BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **35**(3) (2012), 633–649

A Generalized Mixed Quadratic-Quartic Functional Equation

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Abstract. In this paper, we determine the general solution of the functional equation $f(kx+y)+f(kx-y)=g(x+y)+g(x-y)+h(x)+\tilde{h}(y)$ for fixed integers k with $k\neq 0,\pm 1$ without assuming any regularity condition on the unknown functions $f,g,h,\tilde{h}$. The method used for solving these functional equations is elementary but exploits an important result due to Hosszú. The solution of this functional equation can also be determined in certain type of groups using two important results due to Székelyhidi. The results improve and extend some recent results.

2010 Mathematics Subject Classification: Primary: 39B22, 39B82

Keywords and phrases: Quadratic function, quartic function, difference operator, Fréchet functional equation, mixed quadratic-quartic functional equation.

1. Introduction and preliminaries

Rassias [12] (in 2001) introduced the cubic functional equation:

$$f(x+2y)-3f(x+y)+3f(x)-f(x-y)=6f(y) \tag{1.1}$$

and established the solution of the Ulam-Hyers stability problem for this cubic functional equation. Since the function $f(x)=x^3$ satisfies the functional equation (1.1), this equation is called cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function.

Jun and Kim [7] (in 2002) introduced the following cubic functional equation:

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.2}$$

and established the general solution and the generalized Ulam-Hyers stability for the functional equation (1.2). They proved that a function f between real vector spaces X and Y is

Communicated by Mohammad Sal Moslehian.
Received: March 31, 2010; *Revised:* June 3, 2010.

a solution of (1.2) if and only if there exits a unique function $C : X \times X \times X \to Y$ such that $f(x) = C(x,x,x)$ for all $x \in X$, and C is symmetric for each fixed one variable and is additive for fixed two variables. Recently, several further interesting discussions, modifications, extensions, and generalizations of the original problem of Ulam have been proposed (see, e.g., [5, 10, 13, 18–21] and the references therein). This cubic equation (1.2) can be generalized to

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 2f(2x) - 4f(x). \tag{1.3}$$

It is easy to see that the function $f(x) = ax^3 + bx$ is a solution of the functional equation (1.3), which is called a general mixed additive-cubic functional equation. Najati and Eskandani [12] established the general solution of the functional equation (1.3) and investigated the Ulam-Hyers stability of this equation in quasi-Banach spaces. In [20], we determined the general solution of a general mixed additive-cubic functional equation

$$f(kx+y) + f(kx-y) = kf(x+y) + kf(x-y) + 2f(kx) - 2kf(x), \tag{1.4}$$

for a fixed integer k with $k \neq 0, \pm 1$.

Rassias [15] (in 1999) introduced the first quartic functional equation:

$$f(x+2y) - 4f(x+y) + 6f(x) - 4f(x-y) + f(x-2y) = 24f(y) \tag{1.5}$$

and established the solution of the Ulam-Hyers stability problem for the quartic functional equation. Since the function $f(x) = x^4$ satisfies the functional equation (1.5), this equation is called quartic functional equation. Lee, Im and Hwang [9] determined the general solution of the quartic functional equation

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y). \tag{1.6}$$

In [5], Gordji, Abbaszadeh and Park established the general solution of a generalized quadratic and quartic type functional equation

$$\begin{aligned} f(kx+y) + f(kx-y) = {} & k^2 f(x+y) + k^2 f(x-y) \\ & + 2(f(kx) - k^2 f(x)) - 2(k^2-1)f(y), \end{aligned} \tag{1.7}$$

in quasi-Banach spaces for a fixed integer k with $k \neq 0, \pm 1$.

Let k be a fixed integer with $k \neq 0, \pm 1$, X and Y are real vector spaces. The functional equations (1.3)–(1.7) can be generalized to

$$f(kx+y) + f(kx-y) = g(x+y) + g(x-y) + h(x) + \tilde{h}(y). \tag{1.8}$$

for all $x, y \in X$, where $f, g, h, \tilde{h} : X \to Y$ are unknown functions to be determined. In this paper, we determine the general solution of the functional equation (1.8) and some other related functional equations. We will first solve these functional equations using an elementary technique (see [2, 17, 18, 22]) but without using any regularity condition on the unknown functions. The motivation for studying these functional equations came from the fact that recently polynomial equations have found applications in approximate checking, self-testing, and self-correcting of computer programs that compute polynomials. The interested reader should refer to [4, 16] and references therein.

Let X and Y be real vector spaces. A function $A : X \to Y$ is said to be additive if $A(x+y) = A(x) + A(y)$ for all $x, y \in X$. It is easy to see that $A(rx) = rA(x)$ for all $x \in X$ and all $r \in \mathbb{Q}$ (the set of rational numbers).

Let $n \in \mathbb{N}$ (the set of natural numbers). A function $A_n : X^n \to Y$ is called n-additive if it is additive in each of its variables. A function A_n is called symmetric if $A_n(x_1, x_2, \ldots, x_n) = A_n(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$ for every permutation $\{\pi(1), \pi(2), \ldots, \pi(n)\}$ of $\{1, 2, \ldots, n\}$. If $A_n(x_1, x_2, \ldots, x_n)$ is an n-additive symmetric map, then $A^n(x)$ will denote the diagonal $A_n(x, x, \ldots, x)$ for $x \in X$ and note that $A^n(rx) = r^n A^n(x)$ whenever $x \in X$ and $r \in \mathbb{Q}$. Such a function $A^n(x)$ will be called a monomial function of degree n (assuming $A^n \not\equiv 0$). Furthermore the resulting function after substitution $x_1 = x_2 = \cdots = x_l = x$ and $x_{l+1} = x_{l+2} = \cdots = x_n = y$ in $A_n(x_1, x_2, \ldots, x_n)$ will be denoted by $A^{l,n-l}(x, y)$.

A function $p : X \to Y$ is called a generalized polynomial (GP) function of degree $n \in \mathbb{N}$ provided that there exist $A^0(x) = A^0 \in Y$ and i-additive symmetric functions $A_i : X^i \to Y$(for $1 \le i \le n$) such that

$$p(x) = \sum_{i=0}^{n} A^i(x), \text{for all } x \in X$$

and $A^n \not\equiv 0$.

For $f : X \to Y$, let $\triangle_h$ be the difference operator defined as follows:

$$\triangle_h f(x) = f(x+h) - f(x)$$

for $h \in X$. Furthermore, let $\triangle_h^0 f(x) = f(x)$, $\triangle_h^1 f(x) = \triangle_h f(x)$ and $\triangle_h \circ \triangle_h^n f(x) = \triangle_h^{n+1} f(x)$ for all $n \in \mathbb{N}$ and all $h \in X$. Here $\triangle_h \circ \triangle_h^n$ denotes the composition of the operators $\triangle_h$ and $\triangle_h^n$. For any given $n \in \mathbb{N}$, the functional equation $\triangle_h^{n+1} f(x) = 0$ for all $x, h \in X$ is well studied. In explicit form the last functional equation can be written as

$$\triangle_h^{n+1} f(x) = \sum_{j=0}^{n+1} (-1)^{n+1-j} \begin{pmatrix} n+1 \\ j \end{pmatrix} f(x+jh) = 0.$$

The following theorem was proved by Mazur and Orlicz, and in greater generality by Djoković (see [3]).

