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Equivalence Classes of Linear Mappings on $\mathscr{B}(\mathscr{M})$

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Abstract. Let $\mathscr{M}$ be a Hilbert C^*-module over the C^*-algebra $\mathscr{A}$, $\mathscr{B}(\mathscr{M})$ the C^*-algebra of all adjointable operators on $\mathscr{M}$, $\mathscr{L}(\mathscr{B}(\mathscr{M}))$ the algebra of all linear operators on $\mathscr{B}(\mathscr{M})$. For a property $\mathscr{P}$ on $\mathscr{B}(\mathscr{M})$ and $\phi_1, \phi_2 \in \mathscr{L}(\mathscr{B}(\mathscr{M}))$ we say that $\phi_1 \sim_{\mathscr{P}} \phi_2$, whenever for every $T \in \mathscr{B}(\mathscr{M})$, $\phi_1(T)$ has property $\mathscr{P}$ if and only if $\phi_2(T)$ has this property. Each property $\mathscr{P}$ produces an equivalence relation on $\mathscr{L}(\mathscr{B}(\mathscr{M}))$. If $\mathscr{I}$ denotes the identity map on $\mathscr{B}(\mathscr{M})$ it is clear that $\phi \sim_{\mathscr{P}} \mathscr{I}$ means that ϕ preserves and reflects property $\mathscr{P}$. We are going to study the equivalence classes with respect to different properties such as being $\mathscr{A}$-Fredholm, semi-$\mathscr{A}$-Fredholm, compact and generalized invertible.

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1. Introduction

A (right) Hilbert C*-module over the C*-algebra $\mathscr{A}$ is a right $\mathscr{A}$-module $\mathscr{M}$ equipped with an $\mathscr{A}$-valued inner product $\langle , \rangle : \mathscr{M} \times \mathscr{M} \to \mathscr{A}$ satisfying the following properties:

(i) $\langle x,x \rangle \geqslant 0$ for all $x \in \mathscr{M}$ and $\langle x,x \rangle = 0$ iff $x = 0$;
(ii) $\langle x,y \rangle = \langle y,x \rangle^*$ for all $x,y \in \mathscr{M}$;
(iii) $\langle x,y \rangle$ is $\mathscr{A}$-linear in the second variable;
(iv) $\mathscr{M}$ is complete with respect to the norm $\| x \| = \| \langle x,x \rangle \|^{\frac{1}{2}}$.

It is clear that each Hilbert space is a Hilbert C^*-module over $\mathbb{C}$ and if $\mathscr{A}$ is a C^*-algebra then every closed right ideal $\mathscr{J}$ of $\mathscr{A}$ is a Hilbert C^*-module over $\mathscr{A}$ with respect to the inner product $\langle a,b \rangle = a^*b$ $(a,b \in \mathscr{J})$. The reader may find the details on relevant theory of Hilbert C^*-modules in [6, 8, 11].

Let $\mathscr{M}$ be a Hilbert C^*-module over the C^*-algebra $\mathscr{A}$. Throughout the paper $\mathscr{B}(\mathscr{M})$ denotes the C^*-algebra of all adjointable operators on $\mathscr{M}$, see [6, 8, 11]. For each $x,y \in \mathscr{M}$ we define the operator $\theta_{x,y}$ by $\theta_{x,y}(z) = x\langle y,z \rangle \quad (z \in \mathscr{M})$. Operators of this form are called elementary operators. Each finite linear combination of elementary operators is said to be a finite rank operator. The closed linear span of the set $\{\theta_{x,y} : x,y \in \mathscr{M}\}$ is denoted

