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Existence of Periodic Solutions for Second Order Hamiltonian System

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Abstract. Some existence theorems are obtained for periodic solutions of second order Hamiltonian system by using the minimax principle. Our results improve those in some known literatures.

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1. Introduction and main results

In this paper, we consider the following Josephson-type system with unbounded nonlinearities

$$\begin{cases} \ddot{u}(t)+Au(t)-\nabla F(t,u(t))=h(t), \text{a.e. } t\in[0,T], \\ u(0)-u(T)=\dot{u}(0)-\dot{u}(T)=0, \end{cases} \tag{1.1}$$

where A is a $(N\times N)-$symmetric matrix, $h\in L^1(0,T;\mathbb{R}^N)$, $T>0$, and $F:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ satisfies the following assumption:

(A) $F(t,x)$ is measurable in t for every $x\in\mathbb{R}^N$ and continuously differentiable in x for a.e. $t\in[0,T]$, and there exist $a\in C(\mathbb{R}^+,\mathbb{R}^+)$ and $b\in L^1([0,T],\mathbb{R}^+)$ such that

$$|F(t,x)|\le a(|x|)b(t),\quad |\nabla F(t,x)|\le a(|x|)b(t)$$

for all $x\in\mathbb{R}^N$ and a.e. $t\in[0,T]$.

Moreover, we assume that

(C1) $\dim N(A)=m\ge 1$ and A has no eigenvalue of the form $k^2\omega^2 (k\in\mathbb{N}/\{0\})$ where $\omega=2\pi/T$;

(C2) There exist linearly independent vectors $\alpha_j\in\mathbb{R}^N (1\le j\le m)$ such that $N(A)=\operatorname{span}\{\alpha_1,\cdots,\alpha_m\}$.

When $A=0$ and $h(t)\equiv 0$, it has been proved that problem (1.1) has at least one solution by the least action principle and the minimax methods (see [1,6–14,19–21,26,27]). Many solvability conditions are given, such as the coercive condition (see [1]); the periodicity

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condition (see [19]); the convexity condition (see [6]); the subadditive condition (see [11]). Recently, by using the variational methods, lots of people have also concerned with the existence of periodic solutions for p-Laplacian systems (see [16,22,24,25]), $p(t)$-Laplacian systems (see [18] and [23]), and discrete p-Laplacian systems (see [3] and [5]).

For the case that $A \neq 0$ and $h(t) \neq 0$, Mawhin and Willem [7] obtained that system (1.1) has at least one solution by using the saddle point theorem under the following bounded condition: there exists $g \in L^1(0,T;\mathbb{R}^+)$ such that

$$|F(t,u)| \leq g(t), |\nabla F(t,u)| \leq g(t), \forall\, u \in \mathbb{R}^N, \text{a.e. } t \in [0,T]. \tag{1.2}$$

They obtained the following result:

Theorem 1.1. [7, Theorem 4.9] *Suppose that F satisfies (C1), (C2) with* $\int_0^T (h(t),\alpha_j)dt = 0 (m \geq 1)$*, (1.2) and*

(F1) *there exist* $T_j > 0$ *such that* $F(t,u+T_j\alpha_j) = F(t,u) (1 \leq j \leq m)$, $\forall\, u \in \mathbb{R}^N$, *a.e.* $t \in [0,T]$.

Then system (1.1) has at least one solution.

In 2006, Feng and Han generalized Mawhin and Willem's result and they obtained the following results:

Theorem 1.2. [2, Theorem 2.1] *Suppose that F satisfies (C1), (C2) with* $\int_0^T (h(t),\alpha_j)dt = 0 (m \geq 1)$*, (F1) and the following conditions: there exist* $a,b \in L^1(0,T;\mathbb{R}^+), 0 \leq \alpha < 1$ *such that*

$$|\nabla F(t,u)| \leq a(t)|u|^\alpha + b(t). \tag{1.3}$$

Then system (1.1) has at least one solution.

Theorem 1.3. [2, Theorem 2.2] *Suppose that F satisfies (C1), (C2) with* $\int_0^T (h(t),\alpha_j)dt = 0 (m \geq 1)$*, (1.3) and*

$$\|u\|^{-2\alpha} \int_0^T F(t,u)dt \to +\infty, \text{ as } \|u\| \to \infty, u \in H^0$$

or

$$\|u\|^{-2\alpha} \int_0^T F(t,u)dt \to -\infty, \text{ as } \|u\| \to \infty, u \in H^0,$$

where H^0 *is defined before (2.2). Then system (1.1) has at least one solution.*

Condition (1.3) is usually called sublinear growth condition. Such condition has been used extensively (see [4, 11–15, 17, 26, 27]). In 2010, Wang and Zhang [17] generalized the condition (1.3). They assumed that

(f1) There exists constants $C_0 > 0$, $K_1 > 0$, $K_2 > 0$, $\alpha \in [0,1)$, $a \in L^1(0,T;\mathbb{R}^+)$ and $b \in L^1(0,T;\mathbb{R}^+)$ and a nonnegative function $w \in C([0,+\infty),[0,+\infty))$ with the properties:
  (i) $w(s) \leq w(t), \forall\, s \leq t, s,t \in [0,+\infty)$,
  (ii) $w(s+t) \leq C_0(w(s)+w(t)), \forall\, s,t \in [0,+\infty)$,
  (iii) $0 \leq w(t) \leq K_1 t^\alpha + K_2, \forall\, t \in [0,+\infty)$,
  (iv) $w(t) \to +\infty$, as $t \to +\infty$,
  such that
$$|\nabla F(t,x)| \leq a(t)w(|x|) + b(t)$$
  for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

If we let $w(t) = t^{\alpha}$, it is easy to see that (f1) generalizes (1.3). Wang and Zhang considered the special case $A = 0, h(t) \equiv 0$ for (1.1). By using the least action principle and saddle point theorem, they obtained system (1.1) with $A = 0$ and $h(t) \equiv 0$ has at least one solution. In our paper, similarly, we will use the condition (f1) to replace (1.3) and by using saddle point theorem, we will generalize Theorem 1.3. Our main results are the following theorems.