Theorem 1.1. *Let X and Y be real vector spaces, $n \in \mathbb{N}$ and $f : X \to Y$, then the following are equivalent.*

(1) $\triangle_h^{n+1} f(x) = 0$ *for all $x, h \in X$.*

(2) $\triangle_{x_1, \ldots, x_{n+1}} f(x_0) = 0$ *for all $x_0, x_1, \ldots, x_{n+1} \in X$.*

(3) $f(x) = A^n(x) + A^{n-1}(x) + A^2(x) + A^1(x) + A^0(x)$ *for all $x \in X$, where $A^0(x) = A^0$ is an arbitrary element of Y and $A^i(x) (i = 1, 2, \ldots, n)$ is the diagonal of an i-additive symmetric function $A_i : X^i \to Y$.*

2. Solution of equation (1.8) on real vector spaces

In this section, we determine the general solution of the functional equations (1.7) and (1.8) and some other related equations without assuming any regularity condition on the unknown functions.

Theorem 2.1. *Let X and Y be real vector spaces. If the functions $f, g, h, \tilde{h} : X \to Y$ satisfy the functional equation*

$$f(kx+y) + f(kx-y) = g(x+y) + g(x-y) + h(x) + \tilde{h}(y) \tag{2.1}$$

for all $x, y \in X$, then f is a solution of the Fréchet functional equation

$$\triangle_{x_1, x_2, x_3, x_4, x_5} f(x_0) = 0$$

for all $x_0, x_1, x_2, x_3, x_4, x_5 \in X$.

Proof. Replacing $kx+y$ by x_0 and $kx-y$ by y_1 (that is, $x = (x_0/2k) + (y_1/2k)$ and $y = (x_0/2) - (y_1/2)$) in (2.1), respectively, we get

$$\begin{aligned} f(x_0) + f(y_1) = {} & g\left(\frac{k+1}{2k}x_0 - \frac{k-1}{2k}y_1\right) + g\left(\frac{1-k}{2k}x_0 + \frac{k+1}{2k}y_1\right) \\ & + h\left(\frac{1}{2k}x_0 + \frac{1}{2k}y_1\right) + \tilde{h}\left(\frac{1}{2}x_0 - \frac{1}{2}y_1\right). \end{aligned} \tag{2.2}$$

Replacing x_0 by $x_0 + x_1$ in (2.2), we have

$$\begin{aligned} f(x_0+x_1) + f(y_1) = {} & g\left(\frac{k+1}{2k}(x_0+x_1) - \frac{k-1}{2k}y_1\right) \\ & + g\left(\frac{1-k}{2k}(x_0+x_1) + \frac{k+1}{2k}y_1\right) \\ & + h\left(\frac{1}{2k}(x_0+x_1) + \frac{1}{2k}y_1\right) + \tilde{h}\left(\frac{1}{2}(x_0+x_1) - \frac{1}{2}y_1\right). \end{aligned} \tag{2.3}$$

Subtracting (2.2) from (2.3), we get

$$\begin{aligned} f(x_0+x_1) - f(x_0) = {} & g\left(\frac{k+1}{2k}(x_0+x_1) - \frac{k-1}{2k}y_1\right) \\ & + g\left(\frac{1-k}{2k}(x_0+x_1) + \frac{k+1}{2k}y_1\right) \\ & - g\left(\frac{k+1}{2k}x_0 - \frac{k-1}{2k}y_1\right) - g\left(\frac{1-k}{2k}x_0 + \frac{k+1}{2k}y_1\right) \\ & + h\left(\frac{1}{2k}(x_0+x_1) + \frac{1}{2k}y_1\right) - h\left(\frac{1}{2k}x_0 + \frac{1}{2k}y_1\right) \\ & + \tilde{h}\left(\frac{1}{2}(x_0+x_1) - \frac{1}{2}y_1\right) \\ & - \tilde{h}\left(\frac{1}{2}x_0 - \frac{1}{2}y_1\right). \end{aligned} \tag{2.4}$$

Letting $y_2 = \frac{1}{2k}x_0 + \frac{1}{2k}y_1$ (that is, $y_1 = 2ky_2 - x_0$) in (2.4), we have

$$\begin{aligned} f(x_0+x_1) - f(x_0) = {} & g\left(x_0 + \frac{k+1}{2k}x_1 - (k-1)y_2\right) \\ & + g\left(-x_0 + \frac{1-k}{2k}x_1 + (k+1)y_2\right) - g(x_0 - (k-1)y_2) \\ & - g(-x_0 + (k+1)y_2) + h\left(\frac{1}{2k}x_1 + y_2\right) \\ & - h(y_2) + \tilde{h}\left(x_0 + \frac{1}{2}x_1 - ky_2\right) - \tilde{h}(x_0 - ky_2). \end{aligned} \tag{2.5}$$

Replacing x_0 by x_0+x_2 in (2.5), we get

$$
\begin{aligned}
f(x_0+x_1+x_2)-f(x_0+x_2) = {} & g\left(x_0+x_2+\frac{k+1}{2k}x_1-(k-1)y_2\right) \\
& + g\left(-x_0-x_2+\frac{1-k}{2k}x_1+(k+1)y_2\right) \\
& - g(x_0+x_2-(k-1)y_2) \\
& - g(-x_0-x_2+(k+1)y_2) \\
& + h\left(\frac{1}{2k}x_1+y_2\right)-h(y_2) \\
& + \tilde{h}\left(x_0+x_2+\frac{1}{2}x_1-ky_2\right) \\
& - \tilde{h}(x_0+x_2-ky_2).
\end{aligned} \tag{2.6}
$$

Subtracting (2.5) from (2.6), we get

$$
\begin{aligned}
& f(x_0+x_1+x_2)-f(x_0+x_1)-f(x_0+x_2)+f(x_0) \\
& = g\left(x_0+x_2+\frac{k+1}{2k}x_1-(k-1)y_2\right)+g\left(-x_0-x_2+\frac{1-k}{2k}x_1+(k+1)y_2\right) \\
& \quad - g(x_0+x_2-(k-1)y_2)-g(-x_0-x_2+(k+1)y_2)+g(x_0-(k-1)y_2) \\
& \quad - g\left(x_0+\frac{k+1}{2k}x_1-(k-1)y_2\right)-g\left(-x_0+\frac{1-k}{2k}x_1+(k+1)y_2\right) \\
& \quad - g(-x_0+(k+1)y_2)+\tilde{h}\left(x_0+x_2+\frac{1}{2}x_1-ky_2\right) \\
& \quad - \tilde{h}(x_0+x_2-ky_2)-\tilde{h}\left(x_0+\frac{1}{2}x_1-ky_2\right)+\tilde{h}(x_0-ky_2).
\end{aligned} \tag{2.7}
$$

