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Oscillatory Behavior of Solutions of Certain Third-Order Mixed Neutral Functional Differential Equations

[1]ZHENLAI HAN, [2]TONGXING LI, [3]CHENGHUI ZHANG AND [4]SHURONG SUN

[1,2,4]School of Science, University of Jinan, Jinan, Shandong 250022, P. R. China
[1,2,3]School of Control Science and Engineering, Shandong University,
Jinan, Shandong 250061, P. R. China
[4]Department of Mathematics and Statistics, Missouri University of Science and Technology
Rolla, Missouri 65409-0020, U.S.A.
[1]hanzhenlai@163.com, [2]litongx2007@163.com, [3]zchui@sdu.edu.cn, [4]sshrong@163.com

Abstract. The present work is concerned with the oscillation and asymptotic properties of the third-order mixed neutral differential equation

$$\left(a(t)\left(x(t)+p_1(t)x(t-\tau_1)+p_2(t)x(t+\tau_2)\right)''\right)'+q_1(t)x(t-\tau_3)+q_2(t)x(t+\tau_4)=0, \quad t \geq t_0.$$

We establish two theorems which guarantee that every solution x of the above equation oscillates or $\lim_{t\to\infty} x(t)=0$. These results complement some known results obtained in the literature. Some examples are considered to illustrate the main results.

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1. Introduction

In recent years, there has been increasing interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions of the third-order differential equations, we refer the reader to the papers [1, 2, 5, 8, 15, 18–20] and [3, 4, 6, 7, 9, 10, 12–14, 16, 17, 21]. To the best of our knowledge, it seems to have few results for the oscillation of third-order mixed neutral differential equations, see the articles [7, 9, 10, 21]. We note that neutral differential equations have various applications in problems dealing with vibrating masses attached to an elastic bar and in some variational problems. For further applications and questions concerning the existence and uniqueness of solutions of neutral differential equations, see Hale [11].

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This paper is concerned with the following third-order mixed neutral functional differential equation

$$(1.1)\qquad \begin{aligned} &\left(a(t)\left(x(t)+p_1(t)x(t-\tau_1)+p_2(t)x(t+\tau_2)\right)''\right)' \\ &+q_1(t)x(t-\tau_3)+q_2(t)x(t+\tau_4)=0, \quad t\geq t_0. \end{aligned}$$

Throughout this paper, we will assume the following assumptions hold.

(h_1) $a\in C^1([t_0,\infty),\mathbb{R})$, $a(t)>0$ for $t\geq t_0$;
(h_2) $p_i\in C([t_0,\infty),[0,a_i])$, where a_i are constants for $i=1,\ 2$, and $a_1+a_2<1$;
(h_3) $q_j\in C([t_0,\infty),(0,\infty))$, for $j=1,\ 2$;
(h_4) $\tau_i\geq 0$ are constants, for $i=1,\ 2,\ 3,\ 4$.

Regarding third-order differential equations, Hanan [15] studied the third order linear differential equation

$$(1.2)\qquad x'''(t)+p(t)x(t)=0, \quad t\geq t_0,$$

and established some sufficient conditions for oscillation and nonoscillation of equation (1.2). He showed that if

$$\liminf_{t\to\infty} t^3p(t)>\frac{2}{3\sqrt{3}},$$

then every solution of equation (1.2) is oscillatory. Baculíková and Džurina [3] obtained some sufficient conditions which ensure that all nonoscillatory solutions of

$$\left[a(t)\left(\left[x(t)+p(t)x(\delta(t))\right]''\right)^{\gamma}\right]'+q(t)x^{\gamma}(\tau(t))=0, \quad t\geq t_0$$

tend to zero when $t\to\infty$. Han *et al.* [12] considered the oscillation of nth-order neutral delay differential equation

$$(x(t)-p(t)x(\tau(t)))'''+q(t)f(x(\delta(t)))=0,\ t\geq t_0,$$

and the authors established some oscillation criteria for the above equation which generalized the results given in [15].