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by $\mathscr{K}(\mathscr{M})$. The elements of $\mathscr{K}(\mathscr{M})$ are called compact operators. It is easy to see that $\mathscr{K}(\mathscr{M})$ is a closed ideal of $\mathscr{B}(\mathscr{M})$, see [8, Section 2.2]. Compact operators acting on Hilbert modules are not compact operators in the usual sense when they are considered as operators from one Banach space to another, but it must be emphasized that the two concepts coincide when $\mathscr{M}$ is assumed to be a Hilbert space. We recall that in the case of Hilbert C^*-modules (even infinite dimensional) it may happen that $\mathscr{K}(\mathscr{M}) = \mathscr{B}(\mathscr{M})$ [11, p. 244]. The Calkin algebra $\mathscr{C}(\mathscr{M})$ on $\mathscr{M}$ is the quotient C^*-algebra $\mathscr{B}(\mathscr{M})/\mathscr{K}(\mathscr{M})$. Let I denote the identity operator on $\mathscr{M}$. An operator $T \in \mathscr{B}(\mathscr{M})$ is said to be an $\mathscr{A}$-Fredholm operator if for some $S \in \mathscr{B}(\mathscr{M})$ both $I - ST$ and $I - TS$ are in $\mathscr{K}(\mathscr{M})$ [4]. An operator $T \in \mathscr{B}(\mathscr{M})$ is said to be a semi-$\mathscr{A}$-Fredholm operator if there is $S \in \mathscr{B}(\mathscr{M})$ such that either $I - ST$ or $I - TS$ is in $\mathscr{K}(\mathscr{M})$. Obviously each $\mathscr{A}$-Fredholm operator is semi-$\mathscr{A}$-Fredholm and if $\mathscr{K}(\mathscr{M}) = \mathscr{B}(\mathscr{M})$ then each operator in $\mathscr{B}(\mathscr{M})$ is $\mathscr{A}$-Fredholm. Now let $\pi : \mathscr{B}(\mathscr{M}) \to \mathscr{C}(\mathscr{M})$ be the canonical quotient map. Then it is clear that $T \in \mathscr{B}(\mathscr{M})$ is $\mathscr{A}$-Fredholm (resp. semi $\mathscr{A}$-Fredholm) if and only if $\pi(T)$ is invertible (resp. right or left invertible) in the Calkin algebra $\mathscr{C}(\mathscr{M})$. The sets of $\mathscr{A}$-Fredholm operators and semi-$\mathscr{A}$-Fredholm operators on $\mathscr{M}$ are denoted by $\mathscr{F}\mathscr{R}(\mathscr{M})$ and $\mathscr{S}\mathscr{F}(\mathscr{M})$, respectively.

Although in the case of Hilbert spaces this definition of a Fredholm or semi-Fredholm operator implies closeness of range [7, Section 1.4], but the reader must be careful that an $\mathscr{A}$-Fredholm operator or a semi-$\mathscr{A}$-Fredholm operator does not have a closed range in general. The reader may refer to [4, 8, 11] for more about $\mathscr{A}$-Fredholm operators. A brief discussion on semi-$\mathscr{A}$-Fredholm operators may be found in [1].

An operator $T \in \mathscr{B}(\mathscr{M})$ is called generalized invertible if there is $C \in \mathscr{B}(\mathscr{M})$ such that $TCT = T$. The set of generalized invertible operators in $\mathscr{B}(\mathscr{M})$ is denoted by $\mathscr{G}(\mathscr{M})$. It is proved in [1] that $T \in \mathscr{G}(\mathscr{M})$ if and only if $\mathrm{Im}(T)$ is closed, where $\mathrm{Im}(T)$ denotes the range of T. Let

$$\mathscr{R}(\mathscr{M}) = \{T \in \mathscr{B}(\mathscr{M}) : \forall A \in \mathscr{B}(\mathscr{M}) \backslash \mathscr{G}(\mathscr{M}), \quad \exists \lambda \in \mathbb{C} \quad \text{s.t.} \quad A + \lambda T \in \mathscr{G}(\mathscr{M}) \backslash \{0\}\}.$$

When $\mathscr{H}$ is a Hilbert space then $\mathscr{R}(\mathscr{H}) = \mathscr{S}\mathscr{F}(\mathscr{H})$ [5]. In the case of Hilbert C^*-modules we do not have this equality in general, but in certain cases we have a relation between these two sets. In [1] it is shown that for the standard Hilbert C^*-module $\mathscr{H}_{\mathscr{A}}$, where $\mathscr{A}$ is a unital C^*-algebra, every self-adjoint element of $\mathscr{R}(\mathscr{H}_{\mathscr{A}})$ is a semi-$\mathscr{A}$-Fredholm operator but the converse is not true in general.