Letting $y_3=x_0-ky_2$ (that is, $y_2=(x_0-y_3)/k$ in (2.7), we have

$$
\begin{aligned}
& f(x_0+x_1+x_2)-f(x_0+x_1)-f(x_0+x_2)+f(x_0) \\
& = g\left(\frac{1}{k}x_0+\frac{k+1}{2k}x_1+x_2+\frac{k-1}{k}y_3\right)+g\left(\frac{1}{k}x_0+\frac{1-k}{2k}x_1-x_2-\frac{k+1}{k}y_3\right) \\
& \quad - g\left(\frac{1}{k}x_0+x_2+\frac{k-1}{k}y_3\right)-g\left(\frac{1}{k}x_0+\frac{k+1}{2k}x_1+\frac{k-1}{k}y_3\right) \\
& \quad - g\left(\frac{1}{k}x_0+\frac{1-k}{2k}x_1-\frac{k+1}{k}y_3\right)+g\left(\frac{1}{k}x_0+\frac{k-1}{k}y_3\right) \\
& \quad - g\left(\frac{1}{k}x_0-x_2-\frac{k+1}{k}y_3\right)+g\left(\frac{1}{k}x_0-\frac{k+1}{k}y_3\right) \\
& \quad + \tilde{h}\left(\frac{1}{2}x_1+x_2+y_3\right)-\tilde{h}(x_2+y_3)-\tilde{h}\left(\frac{1}{2}x_1+y_3\right)+\tilde{h}(y_3).
\end{aligned} \tag{2.8}
$$

Again replacing x_0 by x_0+x_3 in (2.8) and subtracting (2.8) from the resulting expression, we get

(2.9)
$$\begin{aligned}
&f(x_0+x_1+x_2+x_3)-f(x_0+x_1+x_2)-f(x_0+x_1+x_3)-f(x_0+x_2+x_3)+f(x_0+x_3)\\
&\quad+f(x_0+x_1)+f(x_0+x_2)-f(x_0)=g\left(\frac{1}{k}x_0+\frac{k+1}{2k}x_1+x_2+\frac{1}{k}x_3+\frac{k-1}{k}y_3\right)\\
&\qquad+g\left(\frac{1}{k}x_0+\frac{1-k}{2k}x_1-x_2-\frac{k+1}{k}y_3\right)-g\left(\frac{1}{k}x_0+x_2+\frac{1}{k}x_3+\frac{k-1}{k}y_3\right)\\
&\qquad-g\left(\frac{1}{k}x_0+\frac{k+1}{2k}x_1+\frac{1}{k}x_3+\frac{k-1}{k}y_3\right)-g\left(\frac{1}{k}x_0+\frac{1-k}{2k}x_1+\frac{1}{k}x_3-\frac{k+1}{k}y_3\right)\\
&\qquad+g\left(\frac{1}{k}x_0+\frac{1}{k}x_3+\frac{k-1}{k}y_3\right)-g\left(\frac{1}{k}x_0-x_2+\frac{1}{k}x_3-\frac{k+1}{k}y_3\right)\\
&\qquad+g\left(\frac{1}{k}x_0+\frac{1}{k}x_3-\frac{k+1}{k}y_3\right)-g\left(\frac{1}{k}x_0+\frac{k+1}{2k}x_1+x_2+\frac{k-1}{k}y_3\right)\\
&\qquad-g\left(\frac{1}{k}x_0+\frac{1-k}{2k}x_1-x_2-\frac{k+1}{k}y_3\right)+g\left(\frac{1}{k}x_0+x_2+\frac{k-1}{k}y_3\right)\\
&\qquad+g\left(\frac{1}{k}x_0+\frac{k+1}{2k}x_1+\frac{k-1}{k}y_3\right)+g\left(\frac{1}{k}x_0+\frac{1-k}{2k}x_1-\frac{k+1}{k}y_3\right)\\
&\qquad-g\left(\frac{1}{k}x_0+\frac{k-1}{k}y_3\right)+g\left(\frac{1}{k}x_0-x_2-\frac{k+1}{k}y_3\right)-g\left(\frac{1}{k}x_0-\frac{k+1}{k}y_3\right).
\end{aligned}$$

Putting $y_4=(x_0/k)+(k-1)/ky_3$ (that is, $y_3=ky_4/(k-1)-x_0/(k-1)$) in (2.9), we get

(2.10)
$$\begin{aligned}
&f(x_0+x_1+x_2+x_3)-f(x_0+x_1+x_2)-f(x_0+x_1+x_3)-f(x_0+x_2+x_3)\\
&\quad+f(x_0+x_3)+f(x_0+x_1)+f(x_0+x_2)-f(x_0)=g\left(\frac{k+1}{2k}x_1+x_2+\frac{1}{k}x_3+y_4\right)\\
&\qquad+g\left(\frac{2}{k-1}x_0+\frac{1-k}{2k}x_1-x_2-\frac{k+1}{k-1}y_4\right)-g\left(x_2+\frac{1}{k}x_3+y_4\right)\\
&\qquad-g\left(\frac{k+1}{2k}x_1+\frac{1}{k}x_3+y_4\right)-g\left(\frac{2}{k-1}x_0+\frac{1-k}{2k}x_1+\frac{1}{k}x_3-\frac{k+1}{k-1}y_4\right)\\
&\qquad+g\left(\frac{1}{k}x_3+y_4\right)-g\left(\frac{2}{k-1}x_0-x_2+\frac{1}{k}x_3-\frac{k+1}{k-1}y_4\right)\\
&\qquad+g\left(\frac{2}{k-1}x_0+\frac{1}{k}x_3-\frac{k+1}{k-1}y_3\right)-g\left(\frac{k+1}{2k}x_1+x_2+y_4\right)\\
&\qquad-g\left(\frac{2}{k-1}x_0+\frac{1-k}{2k}x_1-x_2-\frac{k+1}{k-1}y_4\right)+g(x_2+y_4)\\
&\qquad+g\left(\frac{k+1}{2k}x_1+y_4\right)+g\left(\frac{2}{k-1}x_0+\frac{1-k}{2k}x_1-\frac{k+1}{k-1}y_4\right)-g(y_4)\\
&\qquad+g\left(\frac{2}{k-1}x_0-x_2-\frac{k+1}{k-1}y_4\right)-g\left(\frac{2}{k-1}x_0-\frac{k+1}{k-1}y_4\right).
\end{aligned}$$