For the oscillation of nth-order mixed neutral functional differential equations, Grace [9] obtained some oscillation theorems for the odd order neutral differential equation

$$(x(t)+p_1x(t-\tau_1)+p_2x(t+\tau_2))^{(n)}=q_1x(t-\sigma_1)+q_2x(t+\sigma_2), \quad t\geq t_0,$$

where $n\geq 1$ is odd. Grace [10] and Yan [21] established several sufficient conditions for the oscillation of higher order neutral functional differential equation of the form

$$(1.3)\qquad (x(t)+cx(t-h)+Cx(t+H))^{(n)}+qx(t-g)+Qx(t+G)=0, \quad t\geq t_0,$$

where q and Q are nonnegative real constants.

The aim of this paper is to examine the oscillatory behavior of equation (1.1). By a solution of equation (1.1), we mean a function $x\in C([T_x,\infty),\mathbb{R})$ for some $T_x\geq t_0$ which has the properties $[x(t)+p_1(t)x(t-\tau_1)+p_2(t)x(t+\tau_2)]\in C^2([T_x,\infty),\mathbb{R})$ and $a(t)[x(t)+p_1(t)x(t-\tau_1)+p_2(t)x(t+\tau_2)]\in C^1([T_x,\infty),\mathbb{R})$ and satisfying equation (1.1) on $[T_x,\infty)$. As is customary, a solution of equation (1.1) is called oscillatory if it has arbitrarily large zeros on $[t_0,\infty)$, otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions oscillate.

The paper is organized as follows: In Section 2, we give some sufficient conditions which guarantee that every solution x of equation (1.1) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$. In Section 3, two examples are considered to illustrate the main results.

2. Main results

In this section we give some new oscillation criteria for equation (1.1). For the sake of convenience, when we write a functional inequality without specifying its domain of validity we assume that it holds for all sufficiently large t. Before stating our main results, we begin with the following lemmas which are crucial in the proofs of the main results. In the following, we use the notations

$$z(t) = x(t) + p_1(t)x(t-\tau_1) + p_2(t)x(t+\tau_2), \quad Q(t) = Q_1(t) + Q_2(t),$$

$$Q_1(t) = \min\{q_1(t), q_1(t-\tau_1), q_1(t+\tau_2)\}, \quad Q_2(t) = \min\{q_2(t), q_2(t-\tau_1), q_2(t+\tau_2)\},$$

$$\eta(t) = \frac{k(t-\tau_3)}{2} Q(t), \text{ for some } k \in (0,1), \quad \text{and} \quad (\rho'(t))_+ = \max\{0, \rho'(t)\}.$$

Lemma 2.1. *Assume that*

$$\int_{t_0}^{\infty} \frac{1}{a(t)} \mathrm{d}t = \infty. \tag{2.1}$$

Furthermore, assume that x is a positive solution of equation (1.1). *Then there are only the following two cases for $t \geq t_1$ sufficiently large:*

(1) $z(t) > 0,\ z'(t) > 0,\ z''(t) > 0,\ (a(t)z''(t))' < 0$, *or*
(2) $z(t) > 0,\ z'(t) < 0,\ z''(t) > 0,\ (a(t)z''(t))' < 0$.