A linear map $\phi : \mathscr{B}(\mathscr{M}) \to \mathscr{B}(\mathscr{M})$ is said to be surjective up to compact operators if for every $T \in \mathscr{B}(\mathscr{M})$ there exists $S \in \mathscr{B}(\mathscr{M})$ such that $T - \phi(S) \in \mathscr{K}(\mathscr{M})$ or equivalently $\mathscr{B}(\mathscr{M}) = \mathrm{Im}(\phi) + \mathscr{K}(\mathscr{M})$.

Let $\mathscr{L}(\mathscr{B}(\mathscr{M}))$ be the set of all linear maps on $\mathscr{B}(\mathscr{M})$ and $\mathscr{P}$ be a property on $\mathscr{B}(\mathscr{M})$. For $\phi_1, \phi_2 \in \mathscr{L}(\mathscr{B}(\mathscr{M}))$ we say that $\phi_1 \sim_{\mathscr{P}} \phi_2$, whenever for every $T \in \mathscr{B}(\mathscr{M})$, $\phi_1(T)$ has property $\mathscr{P}$ if and only if $\phi_2(T)$ has this property. Let $\mathscr{I}$ denote the identity element of $\mathscr{L}(\mathscr{B}(\mathscr{M}))$ and $\phi \in \mathscr{L}(\mathscr{B}(\mathscr{M}))$ then $\phi \sim_{\mathscr{P}} \mathscr{I}$ means that ϕ preserves and reflects property $\mathscr{P}$, that is for $T \in \mathscr{B}(\mathscr{M}))$, $\phi(T)$ has property $\mathscr{P}$ if and only if T has this property. It is easy to see that each property $\mathscr{P}$ produces an equivalence relation on $\mathscr{L}(\mathscr{B}(\mathscr{M}))$.

Throughout this paper we use the following notations for some specific properties:

(i) "g" is the property of "being generalized invertible";
(ii) "k" is the property of "being compact";
(iii) "fr" is the property of "being $\mathscr{A}$-Fredholm";
(iv) "sf" is the property of "being semi-$\mathscr{A}$-Fredholm";

(v) "r" is the property of "belonging to $\mathscr{R}(\mathscr{M})$".

When $\mathscr{H}$ is a Hilbert space, the linear mappings on $\mathscr{B}(\mathscr{H})$ which preserve and reflect Fredholm, semi-Fredholm or generalized invertible operators have been studied in [9, 10]. In [1] the case $\phi \sim_{sf} \mathscr{I}$ has been considered in the case of Hilbert C^*-modules and some characterizations are obtained in certain cases. Also, the relation between equivalence classes with respect to different properties on $\mathscr{B}(\mathscr{H})$ are discussed in [2]. In the next two sections we are going to prove some results concerning equivalence classes in $\mathscr{B}(\mathscr{M})$ with respect to the above properties.

2. Equivalence classes in the general case

The same reasoning as in [9, Lemma 2.2] and [10, Lemma 2.2] proves the following lemma in the case of Hilbert modules.

Lemma 2.1. *Let $K \in \mathscr{B}(\mathscr{M})$. Then the following are equivalent*

(i) *K is compact;*

(ii) *for every $B \in \mathscr{F}\mathscr{R}(\mathscr{M})$ we have $B+K \in \mathscr{F}\mathscr{R}(\mathscr{M})$;*

(iii) *for every $B \in \mathscr{S}\mathscr{F}(\mathscr{M})$ we have $B+K \in \mathscr{S}\mathscr{F}(\mathscr{M})$.*

Theorem 2.1. *If the linear maps $\phi_1, \phi_2 : \mathscr{B}(\mathscr{M}) \to \mathscr{B}(\mathscr{M})$ are surjective up to compact operators then*

(i) $\phi_1 \sim_{fr} \phi_2 \Rightarrow \phi_1 \sim_k \phi_2$*;*

(ii) $\phi_1 \sim_{sf} \phi_2 \Rightarrow \phi_1 \sim_k \phi_2$*.*

Proof. (i) Suppose that $\phi_1(T)$ is compact. Let S be an arbitrary $\mathscr{A}$-Fredholm operator. Since ϕ_2 is surjective up to compact operators, there exist $A \in \mathscr{B}(\mathscr{M})$ and $K \in \mathscr{K}(\mathscr{M})$ such that $\phi_2(A) = S+K$. By the hypothesis $\phi_1(A)$ is $\mathscr{A}$-Fredholm and since $\phi_1(T)$ is compact so by Lemma 2.1, $\phi_1(T+A) \in \mathscr{F}\mathscr{R}(\mathscr{M})$. Thus $\phi_2(T+A) = \phi_2(T)+S+K$ is also $\mathscr{A}$-Fredholm and Lemma 2.1 implies that $\phi_2(T)$ is compact.

