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Some Generalizations of Small Injective Modules

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Abstract. Let R be a ring. Let m and n be positive integers, a right R-module M is called (m,n)-*small injective*, if every right R-homomorphism from an n-generated submodule of J^m to M extends to one from R^m to M. A ring R is called right (m,n)-small injective if the right R module R_R is (m,n)-small injective. In this paper, we give some properties of (m,n)-small injective modules and right (m,n)-small injective rings.

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1. Introduction

Throughout the paper, R represents an associative ring with identity $1 \neq 0$ and all modules are unitary R-modules. We write M_R (resp. $_RM$) to denote that M is a right (resp. left) R-module. Unless otherwise mentioned, by a module we will mean a right R-module. We recall some concepts and notations which will be used in this paper. We denote the Jacobson radical of a ring (resp. module) R (resp. M) by J (resp. $\text{Rad}(M)$) and the injective hull of M by $E(M)$. If A is a submodule of M, we denote by $A \leq M$.

We write $N \leq^e M$, $N \ll M$ to indicate that N is an essential submodule, a small submodule of M, respectively. A module M is called uniform if $M \neq 0$ and every non-zero submodule of M is essential in M. A module M is called *to have finite uniform dimension*, if M does not contain an infinite direct sum of non-zero submodules. Recall that a module M is called *torsionless*, if given $0 \neq m \in M$, there exists $\alpha \in \text{Hom}(M,R)$ such that $\alpha(m) \neq 0$, equivalently if M can be embedded in a direct product of copies of R. A ring R is called *right Kasch* if every simple right R-module embeds in R_R. A ring R is called *semiregular* if R/J is von Neumann regular and idempotents can be lifted modulo J. Note that if R is semiregular, then for every finitely generated right ideal I of R, $R = H \oplus K$, where $H \leq I$ and $I \cap K \ll R$.

A right R-module M is called (m,n)-*injective*, if every R-homomorphism from an n-generated submodule of R^m to M extends to one from R^m to M. In [2], some characterizations (m,n)-injective modules are given. It is proved that R is right (m,n)-injective (i.e. the right R-module R_R is (m,n)-injective) if and only if every $_RN$ in an exact sequence

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${}_R R^m \to {}_R R^n \to {}_R N \to 0$ is torsionless. This result is similar to the Jain's result–a ring R is right FP-injective if and only if every finitely presented right R-module is torsionless (see [6]). A right R-module M_R is called *small injective*, if every homomorphism from a right small ideal to M_R can be extended to a R-homomorphism from R_R to M_R and a ring R is called right small injective, if R_R is small injective. Yousif and Zhou introduced small injective rings (modules) (see [11]). They proved that a semiperfect ring R with an essential right socle is right self-injective if and only if R is right small injective. From this, some characterizations of QF rings in terms of small injectivity were obtained. Later, in [8], Shen and Chen claimed that if R is semilocal, then R is right self-injective if and only if R is right small injective. Under the small injectivity condition, they gave some new characterizations of QF rings and PF rings. General background material can be found in [1, 3, 10].

In this paper, we use the notation $R^{m\times n}$ for the set of all $m\times n$ matrices over R. For $A\in R^{m\times n}$, A^T will denote the transpose of A. In general, for an R-module N, we write $N^{m\times n}$ for the set of all formal $m\times n$ matrices whose entries are elements of N. If $X\subseteq M^{l\times m}$, $S\subseteq R^{m\times n}$ and $Y\subseteq N^{n\times k}$, define

$$l_{M^{l\times m}}(S)=\{u\in M^{l\times m}|\ us=0,\forall s\in S\}$$
$$r_{N^{n\times k}}(S)=\{v\in N^{n\times k}|\ sv=0;\forall s\in S\}$$
$$r_{R^{m\times n}}(X)=\{r\in R^{m\times n}|\ xr=0,\forall x\in X\}$$
$$l_{R^{m\times n}}(Y)=\{r\in R^{m\times n}|\ rs=0,\forall s\in Y\}$$

We will write $N^n=N^{1\times n}$, $N_n=N^{n\times 1}$.