Theorem 1.4. *Suppose that F satisfies (C1), (C2) and (f1). Assume that one of three following conditions holds:*

(i)

$$\limsup_{\substack{|u|\to\infty\\ u\in N(A)}} \frac{|u|}{w^2(|u|)} < +\infty, \ \lim_{\substack{|u|\to\infty,\\ u\in N(A)}} \frac{1}{w^2(|u|)} \int_0^T F(t,u)dt = -\infty, \tag{1.4}$$

where $|\cdot|$ *is the standard norm defined in* $\mathbb{R}^N$;

(ii)

$$\limsup_{\substack{|u|\to\infty\\ u\in N(A)}} \frac{w^2(|u|)}{|u|} < +\infty, \ \lim_{\substack{|u|\to\infty,\\ u\in N(A)}} \frac{1}{|u|} \int_0^T F(t,u)dt = -\infty, \tag{1.5}$$

or furthermore,

$$\limsup_{\substack{|u|\to\infty\\ u\in N(A)}} \frac{w^2(|u|)}{|u|} = 0, \ \lim_{\substack{|u|\to\infty\\ u\in N(A)}} \frac{1}{|u|} \int_0^T F(t,u)dt < -\int_0^T |h(t)|dt; \tag{1.6}$$

(iii)

$$\int_0^T (h(t),\alpha_j) = 0, (1 \le j \le m), \lim_{|u|\to\infty u\in N(A)} \frac{1}{w^2(|u|)} \int_0^T F(t,u)dt = -\infty. \tag{1.7}$$

Then system (1.1) has at least one solution.

Theorem 1.5. *Suppose that F satisfies (C1), (C2) and (f1). Assume that one of three following conditions holds:*

(i)

$$\limsup_{\substack{|u|\to\infty\\ u\in N(A)}} \frac{|u|}{w^2(|u|)} < +\infty, \ \lim_{\substack{|u|\to\infty,\\ u\in N(A)}} \frac{1}{w^2(|u|)} \int_0^T F(t,u)dt = +\infty; \tag{1.8}$$

(ii)

$$\limsup_{\substack{|u|\to\infty,\\ u\in N(A)}} \frac{w^2(|u|)}{|u|} < +\infty, \ \lim_{\substack{|u|\to\infty\\ u\in N(A)}} \frac{1}{|u|} \int_0^T F(t,u)dt = +\infty, \tag{1.9}$$

or furthermore,

$$\limsup_{\substack{|u|\to\infty\\ u\in N(A)}} \frac{w^2(|u|)}{|u|} = 0, \ \lim_{\substack{|u|\to\infty\\ u\in N(A)}} \frac{1}{|u|} \int_0^T F(t,u)dt > \int_0^T |h(t)|dt. \tag{1.10}$$

(iii)

$$\int_0^T (h(t),\alpha_j) = 0, (1 \le j \le m), \lim_{|u|\to\infty, u\in N(A)} \frac{1}{w^2(|u|)} \int_0^T F(t,u)dt = +\infty. \tag{1.11}$$

Then system (1.1) has at least one solution.

Remark 1.1. Theorem 1.1 and Theorem 1.2 generalize Theorem 1.3 from two aspects. First, obviously, (f1) generalizes (1.3). Second, we consider the case that $\int_0^T (h(t),\alpha_j)dt = 0$ $(m \geq 1)$ in (C2) in Theorem 1.3 is deleted.

2. Preliminaries

Let

$$H_T^1 = \{u : \mathbb{R} \to \mathbb{R}^N | u \text{ is absolutely continuous}, u(t) = u(t+T) \text{ and } \dot{u} \in L^2(0,T;\mathbb{R}^N)\}.$$

Then H_T^1 is a Hilbert space with the inner product and the norm defined by

$$\langle u,v\rangle = \left[\int_0^T (u(t),v(t))dt + \int_0^T (\dot{u}(t),\dot{v}(t))dt\right]^{1/2}$$

and

$$\|u\| = \left[\int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt\right]^{1/2}$$

for each $u,v \in H_T^1$. Let

$$\bar{u} = \frac{1}{T}\int_0^T u(t)dt, \text{and } \tilde{u}(t) = u(t) - \bar{u}.$$

Then one has

$$\|\tilde{u}\|_\infty^2 \leq \frac{T}{12}\int_0^T |\dot{u}(t)|^2 dt, \text{(Sobolev's inequality)}$$

$$\|\tilde{u}\|_{L^2}^2 \leq \frac{T^2}{4\pi^2}\int_0^T |\dot{u}(t)|^2 dt. \text{ (Wirtinger's inequality)}$$

(see Proposition 1.3 in [3]) which implies that

$$\|u\|_\infty \leq C\|u\| \tag{2.1}$$

for some $C > 0$ and all $u \in H_T^1$, where $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$. It follows from assumption (A) that the functional φ on H_T^1 given by

$$\varphi(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \frac{1}{2}\int_0^T (A(t)u(t),u(t))dt + \int_0^T F(t,u(t))dt + \int_0^T (h(t),u(t))dt$$

is continuously differentiable. Moreover, one has

$$\langle\varphi'(u),v\rangle = \int_0^T [(\dot{u}(t),\dot{v}(t)) - (A(t)u(t),v(t)) + (\nabla F(t,u(t)),v(t)) + (h(t),v(t))]dt$$

for $u,v \in H_T^1$. It is well known that the solutions of system (1.1) correspond to the critical points of φ (see [3]).

Let

$$q(u) = \frac{1}{2}\int_0^T \left[|\dot{u}(t)|^2 - (A(t)u(t),u(t))\right] dt.$$

Then it is easy to see that

$$q(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_0^T ((A(t)+I)u(t),u(t))dt = \frac{1}{2}\langle(I-K)u,u)\rangle$$

where $K : H^1_T \to H^1_T$ is the self-adjoint operator defined, using Riesz representation theorem, by

$$\int_0^T ((A(t)+I)u(t), v(t))dt = \langle (Ku, v)\rangle, \forall\, u, v \in H^1_T.$$

The compact imbedding of H^1_T into $C(0,T;\mathbb{R}^N)$ implies that K is compact. By classical spectral theory, we can decompose H^1_T into the orthogonal sum of invariant subspaces for $I-K$

$$H^1_T = H^- \oplus H^0 \oplus H^+,$$

where $H^0 = \operatorname{Ker}(I-K)$ and H^- and H^+ are such that, for some $\delta > 0$,

$$q(u) \le -\frac{\delta}{2}\|u\|^2 \text{if } u \in H^-, \tag{2.2}$$

$$q(u) \ge \frac{\delta}{2}\|u\|^2 \text{if } u \in H^+. \tag{2.3}$$

Moreover, by (C1), it is known that $H^0 = \operatorname{Ker}(I-K) = N(A)$ (see [3]).

We will use the following lemma to obtain the critical points of φ.