(ii) Let $\phi_1(T)$ be compact and let S be an arbitrary semi-$\mathscr{A}$-Fredholm operator. There exist $A \in \mathscr{B}(\mathscr{M})$ and $K \in \mathscr{K}(\mathscr{M})$ such that $\phi_2(A) = S+K$. We have $\phi_1 \sim_{sf} \phi_2$ so $\phi_1(A) \in \mathscr{S}\mathscr{F}(\mathscr{M})$. Since $\phi_1(T)$ is compact, Lemma 2.1 implies that $\phi_1(T+A) \in \mathscr{S}\mathscr{F}(\mathscr{M})$. Therefore $\phi_2(T+A) = \phi_2(T)+S+K$ is also semi-$\mathscr{A}$-Fredholm and again by Lemma 2.1, $\phi_2(T)$ is compact. ∎

As a consequence if $\phi \in \mathscr{L}(\mathscr{B}(\mathscr{M}))$ is surjective up to compact operators and if it preserves and reflects $\mathscr{A}$-Fredholm or semi-$\mathscr{A}$-Fredholm operators then $\phi(\mathscr{K}(\mathscr{M})) = \mathscr{K}(\mathscr{M})$.

Let $\mathscr{F}(\mathscr{H})$ denote the ideal of all finite rank operators on a Hilbert space $\mathscr{H}$. In [2] it is proved that if $\phi_1, \phi_2 \in \mathscr{L}(\mathscr{B}(\mathscr{H}))$ are surjective up to finite rank operators, that is $\mathscr{B}(\mathscr{H}) = \mathrm{Im}(\phi_j) + \mathscr{F}(\mathscr{H})$, $j = 1,2$ then $\phi_1 \sim_g \phi_2$ implies that $\phi_1 \sim_{sf} \phi_2$. In the case of Hilbert C^*-modules we do not have any proof for the same result but we have the next theorem.

Theorem 2.2. *If $\phi_1, \phi_2 : \mathscr{B}(\mathscr{M}) \to \mathscr{B}(\mathscr{M})$ are surjective linear mappings then $\phi_1 \sim_g \phi_2 \Rightarrow \phi_1 \sim_r \phi_2$.*

Proof. Suppose that $\phi_1(T) \in \mathscr{R}(\mathscr{M})$ and consider an operator $A \in \mathscr{B}(\mathscr{M}) \backslash \mathscr{G}(\mathscr{M})$. Since ϕ_2 is surjective there exists an operator $B \in \mathscr{B}(\mathscr{M})$ such that $\phi_2(B) = A$. We have $\phi_1(T) \in \mathscr{R}(\mathscr{M})$, so there exists $\alpha \in \mathbb{C}$ such that $\alpha\phi_1(T) + \phi_1(B)$ is generalized invertible. Since

$\phi_1 \sim_g \phi_2$, $\alpha\phi_2(T)+\phi_2(B)$ is also in $\mathscr{G}(\mathscr{M})$. Note that $\alpha\phi_2(T)+\phi_2(B) \neq 0$, otherwise $\mathrm{Im}(\phi_2(T)) = \mathrm{Im}(\phi_2(B))$ which is not closed. Therefore $\phi_2(T) \in \mathscr{R}(\mathscr{M})$. ∎

3. Equivalence classes in the case of standard Hilbert C^*-modules

Before proving the next results we need some preliminaries. Let $\mathscr{A}$ be a C^*-algebra and consider

$$\mathscr{H}_{\mathscr{A}} := \{(x_k) \in \prod_1^{\infty} \mathscr{A} \mid \sum x_k{}^* x_k \quad \text{converges in norm in} \quad \mathscr{A}\}.$$

It is easy to see that $\mathscr{H}_{\mathscr{A}}$ with pointwise addition, module operation defined by

$$(x_k)a := (x_k a),$$

and the inner product

$$\langle (x_k),(y_k)\rangle := \sum x_k{}^* x_k,$$

for $(x_k) \in \mathscr{H}_{\mathscr{A}}, a \in \mathscr{A}$ becomes a Hilbert C^*-module which is called the standard Hilbert C^*-module over $\mathscr{A}$, see [11] for more details.