2. Main results

Definition 2.1. *A right R-module M is called (m,n)-small injective, if every R-homomorphism from an n-generated submodule of J^m (or J_m) to M can be extended to one from R^m(or R_m) to M. A ring R is called right (m,n)-small injective, if R_R is (m,n)-small injective.*

Example 2.1.

(i) $\mathbb{Z}$ is (m,n)-small injective as a $\mathbb{Z}$-module, but it is not (m,n)-injective.

(ii) Let $R=\left\{\left(\begin{smallmatrix} n & x\\ 0 & n\end{smallmatrix}\right)|\ n\in\mathbb{Z}, x\in\mathbb{Z}_2\right\}$ (see [11, Example 1.6]). Then R is a commutative ring and $J=S_r=\{\left(\begin{smallmatrix} 0 & x\\ 0 & 0\end{smallmatrix}\right)|\ x\in\mathbb{Z}_2\}$. Therefore R is small injective but not self-injective. Thus R is (m,n)-small injective for all m and n. But it is easy to see that R is not $(1,1)$-injective.

(iii) Let $R=F[x_1,x_2,...]$, where F is a field and x_i are commuting indeterminants satisfying the relations: $x_i^3=0$ for all i, $x_ix_j=0$ for all $i\neq j$, and $x_i^2=x_j^2$ for all i and j. Then R is a commutative, FP-injective, local ring. We have R is $(1,n)$-injective, but R is not a self-injective ring (see [7, Example 5.45]). Therefore R is (m,n)-injective for all m and n but R is not small injective.

We next consider some properties of (m,n)-small injective modules. By an argument similar to the one given in the proof of [2, Theorem 2.4], we have:

Proposition 2.1. *The following statements are equivalent for a right R-module M:*

(1) *M is (m,n)-small injective;*

(2) $l_{M^n}r_{R_n}(\alpha_1,\alpha_2,\ldots,\alpha_m)=M\alpha_1+M\alpha_2+\cdots+M\alpha_m$ *for any m-element subset* $\{\alpha_1, \alpha_2, \ldots,\alpha_m\}$ *of* J^n.

The following result is a slight modification of [2, Theorem 2.9].

Proposition 2.2. *The following statements are equivalent for a right R-module M:*

(1) *M is (m,n)-small injective;*
(2) *M is $(m,1)$-small injective and $l_{M^n}(I \cap K) = l_{M^n}(I) + l_{M^n}(K)$, where I and K are submodules of $(J_m)_R$ such that $I+K$ is n-generated;*
(3) *M is $(m,1)$-small injective and $l_{M^n}(I \cap K) = l_{M^n}(I) + l_{M^n}(K)$, where I and K are submodules of $(J_m)_R$ such that I is cyclic and K is $(n-1)$-generated (if $n=1$, then $K=0$).*

The next characterization of (m,n)-small injective module is motivated by [2, Theorem 2.15]. One can prove it by Proposition 2.1 and an similar argument in the proof of [2, Theorem 2.15].

Proposition 2.3. *The following statements are equivalent for a module M_R:*

(1) *M is (m,n)-small injective;*
(2) *If $m = (m_1, m_2, \ldots, m_n) \in M^n$ and $A \in J^{m\times n}$ satisfy $r_{R_n}(A) \le r_{R_n}(m)$, then $m = yA$ for some $y \in M^m$.*

Corollary 2.1. *A right R-module M is (m,n)-small injective if and only if for every $A \in J^{m\times n}$, $l_{M^n} r_{R_n}(A) = M^m A$.*

Corollary 2.2. *Let R be a right (m,n)-small injective ring and M be a left R-module. If $R^m \to J^n \to_R M \to 0$ is exact, then M is torsionless.*