Lemma 2.1. [10, Theorem 4.6] *Let $X = X_1 \oplus X_2$, where X is a real Banach space and $X_1 \neq \{0\}$ and is finite dimensional. Suppose $I \in C^1(X,\mathbb{R})$, satisfies (PS), and*

- (I1) *there is a constant α and a bounded neighborhood D of 0 in X_1 such that $I|_{\partial D} \le \alpha$ and*
- (I2) *there is a constant $\beta > \alpha$ such that $I|_{X_2} \ge \beta$.*

Then I possesses a critical value $c \ge \beta$. Moreover, c can be characterized as

$$c = \inf_{h\in\Gamma} \max_{u\in\bar{D}} I(h(u)),$$

where

$$\Gamma = \{h \in C(\bar{D}, X) | h = id \text{ on } \partial D\}.$$

3. Proofs of theorems

For convenience, we will denote various positive constants as $C_i, i = 1,2,\cdots$, or $D_i, i = 1,2,\cdots$, or $E_i, i = 1,2,\cdots$, or $G_i, i = 1,2,\cdots$.

Lemma 3.1. *Assume that (f1) holds. Then for any (PS) sequence $\{u_n\} \subset H^1_T$ of the functional φ, there exists $C_1 > 0$ such that*

$$\|u_n^+\|^2 \le C_1 w^2(|u_n^0|) + C_1,$$
$$\|u_n^-\|^2 \le C_1 w^2(|u_n^0|) + C_1.$$

Proof. Assume that $\{u_n\}$ is a (PS) sequence in H^1_T. Then there exists a constant $C_2 > 0$ such that

$$|\varphi(u_n)| \le C_2, |\varphi'(u_n)| \le C_2, \forall\, n \in \mathbb{N}.$$

It follows from (f1), (2.1) and Young's inequality that

$$
\begin{aligned}
&\int_0^T (\nabla F(t,u_n(t)),u_n^+(t))dt\\
&\le \int_0^T |\nabla F(t,u_n(t)|\ |u_n^+(t)|dt\\
&\le \int_0^T (a(t)w(|u_n(t)|)+b(t))\ |u_n^+(t)|dt\\
&= \int_0^T (a(t)w(|u_n^+(t)+u_n^-(t)+u_n^0|)+b(t))\ |u_n^+(t)|dt\\
&\le \int_0^T a(t)C_0(C_0+1)(w(|u_n^+(t)|)+w(|u_n^-(t)|)+w(|u_n^0|))\ |u_n^+(t)|dt+\int_0^T b(t)|u_n^+(t)|dt\\
&\le w(\|u_n^+\|_\infty)\|u_n^+\|_\infty\int_0^T a(t)C_0(C_0+1)dt+w(\|u_n^-\|_\infty)\|u_n^+\|_\infty\int_0^T a(t)C_0(C_0+1)dt\\
&\quad +w(|u_n^0|)\|u_n^+\|_\infty\int_0^T C_0(C_0+1)a(t)dt+\|u_n^+\|_\infty\int_0^T b(t)dt\\
&\le (K_1\|u_n^+\|_\infty^\alpha+K_2)\|u_n^+\|_\infty\int_0^T a(t)C_0(C_0+1)dt\\
&\quad +(K_1\|u_n^-\|_\infty^\alpha+K_2)\|u_n^+\|_\infty\int_0^T a(t)C_0(C_0+1)dt\\
&\quad w(|u_n^0|)\|u_n^+\|_\infty\int_0^T C_0(C_0+1)a(t)dt+\|u_n^+\|_\infty\int_0^T b(t)dt\\
&\le C_3\|u_n^+\|^{\alpha+1}+C_4\|u_n^+\|+C_5\|u_n^-\|^\alpha\|u_n^+\|+w(|u_n^0|)\|u_n^+\|CC_0(C_0+1)\int_0^T a(t)dt\\
&\le \varepsilon\|u_n^+\|^2+C_3(\varepsilon)+\varepsilon\|u_n^+\|^2+C_4(\varepsilon)+C_5(\varepsilon)\|u_n^-\|^{2\alpha}+\varepsilon\|u_n^+\|^2+C_6(\varepsilon)w^2(|u_n^0|)+\varepsilon\|u_n^+\|^2\\
&\le 4\varepsilon\|u_n^+\|^2+C_5(\varepsilon)\|u_n^-\|^{2\alpha}+C_6(\varepsilon)w^2(|u_n^0|)+C_7(\varepsilon),
\end{aligned}
$$

where $\varepsilon>0$ and $C_i(\varepsilon)>0$ $(i=1,\cdots,7)$ are constants dependent on ε. Thus, we have

$$
\begin{aligned}
C_2\|u_n^+\| &\ge \langle(\varphi'(u_n),u_n^+)\rangle\\
&= \langle(I-K)u_n,u_n^+\rangle+\int_0^T (\nabla F(t,u_n(t))+w(t),u_n^+(t))dt\\
&\ge \delta\|u_n^+\|^2-4\varepsilon\|u_n^+\|^2-C_5(\varepsilon)\|u_n^-\|^{2\alpha}-C_6(\varepsilon)w^2(|u_n^0|)-C_7(\varepsilon)-C_8\|u_k^+\|\\
&\ge (\delta-5\varepsilon)\|u_n^+\|^2-C_5(\varepsilon)\|u_n^-\|^{2\alpha}-C_6(\varepsilon)w^2(|u_n^0|)-C_9(\varepsilon).
\end{aligned}
$$

If we fix $\varepsilon<\delta/5$, then

$$
\|u_n^+\|^2\le C_{10}w^2(|u_n^0|)+C_{11}\|u_n^-\|^{2\alpha}+C_{12}. \tag{3.1}
$$

Similarly, we can get

$$
\|u_n^-\|^2\le C_{13}w^2(|u_n^0|)+C_{14}\|u_n^+\|^{2\alpha}+C_{15}. \tag{3.2}
$$

It follows from (3.1) and (3.2) that

$$
\|u_n^+\|^2\le C_{10}w^2(|u_n^0|)+C_{16}w^{2\alpha}(|u_n^0|)+C_{17}\|u_n^+\|^{2\alpha^2}+C_{18}.
$$

Since $2\alpha^2 < 2\alpha < 2$, by using Young's inequality again, we have

$$\|u_n^+\|^2 \le C_{19} w^2(|u_n^0|) + C_{20}.$$

Similarly, we can get

$$\|u_n^-\|^2 \le C_{21} w^2(|u_n^0|) + C_{22}.$$

Let $C_1 = \max\{C_{19}, C_{20}, C_{21}, C_{22}\}$. Then we complete the proof. ∎