Proof. Let x be a positive solution of equation (1.1). Then there exists a $t_1 \geq t_0$ such that $x(t) > 0,\ x(t-\tau_1) > 0,\ x(t+\tau_2) > 0,\ x(t-\tau_3) > 0$ and $x(t+\tau_4) > 0$ for all $t \geq t_1$. Then $z(t) > 0$ for all $t \geq t_1$. It follows from equation (1.1) that

$$(a(t)z''(t))' = -q_1(t)x(t-\tau_3) - q_2(t)x(t+\tau_4) < 0,\ t \geq t_1. \tag{2.2}$$

Hence, $a(t)z''(t)$ is strictly decreasing on $[t_1, \infty)$. We claim that $z''(t) > 0$ for $t \geq t_1$. If not, then there exist $t_2 \geq t_1$ and $c_1 < 0$ such that

$$a(t)z''(t) \leq a(t_2)z''(t_2) \leq c_1,\ t \geq t_2.$$

Integrating the above inequality from t_2 to t, we have

$$z'(t) \leq z'(t_2) + c_1 \int_{t_2}^{t} \frac{1}{a(s)} \mathrm{d}s.$$

Letting $t \to \infty$, then $z'(t) \to -\infty$. Thus, there exist $t_3 \geq t_2$ and $c_2 < 0$ such that for $t \geq t_3$,

$$z'(t) \leq c_2.$$

Integrating the above inequality from t_3 to t, we obtain

$$z(t) - z(t_3) \leq c_2(t - t_3).$$

Then $\lim_{t\to\infty} z(t) = -\infty$, which is a contradiction. The proof is complete. ∎

Lemma 2.2. [3, Lemma 4] *Let $z(t)>0,\ z'(t)>0,\ z''(t)>0,\ z'''(t)\le 0$ on (T_l,∞). Then, for any $k\in(0,1)$, and for some t_k, one has*

$$\frac{z(t)}{z'(t)}\ge\frac{t-T_l}{2}\ge\frac{kt}{2}\quad for\quad t\ge t_k\ge T_l. \tag{2.3}$$

Lemma 2.3. *Let x be a positive solution of equation* (1.1) *and the corresponding z satisfy* (2) *in Lemma 2.1. If*

$$\int_{t_0}^{\infty}\int_{v}^{\infty}\frac{1}{a(u)}\int_{u}^{\infty}[q_1(s)+q_2(s)]\mathrm{d}s\mathrm{d}u\mathrm{d}v=\infty \tag{2.4}$$

holds, then $\lim_{t\to\infty}x(t)=0$.

Proof. Note that $a_1+a_2<1$. The proof of Lemma 2.3 is similar to that of [3, Lemma 2]. ∎

Next, we will give some oscillation results which guarantee that every solution x of equation (1.1) oscillates or $\lim_{t\to\infty}x(t)=0$.

Theorem 2.1. *Assume that* (2.1) *holds. Suppose further that* (2.4) *holds, $a'(t)\ge 0$ and $\tau_3\ge\tau_1$. Assume also that there exists $\rho\in C^1([t_0,\infty),(0,\infty))$ such that*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\rho(s)\eta(s)-\frac{1+a_1+a_2}{4}\frac{a(s-\tau_3)((\rho'(s))_+)^2}{\rho(s)}\right]\mathrm{d}s=\infty \tag{2.5}$$

holds for all sufficiently large t_1. Then every solution x of equation (1.1) *oscillates or* $\lim_{t\to\infty}x(t)=0$.

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists a $t_1\ge t_0$ such that $x(t)>0$, $x(t-\tau_1)>0$, $x(t+\tau_2)>0$, $x(t-\tau_3)>0$ and $x(t+\tau_4)>0$ for all $t\ge t_1$. Then we have $z(t)>0$ and (2.2) for $t\ge t_1$. By applying (1.1), for all sufficiently large t, we obtain

$$\begin{aligned}&(a(t)z''(t))'+q_1(t)x(t-\tau_3)+q_2(t)x(t+\tau_4)\\&+a_1(a(t-\tau_1)z''(t-\tau_1))'+a_1q_1(t-\tau_1)x(t-\tau_1-\tau_3)+a_1q_2(t-\tau_1)x(t+\tau_4-\tau_1)\\&+a_2(a(t+\tau_2)z''(t+\tau_2))'+a_2q_1(t+\tau_2)x(t+\tau_2-\tau_3)+a_2q_2(t+\tau_2)x(t+\tau_2+\tau_4)=0.\end{aligned}$$

Thus

$$\begin{aligned}&(a(t)z''(t))'+a_1(a(t-\tau_1)z''(t-\tau_1))'+a_2(a(t+\tau_2)z''(t+\tau_2))'\\&\qquad+Q_1(t)z(t-\tau_3)+Q_2(t)z(t+\tau_4)\le 0.\end{aligned} \tag{2.6}$$

By Lemma 2.1, there are two cases for z.

