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The Borel Radius and the *S* Radius of the K-Quasimeromorphic Mapping in the Unit Disc

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Abstract. By using Ahlfors' theory of covering surface, a fundamental inequality for the K-quasimeromorphic mapping in the unit disc is established. As an application, some results on the Borel radius and the *S* radius dealing with multiple values of the K-quasimeromorphic mapping in the unit disc are obtained.

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1. Introduction

In 1997, the value distribution theory of meromorphic functions due to Nevanlinna (see [11, 12, 15] for standard references) has been extended to the corresponding theory of the K-quasimeromorphic mapping by Sun and Yang [10]. The K-quasimeromorphic mapping is a more widespread function than the meromorphic function, but it has no derivative, even the partial derivative does not exist everywhere. They established a fundamental inequality on the complex plane and used it to prove the existence theorem of the Borel direction and the filling disc theorem of the K-quasimeromorphic mapping. In 1999, Gao [5] established a fundamental inequality dealing with multiple values on the complex plane and improved some results of [10].

Recently, the singular direction is one of the interesting topics studied in the theory of value distribution of the K-quasimeromorphic mapping on the complex plane such as Julia direction, Borel direction, Nevanlinna direction and S direction, see [2,4,8,13,14,16]. Their existence theorems and some connections between them have also been established, which extends the relative properties of meromorphic function on the complex plane. In 2004, Yang and Liu [16] used a fundamental inequality of an angular domain on the complex plane to confirm the existence of a Borel direction of the K-quasimeromorphic mappings

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of the zero order. Later, Wu and Sun [14] proved the existence of a S direction for the K-quasimeromorphic mapping on the complex plane, which was inspired by the idea of the T-direction [6] for the meromorphic function.

Theorem 1.1. *Let $f(z)$ be K-quasimeromorphic mapping on the complex plane and satisfy $\overline{\lim\limits_{r\to\infty}}(S(r,f))/(\ln r)^2 = +\infty$, then there exists a ray $\arg z = \theta$ (namely S direction) such that for any $\varepsilon > 0$,*

$$\overline{\lim_{r\to\infty}}\frac{\overline{n}(\Omega(\theta-\varepsilon,\theta+\varepsilon),r,a)}{S(r,f)} > 0$$

holds for all $a \in \mathbf{C}_\infty := \mathbf{C}\cup\infty$, except for two possible exceptional values.

It is well known that if f is a transcendental meromorphic function defined in $|z| < 1$, it will share some properties with the one on the complex plane, see [1, 7, 11]. Thus a natural question is: Is there a Borel radius or a S radius for the K-quasimeromorphic mapping in $|z| < 1$? However, until now only a few results on the singular radius of the K-quasimeromorphic mapping have been discussed, see [3, 9]. So in this paper, we establish a more precise fundamental inequality for the K-quasimeromorphic mapping in the unit disc and confirm the existence of the Borel radius and the S radius (dealing with multiple values) for the K-quasimeromorphic mapping in the unit disc, which develop some results of [4, 10, 13]. To do so, we recall some definitions and notations, which can be found in [8, 10].

Definition 1.1. [10] *Let $f(s)$ be a homeomorphism from D to D'. If for any rectangle $\{z = x+iy; a < x < b, c < y < d\}$ in D,*

(i) *$f(x+iy)$ is absolutely continuous of y for almost every fixed $x \in (a,b)$ and $f(x+iy)$ is absolutely continuous of x for almost every fixed $y \in (c,d)$;*

(ii) *there exists a constant $K \geq 1$ such that*

$$|f_z(z)| + |f_{\bar{z}}(z)| \leq K(|f_z(z)| - |f_{\bar{z}}(z)|)$$

holds almost everywhere in D; then f is named an univalent K-quasimeromorphic mapping in D.

Definition 1.2. [10] *Let f be a complex and continuous function in the region D. For a point z_0 in D, if there is a neighborhood $U(\subset D)$ and a positive integer n depending on z_0, such that*

$$F(z) = \begin{cases} (f(z))^{1/n}, & f(z_0) = \infty \\ (f(z)-f(z_0))^{1/n} + f(z_0), & f(z_0) \neq \infty \end{cases}$$

is an univalent K-quasimeromorphic mapping, then f is named n-valent K-quasimeromorphic mappings at point z_0. If f is n-valent K-quasimeromorphic at every point of D, then f is called a K-quasimeromorphic mapping in D.