A Hilbert $\mathscr{A}$-module $\mathscr{N}$ is called finitely generated if there is a finite subset $\{x_1,....,x_n\}$ of $\mathscr{N}$ such that $\mathscr{N}$ equals the linear span (over $\mathbb{C}$, and $\mathscr{A}$) of this set. These modules sometimes are similar to finite dimensional Hilbert spaces.

We recall that a closed submodule $\mathscr{N}$ of a Hilbert C^*-module $\mathscr{M}$ is said to be complemented if $\mathscr{M} = \mathscr{N} \oplus \mathscr{N}^{\perp}$ where $\mathscr{N}^{\perp} = \{x \in \mathscr{M} \mid \langle x,y\rangle = 0 \quad \forall y \in \mathscr{N}\}$. The following theorem is [6, Theorem 3.2].

Theorem 3.1. *If $T \in \mathscr{B}(\mathscr{M})$ has a closed range then both* $\mathrm{Im}(T)$ *and* $\mathrm{Ker}(T)$ *are complemented. More precisely,* $\mathscr{M} = \mathrm{Ker}(T) \oplus (\mathrm{Ker}(T))^{\perp} = \mathrm{Im}(T) \oplus (\mathrm{Im}(T))^{\perp}$.

Remark 3.1. Suppose $\mathscr{M}$ is a Hilbert C^*-module over $\mathscr{A}$ and $A \in \mathscr{B}(\mathscr{M})$ has a closed range. Then by [11, Theorem 15.3.8], $|A|$ and A^* have closed ranges and A has a polar decomposition $A = V|A|$, where V is a partial isometry satisfying $\mathrm{Ker}(V) = \mathrm{Ker}(A)$, $\mathrm{Ker}(V^*) = \mathrm{Ker}(A^*) = \mathrm{Im}(A)^{\perp}$, $\mathrm{Im}(V) = \mathrm{Im}(A)$, $\mathrm{Im}(V^*) = \mathrm{Im}(A^*) = \mathrm{Ker}(A)^{\perp}$. Let $P = I - V^*V$, $Q = I - VV^*$, then P and Q are projections onto $\mathrm{Ker}(V) = \mathrm{Ker}(A)$ and $\mathrm{Ker}(V^*) = \mathrm{Im}(A)^{\perp}$, respectively. Furthermore, $|A|+P$ is invertible.

Let $L_A(T) = AT$ and $R_A(T) = TA$ for all $A,T \in \mathscr{B}(\mathscr{M})$. The next two results, in the case of standard Hilbert C^*-modules, provide some sufficient conditions under which $\phi_1 \sim_{sf} \phi_2$ and $\phi_1 \sim_{fr} \phi_2$.

Theorem 3.2. *Let $\mathscr{A}$ be a unital C^*-algebra and $\phi_1,\phi_2 \in \mathscr{L}(\mathscr{B}(\mathscr{H}_{\mathscr{A}}))$. Suppose that ϕ_1 is surjective, ϕ_2 is surjective up to compact operators and $\phi_1 \sim_g \phi_2$. If there exists $A \in \mathscr{B}(\mathscr{H}_{\mathscr{A}})$ such that $\phi_2 = L_A\phi_1$ or $\phi_2 = R_A\phi_1$ then A is an $\mathscr{A}$-Fredholm operator with a closed range, and moreover $\phi_1 \sim_{sf} \phi_2$ and $\phi_1 \sim_{fr} \phi_2$.*

Proof. Suppose that $\phi_2 = L_A\phi_1$. Let I denote the identity operator on $\mathscr{H}_{\mathscr{A}}$. Since ϕ_1 is surjective there exists $T \in \mathscr{B}(\mathscr{H}_{\mathscr{A}})$ such that $\phi_1(T) = I$. Thus $\phi_2(T) = A$ and the range of A is closed, because $\phi_1 \sim_g \phi_2$. Since ϕ_2 is surjective up to compact operators, $I = A\phi_1(T)+K$ for some $T \in \mathscr{B}(\mathscr{H}_{\mathscr{A}})$ and some compact operator K. Thus $\pi(A)$ has a right inverse in $\mathscr{C}(\mathscr{H}_{\mathscr{A}})$. We show that $\mathrm{Ker}(A)$ is finitely generated. If $\mathrm{Ker}(A)$ is not finitely generated then it is not algebraically finitely generated. Therefore $\mathscr{B}(\mathrm{Ker}(A))$ is an infinite dimensional C^*-algebra and hence it contains an element S which is not generalized invertible [3, Theorem