Proof. Let $R^m \xrightarrow{f} J^n \to_R M \to 0$ be exact. There exists $A \in J^{m\times n}$ such that $f(z) = zA$ for all $z \in R^m$ and so $Imf = R^m A$. We claim that $J^n/R^m A$ is torsionless. In fact, let $0 \neq \bar{z} \in J^n/R^m A$, where $z = (z_1, z_2, ..., z_n) \in J^n$. By Proposition 2.3, $r_{R^n}(A) \nsubseteq r_{R^n}(z)$. Therefore there exists $a = (a_1, a_2, ..., a_n)^T \in J_n$ such that $Aa = 0$ but $za \neq 0$. Define $g : J^n/R^m A \to R$ such that $g(\bar{x}) = xa$ for every $x \in J^n$. Clearly, g is well-defined, and $g(\bar{z}) = za \neq 0$. So $M \simeq J^n/R^m A$ is torsionless. ∎

Motivated by [7, Lemma 5.1], we give the following characterization of right (m,n)-small injective ring.

Theorem 2.1. *The following statements are equivalent for a ring R:*

(1) *R is right (m,n)-small injective;*
(2) *$l_{R^n}(BR_n \cap r_{R_n}(A)) = l_{R^n}(B) + R^m A$ for all $A \in J^{m\times n}$ and $B \in R^{n\times n}$;*
(3) *If $r_{R_n}(A) \le r_{R_n}(B)$ with $A \in J^{m\times n}$ and $B \in R^{m\times n}$, then $R^m B \le R^m A$.*

Proof. $(1) \Rightarrow (2)$. Let $x \in l_{R^n}(BR_n \cap r_{R_n}(A))$. For all $y \in r_{R_n}(AB)$, $ABy = 0$ and so $By \in BR_n \cap r_{R_n}(A)$. It implies that $xBy = 0$ and so $y \in r_{R_n}(xB)$. Therefore $r_{R_n}(AB) \le r_{R_n}(xB)$ or $xB \in l_{R^n} r_{R_n}(xB) \le l_{R^n} r_{R_n}(AB)$. Since $A \in J^{m\times n}$, then $AB \in J^{m\times n}$. By Proposition 2.3, there exists $y \in R^m$ such that $xB = yAB$. Thus $x = (x - yA) + yA \in l_{R^n}(B) + R^m A$. From this, we have $l_{R^n}(BR_n \cap r_{R_n}(A)) = l_{R^n}(B) + R^m A$.

$(2) \Rightarrow (1)$. Let $B = I_n$ (identity matrix), then $l_{R^n} r_{R_n}(A) = R^m A$. Thus R is right (m,n)-small injective by Corollary 2.1.

$(1) \Rightarrow (3)$. Assume that $r_{R_n}(A) \le r_{R_n}(B)$. For each $x \in R^m$, we have $r_{R_n}(B) \le r_{R_n}(xB)$ and so $r_{R_n}(A) \le r_{R_n}(xB)$. It implies that $xB \in l_{R^n} r_{R_n}(xB) \le l_{R^n} r_{R_n}(A)$. By Corollary 2.1, $l_{R^n} r_{R_n}(A) = R^m A$ and hence $xB \in R^m A$ for all $x \in R^n$. Thus $R^m B \le R^m A$.

(3) $\Rightarrow$ (1). Let $A \in J^{m\times n}$. We have $R^m A \leq l_{R^n} r_{R_n}(A)$. For each $x \in l_{R^n} r_{R_n}(A)$, then $r_{R_n}(A) \leq r_{R_n}(x)$. Let $B = \binom{x}{0} \in R^{m\times n}$. Therefore $r_{R_n}(x) = r_{R_n}(B)$ and hence $r_{R_n}(A) \leq r_{R_n}(B)$. By (3), $R^m \binom{x}{0} = R^m B \leq R^m A$. It follows that $x \in R^m A$ and so $R^m A = l_{R^n} r_{R_n}(A)$. Thus R is right (m,n)-small injective. ∎

Proposition 2.4. *The following statements are equivalent for module M:*

1. *M is (m,n)-small injective.*
2. *For every n-generated submodule I of J^m and any $f \in \mathrm{Hom}(I,M)$, if (g,h) is the pushout of (f,i) in the following diagram (with i is the inclusion)*

$$\begin{array}{ccc} I & \xrightarrow{\;i\;} & R^m \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ M & \xrightarrow{\;h\;} & P \end{array}$$

there exists $\alpha \in \mathrm{Hom}(P,M)$ such that $\alpha h = id_M$.