It is obvious that a meromorphic function is a 1-quasimeromorphic mapping. The composition function $g \circ f$ of a meromorphic function g and a K-quasimeromorphic mapping f is still a K-quasimeromorphic mapping. Let $n(r,a)$ be the number of zero points of $f(z)-a$ in disc $|z| \leq r$. If the multiple zeros are counted only once, then we use $\overline{n}(r,a)$. Let $n(\Omega(\varphi-\varepsilon,\varphi+\varepsilon),r,a)$ be the number of zero points $f(z)-a$ in $\{\varphi-\varepsilon < \arg z < \varphi+\varepsilon\}\cap\{|z| \leq r\}$. If the multiple zeros are counted only once, we use $\overline{n}(\Omega(\varphi-\varepsilon,\varphi+\varepsilon),r,a)$. Let $\overline{n}^{l)}(\Omega(\varphi-\varepsilon,\varphi+\varepsilon),r,a)$ be the number of distinct roots with the multiplicity $\leq l$ of $f(z) = a$ in the same region.

Let $\mathbf{V}$ be a Riemann sphere whose diameter is 1. $f(z)=u(x,y)+iv(x,y)$ is a K-quasimeromorphic mapping in the angular domain $E=\Omega(\varphi_1,\varphi_2)\cap\{|z|\le r\}$. Set

$$S(E,f)=S(\Omega(\varphi_1,\varphi_2),r)=\frac{|F_r|}{|V|}=\frac{1}{\pi}\int\int_{z\in E}\frac{u_xv_y-v_xu_y}{(1+|f(z)|^2)^2}rdrd\theta.$$

where $|F_r|$ is the area of the image of E on $\mathbf{V}$ and $|\mathbf{V}|$ is the area of $\mathbf{V}$. If $E=\{|z|\le r\}$, then $S(E,f)$ can be replaced by $S(r,f)$.

Definition 1.3. *Let $f(z)$ be a K-quasimeromorphic mapping defined in the unit disc. If $S(r,f)\to+\infty$ as $r\to1^-$, then we call $f(z)$ transcendental. The order of the transcendental K-quasimeromorphic mapping in the unit disc is defined by*

$$\rho=\overline{\lim_{r\to1^-}}\frac{\ln S(r,f)}{-\ln(1-r)}.$$

If $\rho=\lim\limits_{r\to1^-}(\ln S(r,f))/(-\ln(1-r))$, then $f(z)$ is of regular growth. Especially, when $K=1$, if $S(r,f)$ is replaced by $T(r,f)=\int_0^r S(t,f)/tdt$, then ρ is called the order of the meromorphic function $f(z)$.

Definition 1.4. *Let $f(z)$ be a transcendental K-quasimeromorphic mapping defined in the unit disc. A radius $\Delta(\varphi)=\{z:\arg z=\varphi,|z|<1\}$ is called a Borel radius of the order $\rho\in(0,+\infty)$ for the K-quasimeromorphic mapping $f(z)$ in the unit disc, provided that for any $\varepsilon\in(0,\pi)$,*

$$\lim_{r\to1^-}\frac{\ln\overline{n}(\Omega(\varphi-\varepsilon,\varphi+\varepsilon),r,a)}{\ln\frac{1}{1-r}}\ge\rho$$

holds for all $a\in\mathbf{C}_\infty$, except for two possible exceptional values.

Note that this definition of Borel radius meaningfully characterizes the growth of $f(z)$ only when $0<\rho<\infty$. Inspired by the idea of [14], we give a definition of the S radius in the unit disc.