Assume that case (1) holds for $t\ge t_2\ge t_1$. It follows from $z'(t)>0$ that $z(t+\tau_4)\ge z(t-\tau_3)$. Thus, by (2.6), we obtain

$$(a(t)z''(t))'+a_1(a(t-\tau_1)z''(t-\tau_1))'+a_2(a(t+\tau_2)z''(t+\tau_2))'+Q(t)z(t-\tau_3)\le 0. \tag{2.7}$$

Using the Riccati transformation

$$\omega_1(t)=\rho(t)\frac{a(t)z''(t)}{z'(t-\tau_3)},\ t\ge t_2. \tag{2.8}$$

Then $\omega_1(t)>0$ for $t\ge t_2$. Differentiating (2.8), we see that

$$\omega_1'(t)=\rho'(t)\frac{a(t)z''(t)}{z'(t-\tau_3)}+\rho(t)\frac{(a(t)z''(t))'}{z'(t-\tau_3)}-\rho(t)\frac{a(t)z''(t)z''(t-\tau_3)}{(z'(t-\tau_3))^2}.$$

By (2.2), we have $a(t-\tau_3)z''(t-\tau_3) \geq a(t)z''(t)$. Thus, from (2.8), we get

$$\omega_1'(t) \leq \frac{(\rho'(t))_+}{\rho(t)}\omega_1(t) + \rho(t)\frac{(a(t)z''(t))'}{z'(t-\tau_3)} - \frac{(\omega_1(t))^2}{\rho(t)a(t-\tau_3)}. \tag{2.9}$$

Next, define the function ω_2 by

$$\omega_2(t) = \rho(t)\frac{a(t-\tau_1)z''(t-\tau_1)}{z'(t-\tau_3)},\ t \geq t_2. \tag{2.10}$$

Then $\omega_2(t) > 0$ for $t \geq t_2$. Differentiating (2.10), we obtain

$$\begin{aligned}\omega_2'(t) = {} & \rho'(t)\frac{a(t-\tau_1)z''(t-\tau_1)}{z'(t-\tau_3)} + \rho(t)\frac{(a(t-\tau_1)z''(t-\tau_1))'}{z'(t-\tau_3)} \\ & - \rho(t)\frac{a(t-\tau_1)z''(t-\tau_1)z''(t-\tau_3)}{(z'(t-\tau_3))^2}.\end{aligned}$$

Note that $\tau_3 \geq \tau_1$. By (2.2), we find $a(t-\tau_3)z''(t-\tau_3) \geq a(t-\tau_1)z''(t-\tau_1)$. Hence, by (2.10), we get

$$\omega_2'(t) \leq \frac{(\rho'(t))_+}{\rho(t)}\omega_2(t) + \rho(t)\frac{(a(t-\tau_1)z''(t-\tau_1))'}{z'(t-\tau_3)} - \frac{(\omega_2(t))^2}{\rho(t)a(t-\tau_3)}. \tag{2.11}$$

In the following, we define another function ω_3 by

$$\omega_3(t) = \rho(t)\frac{a(t+\tau_2)z''(t+\tau_2)}{z'(t-\tau_3)},\ t \geq t_2. \tag{2.12}$$