Replacing x_0 by x_0+x_4 in (2.10) to get

$$\begin{aligned}
& f(x_0+x_1+x_2+x_3+x_4) - f(x_0+x_1+x_2+x_4) \\
& \quad - f(x_0+x_1+x_3+x_4) - f(x_0+x_2+x_3+x_4) \\
& \quad + f(x_0+x_3+x_4) + f(x_0+x_1+x_4) \\
& \quad + f(x_0+x_2+x_4) - f(x_0+x_4) \\
& \quad = g\left(\frac{k+1}{2k}x_1 + x_2 + \frac{1}{k}x_3 + y_4\right) \\
& \qquad + g\left(\frac{2}{k-1}x_0 + \frac{1-k}{2k}x_1 - x_2 + \frac{2}{k-1}x_4 - \frac{k+1}{k-1}y_4\right) \\
& \qquad - g(x_2 + \frac{1}{k}x_3 + y_4) - g\left(\frac{k+1}{2k}x_1 + \frac{1}{k}x_3 + y_4\right) \\
& \qquad - g\left(\frac{2}{k-1}x_0 + \frac{1-k}{2k}x_1 + \frac{1}{k}x_3 + \frac{2}{k-1}x_4 - \frac{k+1}{k-1}y_4\right) \\
& \qquad + g\left(\frac{1}{k}x_3 + y_4\right) - g\left(\frac{2}{k-1}x_0 - x_2 + \frac{1}{k}x_3 + \frac{2}{k-1}x_4 - \frac{k+1}{k-1}y_4\right) \\
& \qquad + g\left(\frac{2}{k-1}x_0 + \frac{1}{k}x_3 + \frac{2}{k-1}x_4 - \frac{k+1}{k-1}y_4\right) - g\left(\frac{k+1}{2k}x_1 + x_2 + y_4\right) \\
& \qquad - g\left(\frac{2}{k-1}x_0 + \frac{1-k}{2k}x_1 - x_2 + \frac{2}{k-1}x_4 - \frac{k+1}{k-1}y_4\right) + g(x_2+y_4) \\
& \qquad + g\left(\frac{k+1}{2k}x_1 + y_4\right) + g\left(\frac{2}{k-1}x_0 + \frac{1-k}{2k}x_1 + \frac{2}{k-1}x_4 - \frac{k+1}{k-1}y_4\right) \\
& \qquad - g(y_4) + g\left(\frac{2}{k-1}x_0 - x_2 + \frac{2}{k-1}x_4 - \frac{k+1}{k-1}y_4\right) \\
& \qquad - g\left(\frac{2}{k-1}x_0 + \frac{2}{k-1}x_4 - \frac{k+1}{k-1}y_4\right).
\end{aligned} \tag{2.11}$$

Subtract (2.10) from (2.11), we get

(2.12)
$$\begin{aligned}
&f(x_0+x_1+x_2+x_3+x_4)-f(x_0+x_1+x_2+x_3)-f(x_0+x_1+x_2+x_4)\\
&\quad -f(x_0+x_1+x_3+x_4)-f(x_0+x_2+x_3+x_4)+f(x_0+x_1+x_2)\\
&\quad +f(x_0+x_1+x_3)+f(x_0+x_2+x_3)+f(x_0+x_3+x_4)\\
&\quad +f(x_0+x_1+x_4)+f(x_0+x_2+x_4)-f(x_0+x_1)\\
&\quad -f(x_0+x_2)-f(x_0+x_3)-f(x_0+x_4)+f(x_0)\\
&\quad = g\left(\frac{2}{k-1}x_0+\frac{1-k}{2k}x_1-x_2+\frac{2}{k-1}x_4-\frac{k+1}{k-1}y_4\right)\\
&\qquad - g\left(\frac{2}{k-1}x_0+\frac{1-k}{2k}x_1+\frac{1}{k}x_3+\frac{2}{k-1}x_4-\frac{k+1}{k-1}y_4\right)\\
&\qquad - g\left(\frac{2}{k-1}x_0-x_2+\frac{1}{k}x_3+\frac{2}{k-1}x_4-\frac{k+1}{k-1}y_4\right)\\
&\qquad + g\left(\frac{2}{k-1}x_0+\frac{1}{k}x_3+\frac{2}{k-1}x_4-\frac{k+1}{k-1}y_4\right)\\
&\qquad - g\left(\frac{2}{k-1}x_0+\frac{1-k}{2k}x_1-x_2+\frac{2}{k-1}x_4-\frac{k+1}{k-1}y_4\right)\\
&\qquad + g\left(\frac{2}{k-1}x_0+\frac{1-k}{2k}x_1+\frac{2}{k-1}x_4-\frac{k+1}{k-1}y_4\right)\\
&\qquad + g\left(\frac{2}{k-1}x_0-x_2+\frac{2}{k-1}x_4-\frac{k+1}{k-1}y_4\right)-g\left(\frac{2}{k-1}x_0+\frac{2}{k-1}x_4-\frac{k+1}{k-1}y_4\right)\\
&\qquad - g\left(\frac{2}{k-1}x_0+\frac{1-k}{2k}x_1-x_2-\frac{k+1}{k-1}y_4\right)+g\left(\frac{2}{k-1}x_0+\frac{1-k}{2k}x_1+\frac{1}{k}x_3-\frac{k+1}{k-1}y_4\right)\\
&\qquad + g\left(\frac{2}{k-1}x_0-x_2+\frac{1}{k}x_3-\frac{k+1}{k-1}y_4\right)-g\left(\frac{2}{k-1}x_0+\frac{1}{k}x_3-\frac{k+1}{k-1}y_4\right)\\
&\qquad + g\left(\frac{2}{k-1}x_0+\frac{1-k}{2k}x_1-x_2-\frac{k+1}{k-1}y_4\right)+g\left(\frac{2}{k-1}x_0+\frac{1-k}{2k}x_1-\frac{k+1}{k-1}y_4\right)\\
&\qquad - g\left(\frac{2}{k-1}x_0-x_2-\frac{k+1}{k-1}y_4\right)+g\left(\frac{2}{k-1}x_0-\frac{k+1}{k-1}y_4\right).
\end{aligned}$$

Setting $y_5 = (2/(k-1))x_0 - ((k+1)/(k-1))y_4$ (that is, $y_4 = (2/(k+1))x_0 - ((k-1)/(k+1))y_5$) in (2.12), we have