Definition 1.5. *Let $f(z)$ be a transcendental K-quasimeromorphic mapping defined in the unit disc. A radius $\Delta(\varphi)$ is called a S radius of $f(z)$ dealing with multiple values $l(\ge3)$, provided that for any $\varepsilon>0$,*

$$\overline{\lim_{r\to1^-}}\frac{\overline{n}^{l)}(\Omega(\varphi-\varepsilon,\varphi+\varepsilon),r,a)}{S(r,f)}>0$$

holds for all $a\in\mathbf{C}_\infty$, except for two possible exceptional values. If $l\to+\infty$, $\Delta(\varphi)$ is called the S radius of $f(z)$ in the unit disc (S direction in the case of the complex plane).

2. Preliminary lemmas

Lemma 2.1. [5] *Let $f(z)$ be a K-quasimeromorphic mapping in $|z|<R$ and $\{a_1,a_2,\cdots,a_q\}$ be $q(q\ge3)$ distinct points with the mutual spherical distance no less than $\delta\in(0,1/2)$, then for any $r\in(0,R)$, we have*

$$\left(q-2-\frac{2}{l}\right)S(r,f)\le\sum_{v=1}^{q}\overline{n}^{l)}(R,a_v)+\frac{(960+2q\pi)^2}{\delta^6}\cdot\frac{2^5\pi^2K}{q-2-\frac{2}{l}}\cdot\frac{R}{R-r}.$$

Lemma 2.2. [7] *Let $f(z)$ be a meromorphic function defined in $|z|<1$, then*

$$\overline{\lim_{r\to 1^-}}\frac{\ln T(r,f)}{-\ln(1-r)}=\rho \Longleftrightarrow \overline{\lim_{r\to 1^-}}\frac{\ln S(r,f)}{-\ln(1-r)}=\rho+1.$$

3. Main results and their proofs

Theorem 3.1. *Let $f(z)$ be a K-quasimeromorphic mapping in $|z|<1$. Δ and Δ_0 are two angular domains with the common vertex on the center of the unit disc, where*

$$\Delta:=\Omega(\varphi-\eta,\varphi+\eta)\subset\Delta_0:=\Omega(\varphi-\eta_0,\varphi+\eta_0),\ 0<\eta<\eta_0,\ 0\le\varphi<2\pi.$$

Then,

$$\left(q-2-\frac{2}{l}\right)S(\Delta,r)\le\left(1+\frac{2\ln 2}{\ln\frac{1}{1-r}}\right)\sum_{v=1}^{q}\overline{n}^{l)}\left(\Delta_0,\frac{3+r}{4},a_v\right)+O\left(\ln\frac{1}{1-r}\right), \tag{3.1}$$

$$(q-2)S(\Delta,r)\le\left(1+\frac{2\ln 2}{\ln\frac{1}{1-r}}\right)\sum_{v=1}^{q}\overline{n}\left(\Delta_0,\frac{3+r}{4},a_v\right)+O\left(\ln\frac{1}{1-r}\right), \tag{3.2}$$

where $a_1,a_2,\cdots,a_q$ are q distinct points in $\mathbf{V}$ with the mutual spherical distance not less than $\delta\in(0,1/2)$, $r_0\in(1/2,1)$ and $r\in(r_0,1)$.

Proof. Let $r_i=1-(1-r_0)/2^i\quad(i=0,1,\cdots)$, then $r_1=(1+r_0)/2$. For any $r\in(r_1,1)$, there exists $n\in N_+$ such that $r_n\le r\le r_{n+1}$. So we set

$$r_{i,j}=r_i+\frac{j(r_{i+1}-r_i)}{n}\quad(j=0,1,\cdots n-1),$$

then $r_{i,0}=r_i$ and $r_{i,n}=r_{i+1}$. For any positive integer $i\ge 2$, we set

$$\Delta_0(r_{i,j},r_{i+1,j+1}):=\Delta_0\cap\{r_{i,j}\le|z|\le r_{i+1,j+1}\}.$$

So we can easily see that there exists an integer $j_0\in[0,n-1]$ such that

$$\sum_{v=1}^{q}\overline{n}^{l)}\Big(\bigcup_{i=0}^{n+1}\Delta_0(r_{i,j_0},r_{i,j_0+1}),a_v\Big)\le\frac{1}{n}\sum_{v=1}^{q}\overline{n}^{l)}(\Delta_0,r_{n+2},a_v),$$