7]. By Theorem 3.1, $\mathscr{H}_{\mathscr{A}} = \mathrm{Ker}(A) \oplus \mathrm{Ker}(A)^{\perp}$ and we may extend S to $\tilde{S}$ on $\mathscr{H}_{\mathscr{A}}$ by $\tilde{S}(\mathrm{Ker}(A))^{\perp} = \{0\}$. Clearly $\mathrm{Im}(\tilde{S}) = \mathrm{Im}(S)$ is not closed. By surjectivity of ϕ_1 there exists $U \in \mathscr{B}(\mathscr{H}_{\mathscr{A}})$ such that $\phi_1(U) = \tilde{S}$. We have $\phi_2(U) = 0$ and this contradicts $\phi_1 \sim_g \phi_2$. It follows that $\mathrm{Ker}(A)$ is finitely generated. We show that $\pi(A)$ has a left inverse in $\mathscr{C}(\mathscr{H}_{\mathscr{A}})$. Consider P to be the projection onto $\mathrm{Ker}(A)$. Then P is a compact operator [11, Remark 15.4.3]. Let $A = V|A|$ be the polar decomposition of A. If $W = (|A| + P)^{-1}$ then $W = W^*$ and $(|A| + P)W = I$, so $I - |A|W$ is compact. Multiplying by V on the left implies that $V - AW$ and hence $V^* - WA^*$ are compact. This shows that $W^{-1}V^*V - A^*V$ and therefore $I - A^*VW$ are in $\mathscr{K}(\mathscr{H}_{\mathscr{A}})$. Now, since $\pi(A)$ has a left and a right inverse in $\mathscr{C}(\mathscr{H}_{\mathscr{A}})$, it is an invertible element of $\mathscr{C}(\mathscr{H}_{\mathscr{A}})$ and thus A is an $\mathscr{A}$-Fredholm operator. Finally, if $\phi_2(T) = \phi_1(T)A \quad (T \in \mathscr{B}(\mathscr{H}_{\mathscr{A}}))$, then the result follows by considering the same reasoning for $\phi_2(T)^* = A^*\phi_1(T)^*$ to show that A^* and hence A is an $\mathscr{A}$-Fredholm operator with a closed range. The last assertion is now obvious. ∎

Remark 3.2. We recall from [4] that a Hilbert C^*-module $\mathscr{M}$ over a C^*-algebra $\mathscr{A}$ is said to be $\mathscr{A}$-finite rank if the identity operator on $\mathscr{M}$ is compact. By [4, Proposition 2.3] if $\mathscr{A}$ is unital then each $\mathscr{A}$-finite rank module is finitely generated.

Theorem 3.3. *Let $\mathscr{A}$ be a unital C^*-algebra and $\phi_1, \phi_2 \in \mathscr{L}(\mathscr{B}(\mathscr{H}_{\mathscr{A}}))$ be surjective up to compact operators. If $\phi_1 \sim_k \phi_2$ and there exist $A, B \in \mathscr{G}(\mathscr{H}_{\mathscr{A}})$ such that $\phi_2 = L_A R_B \phi_1$ then A, B are $\mathscr{A}$-Fredholm operators and we have $\phi_1 \sim_{sf} \phi_2$, $\phi_1 \sim_{fr} \phi_2$.*

Proof. Let I be the identity element in $\mathscr{B}(\mathscr{H}_{\mathscr{A}})$ and $\phi_2(T) = L_A R_B \phi_1(T)$. Since ϕ_2 is surjective up to compact operators, $I = A\phi_1(T)B + K$ for some $T \in \mathscr{B}(\mathscr{H}_{\mathscr{A}})$ and $K \in \mathscr{K}(\mathscr{H}_{\mathscr{A}})$. Thus A has a right inverse in the Calkin algebra $\mathscr{C}(\mathscr{H}_{\mathscr{A}})$.