(2.13)
$$\begin{aligned}
&f(x_0+x_1+x_2+x_3+x_4) - f(x_0+x_1+x_2+x_3) - f(x_0+x_1+x_2+x_4)\\
&\quad - f(x_0+x_1+x_3+x_4) - f(x_0+x_2+x_3+x_4) + f(x_0+x_1+x_2) + f(x_0+x_1+x_3)\\
&\quad + f(x_0+x_2+x_3) + f(x_0+x_3+x_4) + f(x_0+x_1+x_4) + f(x_0+x_2+x_4) - f(x_0+x_1)\\
&\quad - f(x_0+x_2) - f(x_0+x_3) - f(x_0+x_4) + f(x_0) = g\left(\frac{1-k}{2k}x_1 - x_2 + \frac{2}{k-1}x_4 + y_5\right)\\
&\qquad - g\left(\frac{1-k}{2k}x_1 + \frac{1}{k}x_3 + \frac{2}{k-1}x_4 + y_5\right) - g\left(-x_2 + \frac{1}{k}x_3 + \frac{2}{k-1}x_4 + y_5\right)\\
&\qquad + g\left(\frac{1}{k}x_3 + \frac{2}{k-1}x_4 + y_5\right) - g\left(\frac{1-k}{2k}x_1 - x_2 + \frac{2}{k-1}x_4 + y_5\right)\\
&\qquad + g\left(\frac{1-k}{2k}x_1 + \frac{2}{k-1}x_4 + y_5\right) + g\left(-x_2 + \frac{2}{k-1}x_4 + y_5\right) - g\left(\frac{2}{k-1}x_4 + y_5\right)\\
&\qquad - g\left(\frac{1-k}{2k}x_1 - x_2 + y_5\right) + g\left(\frac{1-k}{2k}x_1 + \frac{1}{k}x_3 + y_5\right) + g\left(-x_2 + \frac{1}{k}x_3 + y_5\right)\\
&\qquad - g\left(\frac{1}{k}x_3 + y_5\right) + g\left(\frac{1-k}{2k}x_1 - x_2 + y_5\right) + g\left(\frac{1-k}{2k}x_1 + y_5\right) - g(-x_2+y_5) + g(y_5).
\end{aligned}$$

Replacing x_0 by $x_0 + x_5$ in (2.13) to get

(2.14)
$$\begin{aligned}
&f(x_0+x_1+x_2+x_3+x_4+x_5) - f(x_0+x_1+x_2+x_3+x_5) - f(x_0+x_1+x_2+x_4+x_5)\\
&\quad - f(x_0+x_1+x_3+x_4+x_5) - f(x_0+x_2+x_3+x_4+x_5) + f(x_0+x_1+x_2+x_5)\\
&\quad + f(x_0+x_1+x_3+x_5) + f(x_0+x_2+x_3+x_5) + f(x_0+x_3+x_4+x_5)\\
&\quad + f(x_0+x_1+x_4+x_5) + f(x_0+x_2+x_4+x_5) - f(x_0+x_1+x_5) - f(x_0+x_2+x_5)\\
&\quad - f(x_0+x_3+x_5) - f(x_0+x_4+x_5) + f(x_0+x_5) = g\left(\frac{1-k}{2k}x_1 - x_2 + \frac{2}{k-1}x_4 + y_5\right)\\
&\qquad - g\left(\frac{1-k}{2k}x_1 + \frac{1}{k}x_3 + \frac{2}{k-1}x_4 + y_5\right) - g\left(-x_2 + \frac{1}{k}x_3 + \frac{2}{k-1}x_4 + y_5\right)\\
&\qquad + g\left(\frac{1}{k}x_3 + \frac{2}{k-1}x_4 + y_5\right) - g\left(\frac{1-k}{2k}x_1 - x_2 + \frac{2}{k-1}x_4 + y_5\right)\\
&\qquad + g\left(\frac{1-k}{2k}x_1 + \frac{2}{k-1}x_4 + y_5\right) + g\left(-x_2 + \frac{2}{k-1}x_4 + y_5\right) - g\left(\frac{2}{k-1}x_4 + y_5\right)\\
&\qquad - g\left(\frac{1-k}{2k}x_1 - x_2 + y_5\right) + g\left(\frac{1-k}{2k}x_1 + \frac{1}{k}x_3 + y_5\right) + g\left(-x_2 + \frac{1}{k}x_3 + y_5\right)\\
&\qquad - g\left(\frac{1}{k}x_3 + y_5\right) + g\left(\frac{1-k}{2k}x_1 - x_2 + y_5\right) + g\left(\frac{1-k}{2k}x_1 + y_5\right) - g(-x_2+y_5) + g(y_5).
\end{aligned}$$

Subtract (2.13) from (2.14), we get

$$\begin{aligned}
&f(x_0+x_1+x_2+x_3+x_4+x_5) - f(x_0+x_1+x_2+x_3+x_4)\\
&\quad - f(x_0+x_1+x_2+x_3+x_5) - f(x_0+x_1+x_2+x_4+x_5)\\
&\quad - f(x_0+x_1+x_3+x_4+x_5) - f(x_0+x_2+x_3+x_4+x_5)\\
&\quad + f(x_0+x_1+x_2+x_3) + f(x_0+x_1+x_2+x_4) + f(x_0+x_1+x_3+x_4)\\
&\quad + f(x_0+x_2+x_3+x_4) + f(x_0+x_1+x_2+x_5) + f(x_0+x_1+x_3+x_5)\\
&\quad + f(x_0+x_2+x_3+x_5) + f(x_0+x_3+x_4+x_5) + f(x_0+x_1+x_4+x_5)\\
&\quad + f(x_0+x_2+x_4+x_5) - f(x_0+x_1+x_5) - f(x_0+x_2+x_5) - f(x_0+x_3+x_5)\\
&\quad - f(x_0+x_4+x_5) - f(x_0+x_1+x_2) - f(x_0+x_1+x_3) - f(x_0+x_2+x_3)\\
&\quad - f(x_0+x_3+x_4) - f(x_0+x_1+x_4) - f(x_0+x_2+x_4) + f(x_0+x_5)\\
&\quad + f(x_0+x_1) + f(x_0+x_2) + f(x_0+x_3) + f(x_0+x_4) - f(x_0) = 0
\end{aligned}$$

which is $\triangle_{x_1,x_2,x_3,x_4,x_5} f(x_0) = 0$ for all $x_0,x_1,x_2,x_3,x_4,x_5 \in X$. ∎

As an application of Theorem 2.1, we can get the following theorem which is proved in [5, Theorem 2.2].

Theorem 2.2. *If X and Y are real vector spaces, then the function $f : X \to Y$ satisfies the functional equation*

(2.15)
$$f(kx+y)+f(kx-y) = k^2 f(x+y)+k^2 f(x-y)+2(f(kx)-k^2 f(x))-2(k^2-1)f(y),$$

for all $x,y \in X$ if and only if f is of the form

$$f(x) = A^4(x)+A^2(x), \text{ for all } x \in X,$$

where $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i : X^i \to Y$ for $i = 2,4$.

Proof. Assume that f satisfies the functional equation (2.15). By Theorem 2.1 we see that f is a solution of the Fréchet functional equation $\triangle_{x_1,x_2,x_3,x_4,x_5} f(x_0) = 0$ for all $x_0,x_1,x_2,x_3,x_4 \in X$. Thus from Theorem 1.1 we have

$$f(x) = A^4(x)+A^3(x)+A^2(x)+A^1(x)+A^0(x), \text{for all } x \in X, \tag{2.16}$$

where $A^0(x) = A^0$ is an arbitrary element of Y, and $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i : X^i \to Y$ for $i = 1,2,3,4$.