where $\Delta_0(r_{i,j_0},r_{i,j_0+1})=\Delta_0\cap\{r_{i,j_0}\le|z|\le r_{i,j_0+1}\}$. Then, we set

$$\Delta(r'_i,r'_{i+1}):=\Delta\cap\{r'_i\le|z|\le r'_{i+1}\}\subset\Delta_0(r_{i,j_0},r_{i+1,j_0+1}),$$

where

$$r'_i=\frac{r_{i,j_0}+r_{i,j_0+1}}{2},\quad r'_{i+1}=\frac{r_{i+1,j_0}+r_{i+1,j_0+1}}{2}.$$

Without loss of generality, we suppose that $\varphi=0$. Since

$$r_{i+1,j_0+1}-r_{i,j_0}=\frac{1-r_0}{2^{i+2}}\left(2+\frac{1-j_0}{n}\right),\qquad r'_{i+1}-r'_i=\frac{1-r_0}{2^{i+2}}\left(2-\frac{2j_0+1}{2n}\right),$$

then $\Delta_0(r_{i,j_0},r_{i+1,j_0+1})$ $(i=2,3,\cdots)$ can map mutually by some transforms such that their sub domains $\Delta(r'_i,r'_{i+1})$ and whose centers $((r'_i+r'_{i+1})/2,0)$ map each other, respectively. Through the Riemann mapping theorem, for any fixed i, we can map the $\Delta_0(r_{i,j_0},r_{i+1,j_0+1})$ on $|\xi|<1$ by a conformal mapping g such that the point $((r'_i+r'_{i+1})/2,0)$ of $\Delta_0(r_{i,j_0},r_{i+1,j_0+1})$

becomes $\xi = 0$, then the image of $\Delta(r'_i, r'_{i+1})$ is contained in $|\xi| \le c < 1$, where $c(>0)$ is a constant defined by η, η_0 and r_0, independent of i. Hence by Lemma 2.1, we have

$$(q-2-\frac{2}{l})S(c, f\circ g^{-1}) \le \sum_{v=1}^{q} \overline{n}^{l)}(1, f\circ g^{-1} = a_v) + \frac{H}{1-c},$$

where H is a constant. Then,

$$\left(q-2-\frac{2}{l}\right) S(\Delta(r'_i, r'_{i+1}), f) \le \sum_{v=1}^{q} \overline{n}^{l)}(\Delta_0(r_{i,j_0}, r_{i+1,j_0+1}), a_v) + \frac{H}{1-c}, \quad i = 0, 1, \cdots, n.$$

Adding two sides of the above expression from $i = 0$ to n, we obtain

$$\left(q-2-\frac{2}{l}\right) \sum_{i=0}^{n} S(\Delta(r'_i, r'_{i+1}), f) \le \sum_{i=0}^{n}\sum_{v=1}^{q} \overline{n}^{l)}(\Delta_0(r_{i,j_0}, r_{i+1,j_0+1}), a_v) + \frac{H}{1-c}(n+1).$$

Since $r_n \le r \le r_{n+1}$, then

$$r_{n+2} = \frac{3+r_n}{4} \le \frac{3+r}{4}, \quad \frac{1-r}{4} \le 1 - r_{n+2} \le \frac{1}{2^{n+2}}, \quad \frac{1}{1-r} \le \frac{1}{1-r_{n+1}} = \frac{2^{n+1}}{1-r_0}.$$

Hence

$$2^n \le \frac{1}{1-r}, \quad n+1 \le 1 + \frac{1}{\ln 2}\ln\frac{1}{1-r}, \quad \frac{1}{n} \le \frac{2\ln 2}{\ln\frac{1}{1-r}} \quad (r \to 1^-).$$