Proof. Similar to [12, Theorem 2.5]. ∎

The dual module of P is denoted by $P^* = \mathrm{Hom}(P,R)$.

Proposition 2.5. *The following conditions are equivalent for a ring R:*

1. *R is right (m,n)-small injective;*
2. *If I is a m-generated and small submodule of a n-generated projective left R-module P, then $I = l_P r_{P^*}(I)$.*

Proof. (1) $\Rightarrow$ (2). Let $I = Ra_1 + Ra_2 + \cdots + Ra_m$ be a m-generated and small submodule of a n-generated projective left R-module P. Since ${}_RP$ is projective, there exist $x_1, x_2, ..., x_n \in P$ and $f_1, f_2, ..., f_n \in P^*$ such that $x = \sum\limits_{i=1}^{n} f(x)x_i$ for all $x \in P$. For each $x \in l_P r_{P^*}(I)$, we have $x = \sum\limits_{i=1}^{n} f_i(x)x_i$ and $a_j = \sum\limits_{i=1}^{n} f_i(a_j)x_i$ for each $j = 1,2,...,m$. Note that $a_i \in \mathrm{Rad}(P)$, then $f(a_i) \in J$ for each $i = 1,2,...,m$. Let $\alpha_i = (f_i(a_1), f_i(a_2), ..., f_i(a_m))$, then $\alpha_i \in J^m$, $\forall i = 1,2,...,n$. Let $\varphi : \alpha_1 R + \alpha_2 R + \cdots \alpha_n R \to R$ via $\varphi(\sum\limits_{i=1}^{n} \alpha_i r_i) = \sum\limits_{i=1}^{n} f_i(x) r_i$. It is easy to see that φ is a homomorphism. By the hypothesis, φ can be extended to R^m. There exists $u = (u_1, u_2, \ldots, u_m) \in R^m$ such that $f_i(x) = \varphi(\alpha_i) = u\alpha_i^T = \sum\limits_{j=1}^{n} u_j f_i(a_j)$ for each $i = 1,2,...,n$. Thus $x = \sum\limits_{i=1}^{n} f_i(x)x_i = \sum\limits_{j=1}^{m} u_j a_j \in I$. It implies that $I = l_P r_{P^*}(I)$.

(2) $\Rightarrow$ (1). For each $\alpha_1, \alpha_2, \ldots \alpha_m \in J^n$. Let $I = R\alpha_1 + R\alpha_2 + \cdots + R\alpha_m \leq R^n$. By (2), $I = l_{R^n} r_{(R^n)^*}(I)$. But $(R^n)^* = R_n$ and so $I = l_{R^n} r_{R_n}(I)$. Therefore $R\alpha_1 + R\alpha_2 + \cdots + R\alpha_m = l_{R^n} r_{R_n}(\alpha_1, \alpha_2, \ldots, \alpha_m)$. Thus R is right (m,n)-small injective by Proposition 2.1. ∎

Proposition 2.6. *The following statements are equivalent for a ring R:*

1. *Every n-generated right ideal of J^m is projective;*
2. *Every quotient module of a (m,n)-small injective module is (m,n)-small injective;*
3. *Every quotient module of a (m,n)-injective module is (m,n)-small injective;*
4. *Every quotient module of a small injective module is (m,n)-small injective;*

(5) *Every quotient module of an injective module is* (m,n)*-small injective.*

Proof. Similar to [9, Theorem 2.17]. ∎

Next, we consider a case when the class of (m,n)-small injective modules coincides with that of (m,n)-injective modules.