By letting $x = y = 0$ in (2.15), we get $f(0) = 0$. Hence $A^0(x) = A^0 = 0$. Let us set $x = 0$ in (2.15) to get $f(y) = f(-y)$ for all $y \in X$. So the function f is even. Thus we have $A^3(x) \equiv 0$ and $A^1(x) \equiv 0$. Therefore we have $f(x) = A^4(x)+A^2(x)$. The proof of the converse can be easily checked. ∎

Theorem 2.3. *If X and Y are real vector spaces, then the function $f : X \to Y$ satisfies the functional equation*

$$f(kx+y)+f(kx-y) = k^2 f(x+y)+k^2 f(x-y)+2k^2(k^2-1)f(x)-2(k^2-1)f(y), \tag{2.17}$$

if and only if f is of the form

$$f(x) = A^4(x), \quad \text{for all } x \in X,$$

where $A^4(x)$ is the diagonal of the 4-additive symmetric map $A_4 : X^4 \to Y$.

Proof. Assume that f satisfies the functional equation (2.17). Putting $x=y=0$ in (2.17), we have $f(0)=0$. Putting $x=0$ in (2.17), we have $f(-y)=f(y)$ for all $y\in X$. By Theorem 2.1 we see that f is a solution of the Fréchet functional equation $\triangle_{x_1,x_2,x_3,x_4,x_5}f(x_0)=0$ for all $x_0,x_1,x_2,x_3,x_4\in X$. Thus from Theorem 1.1 we have

$$f(x)=A^4(x)+A^3(x)+A^2(x)+A^1(x)+A^0(x), \text{for all } x\in X, \tag{2.18}$$

where $A^0(x)=A^0$ is an arbitrary element of Y, and $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i: X^i\to Y$ for $i=1,2,3,4$. Hence $f(x)=A^4(x)+A^2(x)$ for all $x\in X$. Putting $f(x)=A^4(x)+A^2(x)$ into (2.17), and noting that

$$\begin{cases} A^4(x+y)+A^4(x-y)=2A^4(x)+2A^4(y)+12A^{2,2}(x,y),\\ A^2(x+y)+A^2(x-y)=2A^2(x)+2A^2(y),\end{cases}$$

and $A^{2,2}(kx,y)=k^2A^{2,2}(x,y)$, we conclude that $A^2(x)=0$ for all $x\in X$. Hence $f(x)=A^4(x)$ for all $x\in X$. The proof of the converse can be easily checked. ∎

Remark 2.1. We observe that in case $k=2$, equation (2.17) yields the quartic functional equation (1.6). Therefore, Theorem 2.3 is a generalized version of the theorem for a solution of quartic functional equations [9, Theorem 2.1].

Theorem 2.4. *If X and Y are real vector spaces, then the functions $f,g,h,\tilde{h}: X\to Y$ satisfy the functional equation*

$$f(kx+y)+f(kx-y)=g(x+y)+g(x-y)+h(x)+\tilde{h}(y) \tag{2.19}$$

for all $x,y\in X$ if and only if

$$\begin{aligned} f(x)&=A^4(x)+A^3(x)+A^2(x)+A^1(x)+A^0(x),\\ g(x)&=k^2A^4(x)+kA^3(x)+B^2(x)+B^0(x)+C^1(x)+D^0(x),\\ h(x)&=(2k^4-2k^2)A^4(x)+(2k^3-2k)A^3(x)+2k^2A^2(x)+2kA^1(x)+2A^0(x)\\ &\quad-2B^2(x)-2C^1(x)-2B^0(x),\\ \tilde{h}(x)&=(2-2k^2)A^4(x)+2A^2(x)-2B^2(x)-2D^0(x), \end{aligned} \tag{2.20}$$

where $A^0(x)=A^0$, $B^0(x)=B^0$ and $D^0(x)=D^0$ are arbitrary elements of Y, and $A^i(x),B^i(x),C^i(x)$ are the diagonal of the i-additive symmetric maps $A_i,B_i,C_i: X^i\to Y$, respectively, for $i=1,2,3,4$.

Proof. Assume that $f,g,h,\tilde{h}$ satisfy the functional equation (2.19). By Theorem 2.1 we see that f is a solution of the Fréchet functional equation $\triangle_{x_1,x_2,x_3,x_4,x_5}f(x_0)=0$ for all $x_0,x_1,x_2,x_3,x_4,x_5\in X$. Hence from Theorem 1.1 we have

$$f(x)=A^4(x)+A^3(x)+A^2(x)+A^1(x)+A^0(x), \text{for all } x\in X, \tag{2.21}$$

where $A^0(x)=A^0$ is an arbitrary element of Y, and $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i: X^i\to Y$ for $i=1,2,3,4$. Putting (2.21) into (2.19), and noting that

$$\begin{aligned} &A^4(x+y)+A^4(x-y)=2A^4(x)+2A^4(y)+12A^{2,2}(x,y),\\ &A^3(x+y)+A^3(x-y)=2A^3(x)+6A^{1,2}(x,y),\\ &A^2(x+y)+A^2(x-y)=2A^2(x)+2A^2(y),\end{aligned}$$

and $A^{2,2}(kx,y) = k^2A^{2,2}(x,y)$, $A^{1,2}(kx,y) = kA^{1,2}(x,y)$, we conclude that

$$\begin{aligned} g(x+y)+g(x-y)+h(x)+\tilde{h}(y) = {} & 2k^4A^4(x)+2A^4(y)+12k^2A^{2,2}(x,y) \\ & + 2k^3A^3(x)+6kA^{1,2}(x,y)+2k^2A^2(x) \\ & + 2A^2(y)+2kA^1(x)+2A^0(x). \end{aligned}$$

Therefore

$$\begin{aligned} & g(x+y)+g(x-y)+h(x)+\tilde{h}(y) \\ & = k^2A^4(x+y)+k^2A^4(x-y)+kA^3(x+y)+kA^3(x-y) \\ & \quad + (2k^4-2k^2)A^4(x)+(2k^3-2k)A^3(x)+2k^2A^2(x) \\ & \quad + 2kA^1(x)+2A^0(x)+(2-2k^2)A^4(y)+2A^2(y). \end{aligned} \tag{2.22}$$

Letting

$$G(x) = g(x)-k^2A^4(x)-kA^3(x), \quad \tilde{H}(x) = -\tilde{h}(x)+(2-2k^2)A^4(x)+2A^2(x) \tag{2.23}$$

and

$$H(x) = -h(x)+(2k^4-2k^2)A^4(x)+(2k^3-2k)A^3(x)+2k^2A^2(x)+2kA^1(x)+2A^0(x). \tag{2.24}$$