Then $\omega_3(t) > 0$ for $t \geq t_2$. Differentiating (2.12), we find

$$\begin{aligned}\omega_3'(t) = {} & \rho'(t)\frac{a(t+\tau_2)z''(t+\tau_2)}{z'(t-\tau_3)} + \rho(t)\frac{(a(t+\tau_2)z''(t+\tau_2))'}{z'(t-\tau_3)} \\ & - \rho(t)\frac{a(t+\tau_2)z''(t+\tau_2)z''(t-\tau_3)}{(z'(t-\tau_3))^2}.\end{aligned}$$

By (2.2), we obtain $a(t-\tau_3)z''(t-\tau_3) \geq a(t+\tau_2)z''(t+\tau_2)$. Then, by (2.12), we get

$$\omega_3'(t) \leq \frac{(\rho'(t))_+}{\rho(t)}\omega_3(t) + \rho(t)\frac{(a(t+\tau_2)z''(t+\tau_2))'}{z'(t-\tau_3)} - \frac{(\omega_3(t))^2}{\rho(t)a(t-\tau_3)}. \tag{2.13}$$

Therefore, by (2.9), (2.11) and (2.13), we obtain

$$\begin{aligned}& \omega_1'(t) + a_1\omega_2'(t) + a_2\omega_3'(t) \\ & \leq \rho(t)\left[\frac{(a(t)z''(t))' + a_1(a(t-\tau_1)z''(t-\tau_1))' + a_2(a(t+\tau_2)z''(t+\tau_2))'}{z'(t-\tau_3)}\right] \\ & \quad + \frac{(\rho'(t))_+}{\rho(t)}\omega_1(t) - \frac{(\omega_1(t))^2}{\rho(t)a(t-\tau_3)} + a_1\frac{(\rho'(t))_+}{\rho(t)}\omega_2(t) - a_1\frac{(\omega_2(t))^2}{\rho(t)a(t-\tau_3)} \\ & \quad + a_2\frac{(\rho'(t))_+}{\rho(t)}\omega_3(t) - a_2\frac{(\omega_3(t))^2}{\rho(t)a(t-\tau_3)}.\end{aligned} \tag{2.14}$$

Thus, from (2.7) and (2.14), we get

$$
\begin{aligned}
&\omega_1'(t)+a_1\omega_2'(t)+a_2\omega_3'(t)\\
&\le -\rho(t)Q(t)\frac{z(t-\tau_3)}{z'(t-\tau_3)}+\frac{(\rho'(t))_+}{\rho(t)}\omega_1(t)-\frac{(\omega_1(t))^2}{\rho(t)a(t-\tau_3)}+a_1\frac{(\rho'(t))_+}{\rho(t)}\omega_2(t)\\
&\quad -a_1\frac{(\omega_2(t))^2}{\rho(t)a(t-\tau_3)}+a_2\frac{(\rho'(t))_+}{\rho(t)}\omega_3(t)-a_2\frac{(\omega_3(t))^2}{\rho(t)a(t-\tau_3)}.
\end{aligned}\tag{2.15}
$$

On the other hand, using $a'(t)\ge 0$, $z''(t)>0$ for $t\ge t_2$, and

$$(a(t)z''(t))'=a'(t)z''(t)+a(t)z'''(t)<0,$$

we have

$$z'''(t)<0\tag{2.16}$$

for $t\ge t_2$. Then, by Lemma 2.2, we find, for any $k\in(0,1)$, and for t sufficiently large,

$$\frac{z(t-\tau_3)}{z'(t-\tau_3)}\ge\frac{k(t-\tau_3)}{2}$$

due to (2.3). Combining the above inequality with (2.15), we get

$$
\begin{aligned}
&\omega_1'(t)+a_1\omega_2'(t)+a_2\omega_3'(t)\\
&\le -\rho(t)\frac{k(t-\tau_3)}{2}Q(t)+\frac{(\rho'(t))_+}{\rho(t)}\omega_1(t)-\frac{(\omega_1(t))^2}{\rho(t)a(t-\tau_3)}+a_1\frac{(\rho'(t))_+}{\rho(t)}\omega_2(t)\\
&\quad -a_1\frac{(\omega_2(t))^2}{\rho(t)a(t-\tau_3)}+a_2\frac{(\rho'(t))_+}{\rho(t)}\omega_3(t)-a_2\frac{(\omega_3(t))^2}{\rho(t)a(t-\tau_3)}
\end{aligned}\tag{2.17}
$$