Proof of Theorem 1.1. We will use Lemma 2.1 to prove this Theorem. First, we prove φ satisfies (PS) condition when one of case (i), case (ii) and case (iii) holds, respectively. Let $\{u_n\} \in H_T^1$ be a (PS) sequence, that is $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$. Then there exists $E_1 > 0$ such that

$$|\varphi(u_n)| \le E_1, \|\varphi'(u_n)\| \le E_1.$$

By Lemma 3.1, we know that

$$\|u_n^+\|^2 \le C_1 w^2(|u_n^0|) + C_1, \tag{3.3}$$

$$\|u_n^-\|^2 \le C_1 w^2(|u_n^0|) + C_1. \tag{3.4}$$

It follows from the above two inequalities and Young's inequality, we can obtain that

(3.5)

$$\begin{aligned}
&\left|\int_0^T F(t, u_n(t))dt - \int_0^T F(t, u_n^0)dt\right| \\
&= \left|\int_0^T\int_0^1 (\nabla F(t, u_n^0 + s(u_n^+(t) + u_n^-(t))), u_n^+(t) + u_n^-(t))dsdt\right| \\
&\le \int_0^T\int_0^1 |\nabla F(t, s(u_n^+(t) + u_n^-(t)) + u_n^0)||u_n^+(t) + u_n^-(t)|dsdt \\
&\le \int_0^T\int_0^1 [a(t)w(|su_n^+(t) + su_n^-(t) + u_n^0|) + b(t)]|u_n^+(t) + u_n^-(t)|dsdt \\
&\le \int_0^T\int_0^1 [a(t)C_0(C_0+1)(w(|su_n^+(t)|) + w(|su_n^-(t)|) + w(|u_n^0|))]\ |u_n^+(t) + u_n^-(t)|dsdt \\
&\int_0^T b(t)|u_n^+(t) + u_n^-(t)|dt \\
&\le w(\|u_n^+\|_\infty)(\|u_n^+\|_\infty + \|u_n^-\|_\infty)\int_0^T a(t)C_0(C_0+1)dt w(\|u_n^-\|_\infty)(\|u_n^+\|_\infty + \|u_n^-\|_\infty) \\
&\quad \int_0^T a(t)C_0(C_0+1)dt w(|u_n^0|)(\|u_n^+\|_\infty + \|u_n^-\|_\infty)\int_0^T a(t)C_0(C_0+1)dt \\
&\quad + (\|u_n^+\|_\infty + \|u_n^-\|_\infty)\int_0^T b(t)dt
\end{aligned}$$

$$
\begin{aligned}
&\le (K_1\|u_n^+\|_\infty^\alpha + K_2)\|u^+\|_\infty \int_0^T a(t)C_0(C_0+1)dt + (K_1\|u_n^+\|_\infty^\alpha + K_2)\|u^-\|_\infty \\
&\int_0^T a(t)C_0(C_0+1)dt(K_1\|u_n^-\|_\infty^\alpha + K_2)\|u^+\|_\infty \int_0^T a(t)C_0(C_0+1)dt \\
&+ (K_1\|u_n^-\|_\infty^\alpha + K_2)\|u^-\|_\infty \int_0^T a(t)C_0(C_0+1)dt(\|u_n^+\|_\infty + \|u_n^-\|_\infty) \\
&\left(\int_0^T b(t)dt + w(|u_n^0|)\int_0^T a(t)C_0(C_0+1)dt\right) \\
&\le D_1\|u_n^+\|^{\alpha+1} + D_2\|u_n^+\| + D_1\|u_n^-\|^{\alpha+1} + D_2\|u_n^-\| \\
&+ D_1\|u_n^+\|^{\alpha}\|u_n^-\|_\infty + D_1\|u_n^-\|^{\alpha}\|u_n^+\|D_3(\|u_n^+\| + \|u_n^-\|)w(|u_n^0|) \\
&\le D_4\|u_n^+\|^2 + D_5\|u_n^-\|^2 + D_6 w^2(|u_n^0|) + D_7 \\
&\le E_2 w^2(|u_n^0|) + E_3.
\end{aligned}
$$

Since $A(t)$ is continuous in t and T-periodic, it is easy to see that there exists $E_4 > 0$ such that

$$
\frac{1}{2}\langle (I-K)u_n^+, u_n^+\rangle = q(u_n^+) \le E_4\|u_n^+\|^2 \tag{3.6}
$$

Hence, by the above inequality, (2.1), (3.3), (3.5), (3.6) and (2.2), we have

$$
\begin{aligned}
-E_1 \le \varphi(u_n) &= \frac{1}{2}\langle (I-K)u_n^+, u_n^+\rangle + \frac{1}{2}\langle (I-K)u_n^-, u_n^-\rangle + \int_0^T F(t,u_n(t))dt - \int_0^T F(t,u_n^0)dt \\
&\int_0^T F(t,u_n^0)dt + \int_0^T (h(t), u_n^+(t) + u_n^-(t) + u_n^0)dt \\
&\le E_4\|u_n^+\|^2 + E_2 w^2(|u_n^0|) + E_3 \\
&\int_0^T F(t,u_n^0)dt + C\int_0^T |h(t)|dt\|u^+\| + C\|u^-\|\int_0^T |h(t)|dt + |u_n^0|\int_0^T |h(t)|dt \\
&\le E_5 w^2(|u_n^0|) + \int_0^T F(t,u_n^0)dt + E_6 w(|u_n^0|) + E_7 + |u_n^0|\int_0^T |h(t)|dt. \qquad (3.7)
\end{aligned}
$$

Case (i): assume that

$$
\limsup_{\substack{|u|\to\infty \\ u\in N(A)}} \frac{|u|}{w^2(|u|)} < +\infty. \tag{3.8}
$$

Note that

$$
\begin{aligned}
&E_5 w^2(|u_n^0|) + \int_0^T F(t,u_n^0)dt + E_6 w(|u_n^0|) + E_7 + |u_n^0|\int_0^T |h(t)|dt \\
&= w^2(|u_n^0|)\left(E_5 + \frac{1}{w^2(|u_n^0|)}\int_0^T F(t,u_n^0)dt + \frac{|u_n^0|\int_0^T |h(t)|dt}{w^2(|u_n^0|)}\right) + E_6 w(|u_n^0|) + E_7.
\end{aligned}
$$

It follows from (1.4) and (3.8) that $\{u_n^0\}$ is bounded. By (3.3) and (3.4), we know that $\{u_n\}$ is bounded in H_T^1. Similar to the argument to [7, Proposition 4.1], φ satisfies the (PS) condition.