Then from (2.22) we have

$$G(x+y)+G(x-y) = H(x)+\tilde{H}(y). \tag{2.25}$$

Let G satisfies (2.25). We decompose G into the even part and odd part by putting

$$G_e(x) = \frac{1}{2}(G(x)+G(-x)), G_o(x) = \frac{1}{2}(G(x)-G(-x))$$

for all $x \in X$. It is clear that $G(x) = G_e(x)+G_o(x)$ for all $x \in X$. Similarly, we have $H(x) = H_e(x)+H_o(x)$ and $\tilde{H}(x) = \tilde{H}_e(x)+\tilde{H}_o(x)$. Thus

$$G_e(x+y)+G_e(x-y) = H_e(x)+\tilde{H}_e(y), \tag{2.26}$$

and

$$G_o(x+y)+G_o(x-y) = H_o(x)+\tilde{H}_o(y). \tag{2.27}$$

Letting $y = 0$ in (2.26), we have $H_e(x) = 2G_e(x)-\tilde{H}_e(0)$. Setting $x = 0$ in (2.26) to get $\tilde{H}_e(y) = 2G_e(y)-H_e(0)$. Hence

$$G_e(x+y)+G_e(x-y) = 2G_e(x)+2G_e(y)-2G_e(0), \tag{2.28}$$

for all $x,y \in X$. Setting $M(x) = G_e(x)-G_e(0)$, we get

$$M(x+y)+M(x-y) = 2M(x)+2M(y) \tag{2.29}$$

which is the quadratic functional equation and its solution is given by

$$M(x) = B^2(x) \text{ for all } x \in X,$$

where $B^2(x)$ is the diagonal of the 2-additive symmetric map $B_2 : X^2 \to Y$. In this case, we obtain

$$G_e(x) = B^2(x)+G(0), \ \ H_e(x) = 2B^2(x)+H_e(0), \ \ \tilde{H}_e(x) = 2B^2(x)+\tilde{H}_e(0). \tag{2.30}$$

Similarly, letting $y = 0$ in (2.27), we have $H_o(x) = 2G_o(x)$. Setting $x = 0$ in (2.27) to get $\tilde{H}_o(y) = 0$. Then from (2.27) we have

$$G_o(x+y)+G_o(x-y) = 2G_o(x), \tag{2.31}$$

which is the Jensen functional equation and its solution is given by

$$G_o(x) = C^1(x), \tag{2.32}$$

where $C^1 : X \to Y$ is an additive function. Thus

$$\begin{aligned} G(x) &= G_e(x) + G_o(x) = B^2(x) + B^0(x) + D^0(x) + C^1(x), \\ H(x) &= H_e(x) + H_o(x) = 2B^2(x) + 2C^1(x) + 2B^0(x), \\ \tilde{H}(x) &= \tilde{H}_e(x) + \tilde{H}_o(x) = 2B^2(x) + 2D^0(x). \end{aligned} \tag{2.33}$$

where $B^0(x) = B^0$ and $D^0(x) = D^0$ are arbitrary elements of Y. Therefore from (2.23), (2.24), (2.29), we obtain the asserted solution (2.20). The proof of the converse can be easily checked. ∎

3. Solution of equation (1.8) on commutative groups

In this section, we solve the functional equation (1.8) on commutative groups with some additional requirements.

A group G is said to be divisible if for every element $b \in G$ and every $n \in \mathbb{N}$, there exists an element $a \in G$ such that $na = b$. If this element a is unique, then G is said to be uniquely divisible. In a uniquely divisible group, this unique element a is denoted by b/n. That the equation $na = b$ has a solution is equivalent to saying that the multiplication by n is surjective. Similarly, that the equation $na = b$ has a unique solution is equivalent to saying that the multiplication by n is bijective. Hence the notions of n-divisibility and n-unique divisibility refer, respectively, to surjectivity and bijectivity of the multiplication by n.

Lemma 3.1. (Hosszú [6]) *Let $n \geq 0$ be an integer, G and S be abelian groups. Furthermore let S be uniquely divisible. The map F from G into S satisfies the functional equation*

$$\triangle_{x_1,\ldots,x_{n+1}} F(x_0) = 0$$

for all $x_0, x_1, \ldots, x_{n+1} \in G$ if and only if F is given by

$$F(x) = A^n(x) + \cdots + A^1(x) + A^0(x), \text{for all } x \in G,$$

where $A^0(x) = A^0$ is an arbitrary element of S and $A^n(x)$ is the diagonal of an n-additive symmetric function $A_n : G^n \to S$.

The solution of the functional equation (2.15) can be determined in certain type of groups by using Lemma 3.1. As the proof is identical with the proof of Theorem 2.2 we simply state the theorem without a proof.

Theorem 3.1. *Let G and S be uniquely divisible abelian groups. Then the function $f : G \to S$ satisfies the functional equation (2.15) for all $x, y \in G$, if and only if f is of the form*

$$f(x) = A^4(x) + A^2(x), \text{for all } x \in G,$$

where $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i : G^i \to S$ for $i = 2, 4$.

Theorem 3.2. *Let G and S be uniquely divisible abelian groups. Then the function $f : G \to S$ satisfies the functional equation (2.17), if and only if f is of the form*

$$f(x) = A^4(x), \text{for all } x \in X,$$

where $A^4(x)$ is the diagonal of the 4-additive symmetric map $A_4 : G^4 \to S$.

Proof. It can be proved in the same manner as in the proof of Theorem 2.3 by using Lemma 3.1. ∎

Similar to the Theorem 2.4, we have the following theorem. As the proof is identical with the proof of Theorem 2.6 we simply state the theorem without a proof.

Theorem 3.3. *Let G and S be uniquely divisible abelian groups. Then the functions $f, g, h, \tilde{h}$: $G \to S$ satisfy the functional equation (1.8) for all $x, y \in G$, if and only if*

$$\begin{aligned} f(x) &= A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0(x), \\ g(x) &= k^2A^4(x) + kA^3(x) + B^2(x) + B^0(x) + C^1(x) + D^0(x), \\ h(x) &= (2k^4 - 2k^2)A^4(x) + (2k^3 - 2k)A^3(x) + 2k^2A^2(x) + 2kA^1(x) + 2A^0(x) \\ &\quad - 2B^2(x) - 2C^1(x) - 2B^0(x), \\ \tilde{h}(x) &= (2 - 2k^2)A^4(x) + 2A^2(x) - 2B^2(x) - 2D^0(x), \end{aligned}$$

where $A^0(x) = A^0$, $B^0(x) = B^0$ and $D^0(x) = D^0$ are arbitrary elements of S, and $A^i(x), B^i(x), C^i(x)$ are the diagonal of the i-additive symmetric maps $A_i, B_i, C_i : G^i \to S$, respectively, for $i = 1, 2, 3, 4$.

Theorems 3.1–3.3 can be further strengthened using two important results due to Székelyhidi [17]. By the use of the two important results, the proofs becomes even shorter but not so elementary any more. The results needed for this improvement are the following (see [17]).

Theorem 3.4. *Let G be a commutative semigroup with identity, S a commutative group and n a nonnegative integer. Let the multiplication by n! be bijective in S. The function $f : G \to S$ is a solution of Fréchet functional equation*

$$\triangle_{x_1,\ldots,x_{n+1}} f(x_0) = 0 \tag{3.1}$$

for all $x_0, x_1, \ldots, x_{n+1} \in G$ if and only if f is a polynomial of degree at most n.