We show that $\mathrm{Ker}(A)$ is finitely generated. If not, by Remark 3.2 the projection P onto $\mathrm{Ker}(A)$ which is the identity operator on $\mathrm{Ker}(A)$ is not compact. Since ϕ_1 is surjective up to compact operators $\phi_1(S) = P + W$ for some $S \in \mathscr{B}(\mathscr{H}_{\mathscr{A}})$ and $W \in \mathscr{K}(\mathscr{H}_{\mathscr{A}})$. So $\phi_1 \nsim_k \phi_2$ because $\phi_1(S) = P + W$ but $\phi_2(S) = AWB + APB = AWB$. Thus $\mathrm{Ker}(A)$ is finitely generated. By the same argument as in the proof of Theorem 3.2, $\pi(A)$ has a left inverse in $\mathscr{C}(\mathscr{H}_{\mathscr{A}})$ and it follows that $\pi(A)$ is invertible in $\mathscr{C}(\mathscr{H}_{\mathscr{A}})$ and A is an $\mathscr{A}$-Fredholm operator. Now consider $\phi_2(T)^* = B^*\phi_1(T)^*A^*$. By the same argument, we have B^* and hence B is an $\mathscr{A}$-Fredholm operator. ∎

It is clear that each counter example in the case of Hilbert spaces works in the case of Hilbert C^*-modules, see [2] for some examples. Here we are going to show that some of the results which hold in the case of Hilbert spaces are not true in the case of Hilbert C^*-modules.

Example 3.1. Let $\mathscr{H}$ be a Hilbert space and $\lambda : \mathscr{B}(\mathscr{H}) \to \mathscr{F}(\mathscr{H})$ a linear mapping. Suppose that A, B are Fredholm operators on $\mathscr{H}$ and $\phi_1, \phi_2 \in \mathscr{L}(\mathscr{B}(\mathscr{H}))$ are related as $\phi_2 = L_A R_B \phi_1 + \lambda$, then it is easy to see that $\phi_1 \sim_g \phi_2$. We show that this is not true in the case of Hilbert C^*-modules.

Consider $\mathscr{A} = C[0,1]$ as the Hilbert $\mathscr{A}$-module over itself. Then $\mathscr{K}(\mathscr{M}) = \mathscr{B}(\mathscr{M})$ [11, p. 244], and each operator in $\mathscr{B}(\mathscr{M})$ is an $\mathscr{A}$-Fredholm operator. Now let $h(x) = \sqrt{x}$ $(x \in [0,1])$, and define $\theta_{h,h} : \mathscr{A} \to \mathscr{A}$ by $\theta_{h,h}(f) = h\langle h, f\rangle$. Then $\theta_{h,h}(f)(x) = xf(x)$ $(f \in \mathscr{A}, x \in [0,1])$, and $\overline{\mathrm{Im}(\theta_{h,h})} = \{f \in C[0,1] : f(0) = 0\}$. To see the latter equality note that if $f \in C[0,1]$ and $f(0) = 0$ then there exists a sequence $\{p_n\}$ of polynomials which converges uniformly to f on $[0,1]$ and $p_n(0) = 0$ for all $n \in \mathbb{N}$. Thus for each $n \in \mathbb{N}$ there exists a polynomial q_n such that $p_n(x) = xq_n(x) \quad (x \in [0,1])$ and this shows that $f \in \overline{\mathrm{Im}(\theta_{h,h})}$. The reverse inclusion is obvious. We have $\sqrt{x} \in \overline{\mathrm{Im}(\theta_{h,h})} \setminus \mathrm{Im}(\theta_{h,h})$ since it is not of the form $xf(x)$ for some continuous function f on $[0,1]$ and it follows that $\mathrm{Im}(\theta_{h,h})$ is not closed. Now take $\phi_1 = \mathscr{I}$ and $\phi_2 = L_{\theta_{h,h}}$ then $\phi_1(I) = I$, $\phi_2(I) = \theta_{h,h}$ and obviously $\phi_1 \nsim_g \phi_2$.

The above example also shows that $\phi_1 \sim_{fr} \phi_2$ does not imply that $\phi_1 \sim_g \phi_2$.

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