for any $k\in(0,1)$. Then, by (2.17), we find that

$$\omega_1'(t)+a_1\omega_2'(t)+a_2\omega_3'(t)\le-\rho(t)\eta(t)+\frac{1+a_1+a_2}{4}\frac{a(t-\tau_3)((\rho'(t))_+)^2}{\rho(t)}.$$

Integrating the above inequality from t_3 ($t_3\ge t_2$) to t, we obtain

$$\int_{t_3}^{t}\left[\rho(s)\eta(s)-\frac{1+a_1+a_2}{4}\frac{a(s-\tau_3)((\rho'(s))_+)^2}{\rho(s)}\right]\mathrm{d}s\le\omega_1(t_3)+a_1\omega_2(t_3)+a_2\omega_3(t_3),$$

which contradicts (2.5).

Assume that case (2) holds. Then, by Lemma 2.3, we can obtain $\lim_{t\to\infty}x(t)=0$. The proof is complete. ∎

Let $\rho(t)=t$. Then we can obtain the following corollary by Theorem 2.4.

Corollary 2.1. *Assume that* (2.1) *holds. Suppose further that* (2.4) *holds,* $a'(t)\ge 0$ *and* $\tau_3\ge\tau_1$. *If*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[s\eta(s)-\frac{1+a_1+a_2}{4}\frac{a(s-\tau_3)}{s}\right]\mathrm{d}s=\infty$$

holds for all sufficiently large t_1, *then every solution* x *of equation* (1.1) *oscillates or* $\lim_{t\to\infty}x(t)=0$.

Theorem 2.2. *Assume that* (2.1) *holds. Suppose further that* (2.4) *holds,* $a'(t) \geq 0$ *and* $\tau_1 \geq \tau_3$. *Assume also that there exists* $\rho \in C^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\rho(s)\eta(s) - \frac{1+a_1+a_2}{4}\frac{a(s-\tau_1)((\rho'(s))_+)^2}{\rho(s)}\right] \mathrm{d}s = \infty \tag{2.18}$$

holds for all sufficiently large t_1. *Then every solution of equation* (1.1) *oscillates or* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(t-\tau_1) > 0$, $x(t+\tau_2) > 0$, $x(t-\tau_3) > 0$ and $x(t+\tau_4) > 0$ for all $t \geq t_1$. Then we have $z(t) > 0$ and (2.2) for $t \geq t_1$. Proceeding as in the proof of Theorem 2.4, we get (2.6).

By Lemma 2.1, there are two cases for z.

Assume that case (1) holds for $t \geq t_2 \geq t_1$. Then, we obtain (2.7). Using the Riccati transformation

$$\omega_1(t) = \rho(t)\frac{a(t)z''(t)}{z'(t-\tau_1)}, \quad t \geq t_2,$$

$$\omega_2(t) = \rho(t)\frac{a(t-\tau_1)z''(t-\tau_1)}{z'(t-\tau_1)}, \quad t \geq t_2$$

and

$$\omega_3(t) = \rho(t)\frac{a(t+\tau_2)z''(t+\tau_2)}{z'(t-\tau_1)}, \quad t \geq t_2,$$

respectively. Similar to the proof of Theorem 2.4, we get

$$\begin{aligned} &\omega_1'(t) + a_1\omega_2'(t) + a_2\omega_3'(t) \\ &\leq -\rho(t)Q(t)\frac{z(t-\tau_3)}{z'(t-\tau_1)} + \frac{(\rho'(t))_+}{\rho(t)}\omega_1(t) - \frac{(\omega_1(t))^2}{\rho(t)a(t-\tau_1)} + a_1\frac{(\rho'(t))_+}{\rho(t)}\omega_2(t) \\ &\quad - a_1\frac{(\omega_2(t))^2}{\rho(t)a(t-\tau_1)} + a_2\frac{(\rho'(t))_+}{\rho(t)}\omega_3(t) - a_2\frac{(\omega_3(t))^2}{\rho(t)a(t-\tau_1)}. \end{aligned} \tag{2.19}$$