Case (ii): assume that

$$\limsup_{\substack{|u|\to\infty\\ u\in N(A)}} \frac{w^2(|u|)}{|u|} < +\infty. \tag{3.9}$$

Then

$$\limsup_{\substack{|u|\to\infty\\ u\in N(A)}} \frac{w(|u|)}{|u|} = 0. \tag{3.10}$$

Note that

$$\begin{aligned} &E_5 w^2(|u_n^0|) + \int_0^T F(t,u_n^0)dt + E_6 w(|u_n^0|) + E_7 + |u_n^0| \int_0^T |h(t)|dt \\ &= |u_n^0| \left(\int_0^T |h(t)|dt + \frac{1}{|u_n^0|} \int_0^T F(t,u_n^0)dt + \frac{w^2(|u_n^0|)}{|u_n^0|} + \frac{E_6 w(|u_n^0|)}{|u_n^0|} \right) + E_7. \end{aligned}$$

It follows from (1.5) and (3.10) that $\{u_n^0\}$ is bounded. Furthermore, if

$$\limsup_{\substack{|u|\to\infty\\ u\in N(A)}} \frac{w^2(|u|)}{|u|} = 0,$$

by (1.6), we also obtain that $\{u_n^0\}$ is bounded. By (3.3) and (3.4), we know that $\{u_n\}$ is bounded in H_T^1. Similar to the argument to [7, Proposition 4.1], φ satisfies the (PS) condition.

Case (iii): if $\int_0^T (h(t),\alpha_j)dt = 0, (1 \le j \le m)$, we have

$$\begin{aligned} -E_1 \le \varphi(u_n) &= \frac{1}{2}\langle (I-K)u_n^+, u_n^+\rangle + \frac{1}{2}\langle (I-K)u_n^-, u_n^-\rangle + \int_0^T F(t,u_n(t))dt - \int_0^T F(t,u_n^0)dt \\ &\quad \int_0^T F(t,u_n^0)dt + \int_0^T (h(t), u_n^+(t) + u_n^-(t) + u_n^0)dt \\ &\le E_4 \|u_n^+\|^2 + E_2 w^2(|u_n^0|) + E_3 \\ &\quad \int_0^T F(t,u_n^0)dt + C\int_0^T |h(t)|dt \|u^+\| + C\|u^-\| \int_0^T |h(t)|dt \\ &\le E_5 w^2(|u_n^0|) + \int_0^T F(t,u_n^0)dt + E_6 w(|u_n^0|) + E_7 \\ &\le w^2(|u_n^0|) \left(E_5 + \frac{1}{w^2(|u_n^0|)} \int_0^T F(t,u_n^0)dt \right) + E_6 w(|u_n^0|) + E_7 \end{aligned}$$

It follows from (1.7) that $\{u_n^0\}$ is bounded. By (3.3) and (3.4), we know that $\{u_n\}$ is bounded in H_T^1. Similar to the argument to [7, Proposition 4.1], φ satisfies the (PS) condition.

Next, we verify φ satisfies (i) in Lemma 2.1. Decompose $H_T^1 = (H^- \oplus H^0) \oplus H^+$. Let $X_1 = (H^- \oplus H^0), X_2 = H^+$. We know that $\dim(H^- \oplus H^0) < +\infty$. For $\forall\, u \in X_1 = H^- \oplus H^0$,

$u=u^0+u^-$, it follows from (f1), (2.1) and Young's inequality that

$$\begin{aligned}
&\left|\int_0^T F(t,u(t))dt-\int_0^T F(t,u^0)dt\right|\\
&=\left|\int_0^T\int_0^1(\nabla F(t,u^0+su^-(t)),u^-(t))dsdt\right|\\
&\le\int_0^T\int_0^1|\nabla F(t,u^0+su^-(t)||u^-(t)|dsdt\\
&\le\int_0^T\int_0^1(a(t)w(|u^0+su^-(t)|)+b(t))|u^-(t)|dsdt\\
&\le w(|u^0|)\|u^-\|_\infty\int_0^T a(t)C_0dt\\
&\quad+w(\|u^-\|_\infty)\|u^-\|_\infty\int_0^T a(t)C_0dt+\|u^-\|_\infty\int_0^T b(t)dt\\
&\le w(|u^0|)\|u^-\|_\infty\int_0^T a(t)C_0dt\\
&\quad+(K_1\|u^-\|_\infty^\alpha+K_2)\|u^-\|_\infty\int_0^T a(t)C_0dt+\|u^-\|_\infty\int_0^T b(t)dt\\
&\le E_8w(|u^0|)\|u^-\|+E_9\|u^-\|^{\alpha+1}+E_{10}\|u^-\|\\
&\le E_{11}w^2(|u^0|)+\varepsilon\|u^-\|^2+E_9\|u^-\|^{\alpha+1}+E_{10}\|u^-\|. \qquad (3.11)
\end{aligned}$$

Case (i): if $\limsup_{|u|\to\infty,u\in N(A)}|u|/(w^2(|u|))<+\infty$, then

$$\begin{aligned}
\varphi(u)&=\frac{1}{2}\langle(I-K)u^-,u^-\rangle+\int_0^T F(t,u(t))dt-\int_0^T F(t,u^0)dt+\int_0^T F(t,u^0)dt+\int_0^T(h(t),u(t))\\
&\le-\frac{\delta}{2}\|u^-\|^2+E_{11}w^2(|u^0|)+\varepsilon\|u^-\|^2+E_9\|u^-\|^{\alpha+1}+E_{10}\|u^-\|\\
&\quad\int_0^T F(t,u^0)dt+C\|u^-\|\int_0^T|h(t)|dt+|u^0|\int_0^T|h(t)|dt\\
&\le\left(-\frac{\delta}{2}+\varepsilon\right)\|u^-\|^2+E_9\|u^-\|^{\alpha+1}+E_{12}\|u^-\|\\
&\quad w^2(|u^0|)\left(E_{11}+\frac{1}{w^2(|u^0|)}\int_0^T F(t,u^0)dt+\frac{|u^0|\int_0^T|h(t)|dt}{w^2(|u^0|)}\right)
\end{aligned}$$