When r is sufficiently close to 1^-, we have

$$\begin{aligned}
&\left(q-2-\frac{2}{l}\right) S(\Delta, r) \le \left(q-2-\frac{2}{l}\right) \sum_{i=0}^{n} S(\Delta(r'_i, r'_{i+1}), f) + \left(q-2-\frac{2}{l}\right) S(\Delta, r_1)\\
&\le \sum_{i=0}^{n}\sum_{v=1}^{q} \overline{n}^{l)}(\Delta_0(r_{i,j_0}, r_{i+1,j_0+1}), a_v) + \frac{H(n+1)}{1-c} + (q-2-\frac{2}{l})S(\Delta, r_1)\\
&\le \sum_{i=0}^{n}\sum_{v=1}^{q} \overline{n}^{l)}(\Delta_0(r_{i,j_0}, r_{i,j_0+1}), a_v) + \sum_{i=0}^{n}\sum_{v=1}^{q} \overline{n}^{l)}(\Delta_0(r_{i,j_0+1}, r_{i+1,j_0+1}), a_v) + \frac{2H(n+1)}{1-c}\\
&\le (1+\frac{1}{n}) \sum_{v=1}^{q} \overline{n}^{l)}(\Delta_0, r_{n+2}, a_v) + \frac{2H(n+1)}{1-c}\\
&\le \left(1 + \frac{2\ln 2}{\ln\frac{1}{1-r}}\right) \sum_{v=1}^{q} \overline{n}^{l)}\left(\Delta_0, \frac{3+r}{4}, a_v\right) + O\left(\ln\frac{1}{1-r}\right).
\end{aligned}$$

So (3.1) follows, the second inequality (3.2) can be obtained by the similar proof. ∎

Remark 3.1. Theorem 3.1 gives a fundamental inequality for the K-quasimeromorphic mapping in an angular domain of the unit disc, which is more precise than that of [3]. If $K = 1$, it is also better than [11, Theorem VII.14, P.291] and [7, Lemma 3].

Theorem 3.2. *Let $f(z)$ be a K-quasimeromorphic mapping in the unit disc with the order $\rho \in (0, +\infty)$, then $f(z)$ has a Borel radius of the order ρ.*

Proof. Otherwise, for any $\varphi \in [0, 2\pi)$, there exists $\varepsilon_\varphi > 0$ and three distinct complex numbers $a_1, a_2, a_3 \in \mathbf{C}_\infty$ such that

$$\varlimsup_{r\to 1^-} \frac{\ln \overline{n}(\Omega(\varphi-\varepsilon_\varphi, \varphi+\varepsilon_\varphi), r, a_i)}{\ln\frac{1}{1-r}} = \rho_0 < \rho, \quad i = 1, 2, 3. \tag{3.3}$$

It is obvious that the open sets $\{(\varphi-\varepsilon_\varphi/4,\varphi+\varepsilon_\varphi/4)|\varphi\in[0,2\pi)\}$ cover the unit disc. From the finite covering theorem, there exists a subsequence

$$\left(\varphi_1-\frac{\varepsilon_{\varphi_1}}{4},\varphi_1+\frac{\varepsilon_{\varphi_1}}{4}\right),\cdots,\left(\varphi_n-\frac{\varepsilon_{\varphi_n}}{4},\varphi_n+\frac{\varepsilon_{\varphi_n}}{4}\right)$$

lying in $(\varphi_k-\varepsilon_{\varphi_k},\varphi_k+\varepsilon_{\varphi_k})\quad(k=1,\cdots,n)$, such that for any $\varepsilon>0$ and each k

$$\sum_{i=1}^{3}\overline{n}(\Omega(\varphi-\varepsilon_\varphi,\varphi+\varepsilon_\varphi),r,a_i)<3\left(\frac{1}{1-r}\right)^{\rho_0+\varepsilon}.$$

By (3.2) of Theorem 3.1, it follows that

$$\begin{aligned}S(r,f)&\le\sum_{k=1}^{n}S(\Omega(\varphi_k-\frac{\varepsilon_{\varphi_k}}{4},\varphi_k+\frac{\varepsilon_{\varphi_k}}{4})\\&\le(1+\frac{2\ln 2}{\ln\frac{1}{1-r}})\sum_{k=1}^{n}\sum_{i=1}^{3}\overline{n}(\Omega(\varphi_k-\frac{\varepsilon_{\varphi_k}}{2},\varphi_k+\frac{\varepsilon_{\varphi_k}}{2}),\frac{r+3}{4},a_i)+O(\ln\frac{1}{1-r}).\end{aligned}$$