On the other hand, we have (2.16) for $t \geq t_2$. Then, by Lemma 2.2, for any $k \in (0,1)$, we find

$$\frac{z(t-\tau_3)}{z'(t-\tau_1)} = \frac{z(t-\tau_3)}{z'(t-\tau_3)}\frac{z'(t-\tau_3)}{z'(t-\tau_1)} \geq \frac{k(t-\tau_3)}{2}\frac{z'(t-\tau_3)}{z'(t-\tau_1)} \geq \frac{k(t-\tau_3)}{2}$$

due to $\tau_1 \geq \tau_3$ and $z''(t) > 0$ for $t \geq t_2$. Combining the above inequality with (2.19), we get

$$\begin{aligned} &\omega_1'(t) + a_1\omega_2'(t) + a_2\omega_3'(t) \\ &\leq -\rho(t)\frac{k(t-\tau_3)}{2}Q(t) + \frac{(\rho'(t))_+}{\rho(t)}\omega_1(t) - \frac{(\omega_1(t))^2}{\rho(t)a(t-\tau_1)} + a_1\frac{(\rho'(t))_+}{\rho(t)}\omega_2(t) \\ &\quad - a_1\frac{(\omega_2(t))^2}{\rho(t)a(t-\tau_1)} + a_2\frac{(\rho'(t))_+}{\rho(t)}\omega_3(t) - a_2\frac{(\omega_3(t))^2}{\rho(t)a(t-\tau_1)}. \end{aligned} \tag{2.20}$$

Then, by (2.20), we find that

$$\omega_1'(t) + a_1\omega_2'(t) + a_2\omega_3'(t) \leq -\rho(t)\eta(t) + \frac{1+a_1+a_2}{4}\frac{a(t-\tau_1)((\rho'(t))_+)^2}{\rho(t)}.$$

Integrating the above inequality from t_2 to t, we obtain

$$\int_{t_2}^{t}\left[\rho(s)\eta(s)-\frac{1+a_1+a_2}{4}\frac{a(s-\tau_1)((\rho'(s))_+)^2}{\rho(s)}\right]\mathrm{d}s \le \omega_1(t_2)+a_1\omega_2(t_2)+a_2\omega_3(t_2),$$

which contradicts (2.18).

Assume that case (2) holds. Then, by Lemma 2.3, we can obtain $\lim_{t\to\infty}x(t)=0$. This completes the proof. ∎

Let $\rho(t)=t$. Then we can obtain the following corollary by Theorem 2.6.

Corollary 2.2. *Assume that* (2.1) *holds. Suppose further that* (2.4) *holds,* $a'(t)\ge 0$ *and* $\tau_1\ge\tau_3$. *If*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[s\eta(s)-\frac{1+a_1+a_2}{4}\frac{a(s-\tau_1)}{s}\right]\mathrm{d}s=\infty$$

holds for all sufficiently large t_1*, then every solution* x *of equation* (1.1) *oscillates or* $\lim_{t\to\infty}x(t)=0$.

Remark 2.1. Adding some restrictions on arguments $\tau_i(t)$, for $i=1,\ 2,\ 3,\ 4$, our results can be extended to the third-order mixed neutral functional equation

$$\big(a(t)\left(x(t)+p_1(t)x(t-\tau_1(t))+p_2(t)x(t+\tau_2(t))\right)''\big)'+q_1(t)x(t-\tau_3(t))+q_2(t)x(t+\tau_4(t))=0$$

for $t\ge t_0$, the details are left to the reader.