Theorem 2.2. *Let* R *be a semiregular ring. Then* M *is* (m,n)*-small injective if and only if* M *is* (m,n)*-injective.*

Proof. Let $f: K \longrightarrow M_R$ be a R-homomorphism, where K is a n-generated submodule of R^m. Since R is semiregular, then R^m is too. There exists a decomposition $R^m = H \oplus L$, where $H \leq K$ and $K \cap L \ll K$. Hence $R^m = K + L$, $K = H \oplus (K \cap L)$ and so $K \cap L$ is a n-generated submodule of J^m. Thus there exists a homomorphism $g: (R^m)_R \longrightarrow M$ such that $g(x) = f(x)$ for all $x \in K \cap L$. We construct a homomorphism $\varphi: (R^m)_R \longrightarrow M$ defined by $\varphi(r) = f(k) + g(l)$ for any $r = k + l,\ k \in K,\ l \in L$. Now we show that φ is well defined. Indeed, if $k_1 + l_1 = k_2 + l_2$, where $k_i \in K,\ l_i \in L,\ i = 1,2$, then $k_1 - k_2 = l_1 - l_2 \in K \cap L$. Hence $f(k_1 - k_2) = g(l_1 - l_2)$, which implies that $\varphi(k_1 + l_1) = \varphi(k_2 + l_2)$. Thus φ is a homomorphism and $\varphi_{|K} = f$. ∎

Corollary 2.3. *Let* R *be a semiregular ring. Then* R *is right* (m,n)*-small injective if and only if* R *is right* (m,n)*-injective.*

Note that a ring R is right (m,n)-small injective for all positive integers m and n if and only if R is right (J,R)-FP-injective in the sense of Yousif and Zhou [11]. We shall conclude this paper with some properties of such rings.

Theorem 2.3. *The following statements are equivalent for a ring* R*:*

(1) R *is right* (m,n)*-small injective for all* $m,n \in \mathbb{N}$.

(2) $R^{n\times n}$ *is right* $(1,1)$*-small injective for all* $n \in \mathbb{N}$.

Proof. The result follows by [11, Lemma 1.3]. ∎

A module M_R is FP-injective, if for every finitely generated submodule K of a free right R-module F, every homomorphism from K to M extends to one from F to M. In [7, Theorem 5.39], they proved that R is right FP-injective if and only if R is right (m,n)-injective for all $m,n \in \mathbb{N}$

From Theorem 2.2 and Theorem 2.3, we have:

Corollary 2.4. *Let* R *be a semiregular ring. Then* R *is right FP-injective if and only if* R *is right* (J,R)*-FP-injective.*

Proposition 2.7. *If* R *is right Kasch and right* (J,R)*-FP-injective, then* R *is left* (J,R)*-FP-injective.*

Proof. By Theorem 2.3, we claim that $R^{n\times n}$ is left $(1,1)$-small injective for all $m \in \mathbb{N}$. Since R is right Kasch, $R^{n\times n}$ is too. Let $T = R^{n\times n}$. For each $x \in J^{n\times n} = J(T)$. Let $y \in r_T l_T(xT)$ we need to show that $y \in xT$. Assume that $y \notin xT$. Let L be a maximal submodule of $xT + yT$ such that $xT \leq L$. Since R is right Kasch, there exists a T-monomorphism φ : $(xT + yT)/L \rightarrow T$. Note that $r_T l_T(J(T)) = J(T)$ and so $y \in J(T)$. Let $\psi: xT + yT \rightarrow T$ via $\psi(z) = \varphi(z + L)$ for all $z \in xT + yT$. By hypothesis, there is $u \in T$ such that $\psi = u\cdot$. Then $\psi(y) = uy \neq 0$ and $\psi(x) = ux = 0$ and so $u \in l_T(xT)$. But $y \in r_T l_T(xT)$ and hence $uy = 0$, this is a contradiction. Thus $r_T l_T(xT) = xT$. ∎

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