Then, there is a positive constant C such that

$$S(r,f)\le C(1+\frac{2\ln 2}{\ln\frac{1}{1-r}})(\frac{1}{1-r})^{\rho_0+\varepsilon}+O(\ln\frac{1}{1-r}),$$

This is in contradiction to that $f(z)$ is of the order ρ. ∎

Corollary 3.1. *Let $f(z)$ be a meromorphic function in the unit disc with the order $\rho\in(0,+\infty)$, then $f(z)$ has a Borel radius of the order $\rho+1$.*

Remark 3.2. Why the Borel radius is of the order $\rho+1$? In fact, from Lemma 2.2

$$\overline{n}(r,a)=O(S(r,f))=O\left(\frac{1}{1-r}T(r,f)\right)=O\left((\frac{1}{1-r})^{1+\rho+\varepsilon}\right)$$

in general comes into existence when $r\to1^-$. Hence for some ray to be a Borel radius for a function f, it means that the function f has a maximal number(relative to its growth) of a-points in an $\varepsilon-$neighborhood of that ray.

Theorem 3.3. *Let $f(z)$ be a K-quasimeromorphic mapping defined in $|z|<1$ and satisfy*

$$\overline{\lim_{r\to1^-}}\frac{S(r,f)}{\ln\frac{1}{1-r}}=+\infty,\tag{3.4}$$

then $f(z)$ can take any complex number infinite times, except for two possible exceptional values.

Proof. Otherwise, for any $\varphi\in[0,2\pi)$ and $r\in(0,1)$, there exists $\varepsilon_0>0$ and three distinct complex numbers $a_1,a_2,a_3\in\mathbf{C}_\infty$ such that

$$\sum_{j=1}^{3}\overline{n}(\Omega(\varphi-2\varepsilon_0,\varphi+2\varepsilon_0),r,a_j)\le\sum_{j=1}^{3}n(r,a_j)=O(1).$$

By (3.2) of Theorem 3.1, we have

$$S(\Omega(\varphi-\varepsilon_0,\varphi+\varepsilon_0),r)\le O\left(1+\frac{2\ln 2}{\ln\frac{1}{1-r}}\right)+O\left(\ln\frac{1}{1-r}\right).$$

Since φ is arbitrary, from the similar proof of Theorem 3.2, we have

$$S(r,f) \le C\left(1+\frac{2\ln 2}{\ln\frac{1}{1-r}}\right)+O\left(\ln\frac{1}{1-r}\right),$$

where C is a positive constant. This is in contradiction to the hypothesis (3.4). ∎

Corollary 3.2. *Let $f(z)$ be a meromorphic function defined in $|z|<1$ and satisfy (3.4), then $f(z)$ can take any complex number infinite times with at most two exceptional values.*

Theorem 3.4. *Let $f(z)$ be a K-quasimeromorphic mapping defined in $|z|<1$. If $f(z)$ satisfies (3.4) and*

$$\overline{\lim_{r\to 1^-}}\frac{S(\frac{3+r}{4},f)}{S(r,f)} < +\infty, \tag{3.5}$$

then $f(z)$ has a S radius (dealing with multiple values $l(\ge 3)$).

Proof. From the condition of (3.4), there exists an increasing sequence $\{r_n\}\uparrow 1$ $(n\to\infty)$ such that $\lim\limits_{n\to\infty} S(r_n,f)/(\ln\frac{1}{1-r_n}) = +\infty$. Using the finite covering theorem on $[0,2\pi)$, there must be some $\varphi_0\in[0,2\pi)$ such that for any $\varepsilon\in(0,\pi/4)$,

$$\overline{\lim_{n\to\infty}}\frac{S((\varphi_0-\varepsilon,\varphi_0+\varepsilon),r_n)}{S(r_n,f)} > 0.$$