Theorem 3.5. *Let G and S be commutative groups, n a nonnegative integer, φ_i, ψ_i additive functions from G into G and $\varphi_i(G) \subseteq \psi_i(G) (i = 1, 2, \ldots, n+1)$. If the functions $f, f_i : G \to S (i = 1, 2, \ldots, n+1)$ satisfy*

$$f(x) + \sum_{i=1}^{n+1} f_i(\varphi_i(x) + \psi_i(y)) = 0, \tag{3.2}$$

then f satisfies Fréchet functional equation $\triangle_{x_1,\ldots,x_{n+1}} f(x_0) = 0$.

Using these two theorems, Theorems 3.1–3.3 can be further improved.

Theorem 3.6. *Let G and S be commutative groups. Let the multiplication by k be surjective in G and let the multiplication by 24 and $2(k^2 - 1)$ be bijective in S. Then the function $f : G \to S$ satisfies the functional equation (2.15) for all $x, y \in G$, if and only if f is of the form*

$$f(x) = A^4(x) + A^2(x), \text{for all } x \in G,$$

where $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i : G^i \to S$ for $i = 2, 4$.

Proof. Assume that f satisfies the functional equation (2.15). Using the unique divisibility of S by $2(k^2-1)$ and interchange x with y in (2.15), we can rewrite the functional equation (2.15) in the form

$$f(x)+\sum_{i=1}^{5} f_i(\varphi_i(x)+\psi_i(y))=0$$

where $f_1(\cdot)=f_2(\cdot)=(1/(2(k^2-1)))f(\cdot)$, $f_3(\cdot)=f_4(\cdot)=((-k^2)/(2(k^2-1)))f(\cdot)$, $f_5(\cdot)=((-2)/(2(k^2-1)))(f(k\cdot)-k^2f(\cdot))$, $\varphi_1(x)=\varphi_3(x)=x$, $\varphi_2(x)=\varphi_4(x)=-x$, $\varphi_5(x)=0$, and $\psi_1(y)=\psi_2(y)=ky$, $\psi_3(y)=\psi_4(y)=\psi_5(y)=y$. From these φ_i and ψ_i we see that $\varphi_i(G)\subseteq\psi_i(G)$ for $i=1,2,3,4,5$. Thus by Theorem 3.5, f satisfies the Fréchet functional equation (3.1). By Theorem 3.4, f is a generalized polynomial function of degree at most 4, that is f is of the form

$$f(x)=A^4(x)+A^3(x)+A^2(x)+A^1(x)+A^0(x), \text{for all } x\in G,$$

where $A^0(x)=A^0$ is an arbitrary element of S, and $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i: G^i\to S$ for $i=1,2,3,4$. By letting $x=y=0$ in (2.15), we get $f(0)=0$. Hence $A^0(x)=A^0=0$. Setting $x=0$ in (2.15) to get $f(y)=f(-y)$ for all $y\in G$. So the function f is even. Thus we have $A^3(x)=0$ and $A^1(x)=0$. Therefore we have $f(x)=A^4(x)+A^2(x)$. The proof of the converse can be easily checked. ∎

Similar to the Theorem 2.3 and Theorem 3.6, we have the following theorem.

Theorem 3.7. *Let G and S be commutative groups. Let the multiplication by* 24 *and* $2k^2(k^2-1)$ *be bijective in S. Then the function* $f: G\to S$ *satisfies the functional equation (2.17) for all* $x,y\in G$, *if and only if f is of the form*

$$f(x)=A^4(x), \textit{for all } x\in G,$$

where $A^4(x)$ *is the diagonal of the* 4*-additive symmetric map* $A_4: G^4\to S$.

Proof. Assume that f satisfies the functional equation (2.17). Using the unique divisibility of S by $2k^2(k^2-1)$, we can rewrite the functional equation (2.17) in the form

$$f(x)-\frac{f(kx+y)}{2k^2(k^2-1)}-\frac{f(kx-y)}{2k^2(k^2-1)}+\frac{k^2f(x+y)}{2k^2(k^2-1)}+\frac{k^2f(x-y)}{2k^2(k^2-1)}-\frac{2(k^2-1)}{2k^2(k^2-1)}f(y)=0.$$

Thus by Theorem 3.5, f satisfies the Fréchet functional equation (3.1). By Theorem 3.4, f is a generalized polynomial function of degree at most 4, that is f is of the form

$$f(x)=A^4(x)+A^3(x)+A^2(x)+A^1(x)+A^0(x), \text{for all } x\in G,$$

where $A^0(x)=A^0$ is an arbitrary element of S, and $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i: G^i\to S$ for $i=1,2,3,4$. The remaining assertion goes through by the similar way to corresponding part of Theorem 2.3. ∎

Theorem 3.8. *Let G and S be commutative groups. Let the multiplication by* $2k^2(k^2-1)$ *be surjective in G and let the multiplication by* 24 *be bijective in S. Then the function* $f: G\to S$

satisfies the functional equation (1.8) for all $x, y \in G$, if and only if f is of the form

$$\begin{aligned} f(x) &= A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0(x), \\ g(x) &= k^2A^4(x) + kA^3(x) + B^2(x) + B^0(x) + C^1(x) + D^0(x), \\ h(x) &= (2k^4 - 2k^2)A^4(x) + (2k^3 - 2k)A^3(x) + 2k^2A^2(x) + 2kA^1(x) + 2A^0(x) \\ &\quad - 2B^2(x) - 2C^1(x) - 2B^0(x), \\ \tilde{h}(x) &= (2 - 2k^2)A^4(x) + 2A^2(x) - 2B^2(x) - 2D^0(x), \end{aligned}$$

where $A^0(x) = A^0$, $B^0(x) = B^0$ and $D^0(x) = D^0$ are arbitrary elements of Y, and $A^i(x), B^i(x), C^i(x)$ are the diagonal of the i-additive symmetric maps $A_i, B_i, C_i : X^i \to Y$, respectively, for $i = 1, 2, 3, 4$.

Proof. Assume that f satisfies the functional equation (1.8). Using the multiplication by $2k^2(k^2 - 1)$ be surjective in G, we can rewrite the functional equation (1.8) in the form

$$f(x) + f(-x + 2k^2y) - f(x + (k - k^2)y) - f(-x + (k + k^2)y) - h(ky) - \tilde{h}(x - k^2y) = 0.$$

Thus by Theorem 3.5, f satisfies the Fréchet functional equation (3.1). By Theorem 3.4, f is a generalized polynomial function of degree at most 4, that is f is of the form

$$f(x) = A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0(x), \text{for all } x \in G,$$

where $A^0(x) = A^0$ is an arbitrary element of S, and $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i : G^i \to S$ for $i = 1, 2, 3, 4$. The remaining assertion goes through by the similar way to corresponding part of Theorem 2.4. ∎

Acknowledgement. The authors would like to thank the referees for giving useful suggestions for the improvement of this paper.

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