Choosing $\varepsilon<\delta/2$, by (1.4) and $\alpha<1$, we have

$$\varphi(u)\to-\infty,\text{as}\|u\|\to\infty,\ u\in X_1.$$

Case (ii): if $\limsup_{\substack{|u|\to\infty\\ u\in N(A)}}w^2(|u|)/(|u|)<+\infty$, then

$$\begin{aligned}
\varphi(u)&\le-\frac{\delta}{2}\|u^-\|^2+E_{11}w^2(|u^0|)+\varepsilon\|u^-\|^2+E_9\|u^-\|^{\alpha+1}+E_{10}\|u^-\|\\
&\quad\int_0^T F(t,u^0)dt+C\|u^-\|\int_0^T|h(t)|dt+|u^0|\int_0^T|h(t)|dt\\
&\le\left(-\frac{\delta}{2}+\varepsilon\right)\|u^-\|^2+E_9\|u^-\|^{\alpha+1}+E_{12}\|u^-\|
\end{aligned}$$

$$|u^0|\left(\int_0^T |h(t)|dt+\frac{1}{|u^0|}\int_0^T F(t,u^0)dt+\frac{E_{11}w^2(|u^0|)}{|u^0|}\right)$$

Choosing $\varepsilon<\delta/2$, by (1.5) and (3.10), we have

$$\varphi(u)\to-\infty,\text{as}\|u\|\to\infty.$$

Furthermore, if

$$\limsup_{\substack{|u|\to\infty\\ u\in N(A)}}\frac{w^2(|u|)}{|u|}=0,$$

then by (1.6), we also obtain that

$$\varphi(u)\to-\infty,\text{as}\|u\|\to\infty.$$

Case (iii): if $\int_0^T(h(t),\alpha_j)dt=0,(1\le j\le m)$, then

$$\begin{aligned}\varphi(u)&=\frac{1}{2}\langle(I-K)u^-,u^-\rangle+\int_0^T F(t,u(t))dt-\int_0^T F(t,u^0)dt+\int_0^T F(t,u^0)dt+\int_0^T(h(t),u(t))\\&\le-\frac{\delta}{2}\|u^-\|^2+E_{11}w^2(|u^0|)+\varepsilon\|u^-\|^2+E_9\|u^-\|^{\alpha+1}+E_{10}\|u^-\|\\&\quad\int_0^T F(t,u^0)dt+C\|u^-\|\int_0^T|h(t)|dt\\&\le\left(-\frac{\delta}{2}+\varepsilon\right)\|u^-\|^2+E_9\|u^-\|^{\alpha+1}+E_{12}\|u^-\|\\&\quad w^2(|u^0|)\left(E_{11}+\frac{1}{w^2(|u^0|)}\int_0^T F(t,u^0)dt\right)\end{aligned}$$

Choosing $\varepsilon<\delta/2$, by (1.7), we have

$$\varphi(u)\to-\infty,\ \text{as}\|u\|\to\infty,\ u\in X_1.$$

Finally, we verify that φ satisfies (ii) in Lemma 2.1. In fact, for $\forall\ u\in X_2=H^+$, $u=u^+$, by (f1) and (2.1), we have

$$\begin{aligned}&\int_0^T F(t,u(t))-\int_0^T F(t,0)dt\\&=\int_0^T\int_0^1(\nabla F(t,su(t)),u(t))dsdt\\&\le\int_0^T\int_0^1|\nabla F(t,su(t)||u(t)|dsdt\\&\le\int_0^T(a(t)w(\|u\|_\infty)+b(t))|u(t)|dsdt\\&\le w(\|u\|_\infty)\|u\|_\infty\int_0^T a(t)dt+\|u\|_\infty\int_0^T b(t)dt\\&\le(K_1\|u\|_\infty^\alpha+K_2)\|u\|_\infty\int_0^T a(t)dt+\|u\|_\infty\int_0^T b(t)dt\\&\le E_{13}\|u\|^{\alpha+1}+E_{14}\|u\|.\end{aligned}\tag{3.12}$$

Hence, for $\forall\ u \in X_2 = H^+$, we have

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\langle (I-K)u,u\rangle + \int_0^T F(t,u(t))dt - \int_0^T F(t,0)dt + \int_0^T F(t,0)dt + \int_0^T (h(t),u(t))dt \\ &\geq \frac{\delta}{2}\|u\|^2 - E_{13}\|u\|^{\alpha+1} - E_{14}\|u\| - C\|u\|\int_0^T |h(t)|dt + \int_0^T F(t,0)dt\end{aligned}$$

it is easy to see φ is bounded from below in X_2. Hence there exists $R > 0$ and $\alpha < \beta$ such that

$$\varphi(u) \leq \alpha, u \in \partial B_R \cap E_1 = \partial D.$$

Thus by Lemma 2.1, we know that φ has at least one critical point. We complete the proof. ∎

Proof of Theorem 1.2. First, we prove φ satisfies (PS) condition when one of case (i), case (ii) and case (iii) holds, respectively. Let $\{u_n\} \subset H_T^1$ be a (PS) sequence, that is $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$. Then there exists $G_1 > 0$ such that

$$|\varphi(u_n)| \leq G_1.$$

Since $A(t)$ is continuous in t and T-periodic, it is easy to see that there exists $G_2 > 0$ such that

$$\frac{1}{2}\langle (I-K)u_n^-, u_n^-\rangle = q(u_n^-) \geq -G_2\|u_n^-\|^2. \tag{3.13}$$

Hence, by the above inequality, (2.1), (2.3), (3.4), (3.5), (3.13) and (2.2), we have

$$\begin{aligned}-G_1 \geq \varphi(u_n) &= \frac{1}{2}\langle (I-K)u_n^+, u_n^+\rangle + \frac{1}{2}\langle (I-K)u_n^-, u_n^-\rangle + \int_0^T F(t,u_n(t))dt - \int_0^T F(t,u_n^0)dt \\ &\quad \int_0^T F(t,u_n^0)dt + \int_0^T (h(t), u_n^+(t) + u_n^-(t) + u_n^0)dt \\ &\geq \frac{\delta}{2}\|u_n^+\|^2 - G_2\|u_n^-\|^2 - E_2 w^2(|u_n^0|) - E_3 \\ &\quad \int_0^T F(t,u_n^0)dt - C\int_0^T |h(t)|dt\|u^+\| - C\|u^-\|\int_0^T |h(t)|dt - |u_n^0|\int_0^T |h(t)|dt \\ &\geq -G_3 w^2(|u_n^0|) + \int_0^T F(t,u_n^0)dt - G_4 w(|u_n^0|) - G_5 - |u_n^0|\int_0^T |h(t)|dt. \qquad (3.14)\end{aligned}$$

Case (i): assume that

$$\limsup_{|u|\to\infty, u \in N(A)} \frac{|u|}{w^2(|u|)} < +\infty. \tag{3.15}$$

Note that

$$\begin{aligned}&-G_3 w^2(|u_n^0|) + \int_0^T F(t,u_n^0)dt - G_4 w(|u_n^0|) - G_5 - |u_n^0|\int_0^T |h(t)|dt \\ &= w^2(|u_n^0|)\left(-G_3 + \frac{1}{w^2(|u_n^0|)}\int_0^T F(t,u_n^0)dt - \frac{|u_n^0|\int_0^T |h(t)|dt}{w^2(|u_n^0|)}\right) - G_4 w(|u_n^0|) - G_5\end{aligned}$$

It follows from (1.8) and (3.15) that $\{u_n^0\}$ is bounded. By (3.3) and (3.4), we know that $\{u_n\}$ is bounded in H_T^1. Similar to the argument to [7, Proposition 4.1], φ satisfies the (PS) condition. For case (ii) and case (iii), combining case (i) with the argument of Theorem 1.1, it is easy to see that φ also satisfies (PS) condition.