Now we can predicatively say that the radius $\Delta(\varphi_0)=\{z:\arg z=\varphi_0,|z|<1\}$ is a S radius of $f(z)$ dealing with multiple values l. Otherwise, there are three distinct complex numbers $a_1,a_2,a_3\in\mathbf{C}_\infty$ and a positive δ such that

$$\overline{\lim_{n\to\infty}}\frac{\sum_{j=1}^{3}\overline{n}^{l)}(\Omega(\varphi_0-\delta,\varphi_0+\delta),r_n,a_j)}{S(r_n,f)} = 0.$$

By (3.1) of Theorem 3.1, when $q=3$, for any $0<\varepsilon<\delta$, we have

$$\begin{aligned}&(1-\frac{2}{l})S(\Omega(\varphi_0-\varepsilon,\varphi_0+\varepsilon),r_n)\\ &\le (1+\frac{2\ln 2}{\ln\frac{1}{1-r_n}})\sum_{j=1}^{3}\overline{n}^{l)}(\Omega(\varphi_0-\delta,\varphi_0+\delta),\frac{3+r_n}{4},a_j)+O(\ln\frac{1}{1-r_n}).\end{aligned}$$

Hence

$$\begin{aligned}&(1-\frac{2}{l})\overline{\lim_{n\to\infty}}\frac{S(\Omega(\varphi_0-\varepsilon,\varphi_0+\varepsilon),r_n)}{S(r_n,f)}\\ &\le \overline{\lim_{n\to\infty}}(1+\frac{2\ln 2}{\ln\frac{1}{1-r_n}})\frac{\sum\limits_{j=1}^{3}\overline{n}^{l)}(\Omega(\varphi_0-\delta,\varphi_0+\delta),\frac{3+r_n}{4},a_j)}{S(\frac{3+r_n}{4},f)}\frac{S(\frac{3+r_n}{4},f)}{S(r_n,f)}+\overline{\lim_{n\to\infty}}\frac{O(\ln\frac{1}{1-r_n})}{S(r_n,f)}.\end{aligned}$$

It follows from (3.4) and (3.5) that $1-\frac{2}{l}\le 0$, we get a contradiction. Hence the radius $\Delta(\varphi_0)$ is a S radius of $f(z)$ dealing with multiple values l. By the similar proof, the radius $\Delta(\varphi_0)$ is also a S radius. ∎

Corollary 3.3. *Let $f(z)$ be a meromorphic function defined in $|z|<1$ and satisfy the conditions of (3.4) and (3.5), then $f(z)$ has a S radius (dealing with multiple values $l(\ge 3)$).*

Theorem 3.5. *Let $f(z)$ be a K-quasimeromorphic mapping in the unit disc with order $\rho \in [0,+\infty)$ and of regular growth, then every S radius (dealing with multiple values) is a Borel radius of the order ρ.*

Proof. Let $\Delta(\varphi_0) = \{z : \arg z = \varphi_0, |z| < 1\}$ be a S radius dealing with multiple values for $f(z)$ in the unit disc, then for any $\varepsilon \in (0,\pi/2)$ and each a (except for two possible exceptional values), we have

$$\overline{\lim_{r\to 1^-}} \frac{\overline{n}^{l)}(\Omega(\varphi_0-\varepsilon,\varphi_0+\varepsilon),r,a)}{S(r,f)} > \delta > 0.$$

Then, there exist $\{r_n\}$ for a sufficiently large n we have

$$\overline{n}^{l)}(\Omega(\varphi_0-\varepsilon,\varphi_0+\varepsilon),r_n,a) > \frac{\delta}{2} S(r_n,f).$$

Since $f(z)$ is of regular growth, it follows that

$$\overline{\lim_{r\to 1^-}} \frac{\ln \overline{n}^{l)}(\Omega(\varphi_0-\varepsilon,\varphi_0+\varepsilon),r,a)}{\ln \frac{1}{1-r}} \geq \overline{\lim_{n\to\infty}} \frac{\ln \overline{n}^{l)}(\Omega(\varphi_0-\varepsilon,\varphi_0+\varepsilon),r_n,a)}{\ln \frac{1}{1-r_n}} \geq \rho$$

holds for all $a \in \mathbf{C}_\infty$, except for two possible exceptional values. Hence every S radius (dealing with multiple values) is a Borel radius of the order ρ. ∎

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