3. Examples

Grace [10] and Yan [21] considered the oscillation of equation

$$(x(t)+cx(t-h)+Cx(t+H))'''+qx(t-g)+Qx(t+G)=0,\qquad t\ge t_0, \tag{3.1}$$

where c, C and Q are nonnegative real constants, and g, G, h, H and Q are positive real constants. They obtained some oscillation theorems for equation (3.1). For example

Theorem 3.1. [10, Theorem 5] *If* $g>h$*, and*

$$\left(\frac{q}{1+c+C}\right)^{1/3}\left(\frac{g-h}{3}\right)c>1,$$

then every solution of equation (3.1) *is oscillatory.*

Theorem 3.2. [21, Theorem 5] *If* $g>h$*, and*

$$q\left(\frac{\mathrm{e}}{3}\right)^3(g-h)^3>\exp\left[-\left(\frac{q}{1+c+C}\right)^{1/3}h\right]+c+C\exp\left[-\left(\frac{q}{1+c+C}\right)^{1/3}(H+h)\right],$$

then every solution of equation (3.1) *is oscillatory.*

We note that Theorem 3.1 and Theorem 3.2 are based on the condition $g>h$, that is, these results cannot be applied to equation (3.1) for the case when $g\le h$. Therefore, Theorem 2.6 obtained in this paper complements results of [10, 21].

In the following, we will give two examples to illustrate the main results.

Example 3.1. Consider the third-order differential equation

(3.2)
$$\left(x(t)+\frac{1}{3}x(t-1)+\frac{1}{3}x(t+1)\right)'''+\left(\mathrm{e}^{-2}+\frac{1}{3}\mathrm{e}^{-1}\right)x(t-2)+\frac{1}{3}x(t+1)=0, \quad t\geq t_0.$$

Let

$$a(t)=1,\ p_1(t)=p_2(t)=\frac{1}{3},\ q_1(t)=\mathrm{e}^{-2}+\frac{\mathrm{e}^{-1}}{3},\ q_2(t)=\frac{1}{3}, \tau_1=\tau_2=\tau_4=1, \tau_3=2.$$

Clearly, (2.1) holds. Take $\rho(t)=1$. Then condition (2.5) holds. On the other hand, we have (2.4). Then, by Theorem 2.4, every solution x of equation (3.2) oscillates or $\lim_{t\to\infty}x(t)=0$. It is easy to find that $x(t)=\mathrm{e}^{-t}$ is a solution of equation (3.2). However, the results established in [10,21] do not apply to equation (3.2).

Example 3.2. Consider the third-order differential equation

(3.3)
$$\left(\alpha t\left(x(t)+p_1(t)x(t-\tau_1)+p_2(t)x(t+\tau_2)\right)''\right)'+\frac{\beta}{t}x(t-\tau_3)+\frac{\gamma}{t}x(t+\tau_4)=0, \quad t\geq t_0,$$

where $\alpha,\ \beta,\ \gamma$ are positive constants, $0\leq p_i(t)\leq a_i$, for $i=1,\ 2$, $a_1+a_2<1$.

Let

$$a(t)=\alpha t,\ p_1(t)=p_2(t)=\frac{1}{3},\ q_1(t)=\frac{\beta}{t},\ q_2(t)=\frac{\gamma}{t}.$$

It is easy to verify that all the conditions of Corollary 2.7 hold. Thus, from Corollary 2.7, every solution x of equation (3.3) oscillates or $\lim_{t\to\infty}x(t)=0$.

Remark 3.1. It remains an open problem to study equation (1.1) when $\int_{t_0}^{\infty}1/a(t)\mathrm{d}t<\infty$ or $a'(t)\leq 0$.

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