Next, we verify φ satisfies (i) and (ii) in Lemma 2.1. we will let $E_1 = H^-, E_2 = H^0 \oplus H^+$, which is different from the decomposition of Theorem 1.1. Obviously, $\dim E_1 < +\infty$. Similar to (3.12), we can obtain that for $u \in H^-$,

$$\int_0^T F(t,u(t)) - \int_0^T F(t,0)dt \leq G_6\|u\|^{\alpha+1} + G_7\|u\|. \tag{3.16}$$

Hence, for $\forall\, u \in H^-$, we have

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\langle (I-K)u,u\rangle + \int_0^T F(t,u(t))dt - \int_0^T F(t,0)dt + \int_0^T F(t,0)dt + \int_0^T (h(t),u(t))dt \\ &\leq -\frac{\delta}{2}\|u\|^2 + G_6\|u\|^{\alpha+1} + G_7\|u\| + C\|u\|\int_0^T |h(t)|dt + \int_0^T F(t,0)dt\end{aligned}$$

Since $\alpha < 1$, we have

$$\varphi(u) \to -\infty, \text{ as} \|u\| \to \infty,\ u \in X_1.$$

Similar to (3.11), we can obtain that for $u \in H^0 \oplus H^+, u = u^0 + u^+$,

(3.17)
$$\left|\int_0^T F(t,u(t))dt - \int_0^T F(t,u^0)dt\right| \leq G_8 w^2(|u^0|) + \varepsilon\|u^+\|^2 + G_9\|u^+\|^{\alpha+1} + G_{10}\|u^+\|.$$

Case (i): if $\limsup_{|u|\to\infty, u\in N(A)} |u|/(w^2(|u|)) < +\infty$, then by (2.1), (2.3) and (3.17), for $\forall\, u \in H^0 \oplus H^+$, $u = u^0 + u^+$,

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\langle (I-K)u^+,u^+\rangle + \int_0^T F(t,u(t))dt - \int_0^T F(t,u^0)dt + \int_0^T F(t,u^0)dt + \int_0^T (h(t),u(t)) \\ &\geq \frac{\delta}{2}\|u^+\|^2 - G_8 w^2(|u^0|) - \varepsilon\|u^+\|^2 - G_9\|u^+\|^{\alpha+1} - G_{10}\|u^+\| \\ &\quad \int_0^T F(t,u^0)dt - C\|u^+\|\int_0^T |h(t)|dt - |u^0|\int_0^T |h(t)|dt \\ &\geq \left(\frac{\delta}{2} - \varepsilon\right)\|u^+\|^2 - G_9\|u^+\|^{\alpha+1} + G_{11}\|u^+\| \\ &\quad w^2(|u^0|)\left(-G_8 + \frac{1}{w^2(|u^0|)}\int_0^T F(t,u^0)dt - \frac{|u^0|\int_0^T |h(t)|dt}{w^2(|u^0|)}\right)\end{aligned}$$

Choosing $\varepsilon < \delta/2$, by (1.8), we have

$$\varphi(u) \to +\infty, \text{ as} \|u\| \to \infty,\ u \in X_2.$$

For case (ii) and case (iii), combining case (i) with the argument of Theorem 1.1, it is easy to see that

$$\varphi(u) \to +\infty, \text{ as} \|u\| \to \infty,\ u \in X_2.$$

Thus we complete the proof. ∎

4. Examples

In this section, we give some examples to verify our theorems. At first, let $T = 1$ and

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\dim N(A) = 2$ and $N(A) = \operatorname{span}\{\alpha_1, \alpha_2\}$, where $\alpha_1 = (0,1,0)$ and $\alpha_2 = (0,0,1)$. So (C1) and (C2) hold.

Example 4.1. (i) Let

$$F(t,x) = (0.4T - t)\,|x|^{7/4},\ \forall\, x \in \mathbb{R}^N,\ t \in [0,T].$$

Then

$$|\nabla F(t,x)| = \frac{7}{4}\,|0.4T - t|\,|x|^{3/4}.$$

Let $w(|x|) = |x|^{3/4}$. Then it is clear that (f1) holds. Moreover,

$$\limsup_{\substack{|u|\to\infty \\ u\in N(A)}} \frac{|u|}{w^2(|u|)} = \limsup_{\substack{|u|\to\infty \\ u\in N(A)}} \frac{|u|}{|u|^{3/2}} = 0 < +\infty,$$

$$\lim_{\substack{|u|\to\infty, \\ u\in N(A)}} \frac{1}{w^2(|u|)} \int_0^T F(t,u)dt = \lim_{\substack{|u|\to\infty, \\ u\in N(A)}} \frac{-0.1T^2|u|^{7/4}}{|u|^{3/2}} = -\infty.$$

Hence, (1.4) holds and then by Theorem 1.1, system (1.1) has at least one solution.

(ii) Let

$$F(t,x) = (0.5T - t)\,|x|^{5/4} - \frac{l(t)|x|^{5/2}}{1 + |x|^2},\quad \forall\, x \in \mathbb{R}^N,\ t \in [0,T],$$

where $l \in C([0,T];\mathbb{R}^+)$ with $\int_0^T l(t)dt > \int_0^T |h(t)|dt$. Then there exists $C > 0$ such that

$$\begin{aligned} |\nabla F(t,x)| &\le \frac{5}{4}\,|0.5T - t|\,|x|^{1/4} + \frac{l(t)(\frac{9}{2}|x|^{7/2} + \frac{5}{2}|x|^{3/2})}{1 + 2|x|^2 + |x|^4} \\ &\le \frac{5}{4}\,|0.5T - t|\,|x|^{1/4} + Cl(t) \end{aligned}$$

Let $w(|x|) = |x|^{1/4}$. Then it is clear that (f1) holds. Moreover,

$$\limsup_{\substack{|u|\to\infty \\ u\in N(A)}} \frac{w^2(|u|)}{|u|} = \limsup_{\substack{|u|\to\infty \\ u\in N(A)}} \frac{|u|^{1/2}}{|u|} = 0 < +\infty,$$

$$\lim_{\substack{|u|\to\infty, \\ u\in N(A)}} \frac{1}{|u|^{1/2}} \int_0^T F(t,u)dt = \lim_{\substack{|u|\to\infty, \\ u\in N(A)}} \frac{-|u|^{5/2}\int_0^T l(t)dt}{|u|^{1/2} + |u|^{5/2}} = -\int_0^T l(t)dt < -\int_0^T |h(t)|dt.$$

Hence, (1.6) holds and then by Theorem 1.1, system (1.1) has at least one solution.

