it follows from Theorem 4.2 that

$$|a_n| \leqslant \sqrt{\frac{1+|b_1|^2}{((n-1)!)^2}\prod_{k=2}^{n}(k-2\alpha)^2-|b_n|^2} < \frac{\sqrt{2}}{(n-1)!}\prod_{k=2}^{n}(k-2\alpha).$$

The proof of second part is similar. ∎

Setting $\alpha=0$ and letting $\mathscr{S}^*\hat{C}_H := \mathscr{S}^*_0\hat{C}_H$, the proof of next corollary follows from Corollary 4.3.

Corollary 4.4. *If a function f given by* (4.1) *is in $\mathscr{K}^*\hat{C}_H$, then*

$$|a_n| < \sqrt{2}n,\qquad |b_n| < \sqrt{2}n$$

for $n=2,3,\ldots$.

Letting $b_1=0$ in the proof of Corollary 4.3 we obtain the following result.

Corollary 4.5. *If a function f given by* (4.1) *is in $\mathscr{S}^*_\alpha\hat{C}^0_H$, then*

$$|a_n| \leqslant \frac{1}{(n-1)!}\prod_{k=2}^{n}(k-2\alpha),\quad |b_n| \leqslant \frac{1}{(n-1)!}\prod_{k=2}^{n}(k-2\alpha)$$

for $n=2,3,\ldots$.

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References

[1] O. P. Ahuja, Planar harmonic univalent and related mappings, *JIPAM. J. Inequal. Pure Appl. Math.* **6** (2005), no. 4, Article 122, 18 pp. (electronic).

[2] O. P. Ahuja, The Bieberbach conjecture and its impact on the developments in geometric function theory, *Math. Chronicle* **15** (1986), 1–28.

[3] J. Clunie and T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **9** (1984), 3–25.

[4] M. Dorff , Harmonic univalent mappings, *Explorations in Complex Variables with Accompanying Applets*, 236–307, Preprint, 2010. (http://www.jimrolf.com/explorationsInComplexVariables.html).

[5] P. Duren, *Harmonic Mappings in the Plane*, Cambridge Tracts in Mathematics, 156, Cambridge Univ. Press, Cambridge, 2004.

[6] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, 259, Springer, New York, 1983.

[7] A. W. Goodman, *Univalent Functions. Vol. I, II*, Mariner, Tampa, FL, 1983.

[8] A. Michalski, Harmonic univalent functions convex in orthogonal directions, *Ann. Univ. Mariae Curie-Skłodowska Sect. A* **60** (2006), 43–56.

[9] M. I. S. Robertson, On the theory of univalent functions, *Ann. of Math. (2)* **37** (1936), no. 2, 374–408.