(iii) Let

$$F(t,x) = (0.4T - t)\ln^{3/2}(1 + |x|^2) + d(t)\ln(1 + |x|^2),\ \forall\, x \in \mathbb{R}^N,\ t \in [0,T], \tag{4.1}$$

and h satisfy $\int_0^T h(t)dt = 0$, where $d \in C([0,T];\mathbb{R}^+)$. Then $\int_0^T (h(t),\alpha_j)dt = 0, j = 1,2$ and

$$|\nabla F(t,x)| \leq \frac{3}{2}\,|0.4T - t|\ln^{1/2}(1+|x|^2) + d(t).$$

Let $w(|x|) = \ln^{1/2}(1+|x|^2)$. Similar to the argument in [18], we know that (f1) holds. Moreover,

$$\lim_{\substack{|u|\to\infty,\\ u\in N(A)}} \frac{1}{w^2(|u|)}\int_0^T F(t,u)dt = \lim_{\substack{|u|\to\infty,\\ u\in N(A)}} -0.1T^2\ln^{1/2}(1+|u|^2) + d(t) = -\infty.$$

Hence, (1.7) holds and then by Theorem 1.1, system (1.1) has at least one solution. Moreover, note that $H^0 = N(A)$ (see [3]) and for any $\alpha \in (0,1)$,

$$\lim_{\substack{|u|\to\infty,\\ u\in H^0}} \frac{1}{|u|^{2\alpha}}\int_0^T F(t,u)dt = 0.$$

So (4.1) does not satisfy Theorem 1.3.

Example 4.2. (i) Let

$$F(t,x) = (0.6T - t)\,|x|^{7/4},\ \forall\, x \in \mathbb{R}^N,\ t \in [0,T].$$

Then

$$|\nabla F(t,x)| = \frac{7}{4}\,|0.6T - t|\,|x|^{3/4}.$$

Let $w(|x|) = |x|^{3/4}$. Then it is clear that (f1) holds. Moreover,

$$\limsup_{\substack{|u|\to\infty\\ u\in N(A)}} \frac{|u|}{w^2(|u|)} = \limsup_{\substack{|u|\to\infty\\ u\in N(A)}} \frac{|u|}{|u|^{3/2}} = 0 < +\infty,$$

$$\lim_{\substack{|u|\to\infty,\\ u\in N(A)}} \frac{1}{w^2(|u|)}\int_0^T F(t,u)dt = \lim_{\substack{|u|\to\infty,\\ u\in N(A)}} \frac{0.1T^2|u|^{7/4}}{|u|^{3/2}} = +\infty.$$

Hence, (1.8) holds and then by Theorem 1.2, system (1.1) has at least one solution.

(ii) Let

$$F(t,x) = (0.5T - t)\,|x|^{5/4} + \frac{l(t)|x|^{5/2}}{1+|x|^2},\ \ \forall\, x \in \mathbb{R}^N,\ t \in [0,T],$$

where $l \in C([0,T];\mathbb{R}^+)$ and $\int_0^T l(t)dt > \int_0^T |h(t)|dt$. Then there exists $C > 0$ such that

$$\begin{aligned}|\nabla F(t,x)| &\leq \frac{5}{4}\,|0.5T - t|\,|x|^{1/4} + \frac{l(t)(\frac{9}{2}|x|^{7/2} + \frac{5}{2}|x|^{3/2})}{1+2|x|^2+|x|^4}\\ &\leq \frac{5}{4}\,|0.5T - t|\,|x|^{1/4} + Cl(t).\end{aligned}$$

Let $w(|x|) = |x|^{1/4}$. Then it is clear that (f1) holds. Moreover,

$$\limsup_{\substack{|u|\to\infty\\ u\in N(A)}} \frac{w^2(|u|)}{|u|} = \limsup_{\substack{|u|\to\infty\\ u\in N(A)}} \frac{|u|^{1/2}}{|u|} = 0 < +\infty,$$

$$\lim_{\substack{|u|\to\infty,\\ u\in N(A)}} \frac{1}{|u|^{1/2}}\int_0^T F(t,u)dt = \lim_{\substack{|u|\to\infty,\\ u\in N(A)}} \frac{|u|^{5/2}\int_0^T l(t)dt}{|u|^{1/2}+|u|^{5/2}} = \int_0^T l(t)dt > \int_0^T |h(t)|dt.$$

Hence, (1.10) holds and then by Theorem 1.2, system (1.1) has at least one solution.

(iii) Let

$$F(t,x) = (0.6T - t)\ln^{3/2}(1+|x|^2) + d(t)\ln(1+|x|^2), \quad \forall\, x \in \mathbb{R}^N,\ t \in [0,T], \tag{4.2}$$

and h satisfy $\int_0^T h(t)dt = 0$, where $d \in C([0,T];\mathbb{R}^+)$. Then $\int_0^T (h(t), \alpha_j)dt = 0, j = 1,2$ and

$$|\nabla F(t,x)| \le \frac{3}{2}|0.6T - t|\ln^{1/2}(1+|x|^2) + d(t).$$

Let $w(|x|) = \ln^{1/2}(1+|x|^2)$. Similar to the argument in [17], we know that (f1) holds. Moreover,

$$\lim_{\substack{|u|\to\infty,\\ u\in N(A)}} \frac{1}{w^2(|u|)}\int_0^T F(t,u)dt = \lim_{\substack{|u|\to\infty,\\ u\in N(A)}} 0.1T^2\ln^{1/2}(1+|u|^2) + d(t) = +\infty.$$

Hence, (1.11) holds and then by Theorem 1.2, system (1.1) has at least one solution. Moreover, similar to Example 4.1 (iii), (4.2) does not satisfy Theorem 1.3.

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