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Oscillation of a Certain Class of Third Order Nonlinear Difference Equations

S. H. SAKER
College of Science Research Centre, King Saud University, Riyadh 11451, P. O. Box 2455, Saudi Arabia
shsaker@mans.edu.eg

Abstract. In this paper, we are concerned with oscillation of the nonlinear difference equation $\Delta(c_n[\Delta(d_n\Delta x_n)]^\gamma) + q_n f(x_{g(n)}) = 0$, $n \geq n_0$, where $\gamma > 0$ is the quotient of odd positive integers, c_n, d_n and q_n are positive sequences of real numbers, $g(n)$ is a sequence of nonnegative integers and $f \in C(\mathbf{R},\mathbf{R})$ such that $uf(u) > 0$ for $u \neq 0$. We establish some new sufficient conditions for oscillation by employing the Riccati substitution and the analysis of the associated Riccati difference inequality. Our results extend and improve some previously obtained ones. Some examples are considered to illustrate the main results.

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1. Introduction

In recent years, the asymptotic properties and oscillation of difference equations and their applications have been and still are receiving intensive attention. In fact, in the last few years several monographs and hundreds of research papers have been written, see for example the monographs [1, 3, 6, 11]. Determination of oscillatory behavior for solutions of first and second order difference equations has occupied a great part of researchers' interest. Compared to the first and second order difference equations, the study of third order difference equations has received considerably less attention in the literature, even though such equations arise in the study of economics, mathematical biology, and other areas of mathematics which discrete models are used (see for example [4]). For contributions, we refer the reader to the papers [2, 5, 7, 8, 13–19] and the references cited therein. For completeness and comparison, we present below some of these results.

In this paper, we are concerned with oscillation of the nonlinear difference equation

$$\Delta(c_n[\Delta(d_n\Delta x_n)]^\gamma) + q_n f(x_{g(n)}) = 0, \quad n \geq n_0, \tag{1.1}$$

where $\gamma > 0$ is quotient of odd positive integers. Throughout this paper, we will assume the following hypotheses:

(h_1). c_n, d_n, q_n are positive sequences of real numbers, $g(n) : \mathbf{N} \to \mathbf{Z}$, $\lim_{n\to\infty} g(n) = \infty$,

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(h_2). $f : \mathbf{R} \to \mathbf{R}$ is continuous, $f(-u) = -f(u)$, for $u \neq 0$, *and* $f(u)/u^{\gamma} \geqslant K > 0$.

Equation (1.1) is called a delay equation if $g(n) < n$ and is called an advanced equation if $g(n) > n$. Since, we are interested in oscillation and asymptotic behavior of solutions near infinity, we make a standing hypothesis that the equation under consideration does possess such solutions and the solutions vanishing in some neighborhood of infinity will be excluded from our consideration. Our attention is restricted to those solutions of (1.1) which exist on $[n_x, \infty)$ and satisfy $\sup\{|x_n| : n > n_1\} > 0$ for any $n_1 \geq n_x$. A solution x_n of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. The equation (1.1) is said to be oscillatory in case there exists at least one oscillatory solution.

Here are a few background details that serve the readers and motivate the contents of this paper. For oscillation of linear difference equation Smith in [16] considered the equation of the form

$$\Delta^3 x_n - p_n x_{n+2} = 0, \qquad n \geq n_0, \tag{1.2}$$

and proved that if

$$\sum_{n=n_0}^{\infty} p_n = \infty, \tag{1.3}$$

then (1.2) is oscillatory. The main investigation depends on the value of the functional $G(x_n) = (\Delta x_n)^2 - 2x_{n+1}\Delta^2 x_n$, which is the discrete analogy of the function defined by Lazer [12] for third order differential equations. Further in [16] the author considered the quasi-adjoint difference equation

$$\Delta^3 x_n + p_n x_{n+1} = 0, \qquad n \geq n_0, \tag{1.4}$$

where $p_n > 0$ for $n \geq n_0$ and proved that (1.2) is oscillatory if and only if (1.4) is oscillatory. But one can easily see that the results cannot be applied if $p_n = n^{-\alpha}$ for $\alpha > 1$.

In [14] the authors considered the difference equation of the form

$$\Delta^3 x_n + p_n x_n = 0, \qquad n \geq n_0, \tag{1.5}$$

and proved that if p_n is a positive sequence and

$$p_n > 1, \quad \text{for} \quad n \geq n_0, \tag{1.6}$$

then (1.5) is oscillatory. In [15] the author considered the equation (1.4) where $p_n > 0$ for $n \geq n_0$ and proved that if

$$\sum_{l=n_0}^{\infty} \left[\sum_{t=n_0}^{l-1} \sum_{s=n_0}^{t-1} p_s \right] = \infty, \tag{1.7}$$

and there exists a positive sequence ρ_n such that,

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n} \left[\rho_s p_s - \frac{(\Delta \rho_s)^2}{4\rho_s (s - n_0)} \right] = \infty, \tag{1.8}$$

then the solution x_n of (1.4) is oscillatory or satisfies $\lim_{n\to\infty} x_n = 0$. One can easily see that the results established in [15] provided substantial improvement for those obtained in [16] and [14].

In [17] the author considered the linear difference equation

$$\Delta^3 x_n + p_{n+1}\Delta x_{n+2} + q_n x_{n+2} = 0, \qquad n \geq n_0, \tag{1.9}$$

where p_n and q_n are real sequences satisfying

$$p_n \geq 0,\ q_n < 0 \text{ and } \sum_{s=n_0}^{\infty} (\Delta p_n - 2q_n) = \infty, \tag{1.10}$$

and proved that if $p_{n+1} + q_n \leq 0$ for $n \geq n_0$, then (1.9) has both oscillatory and nonoscillatory solutions. Further it was proved that if there is a solution x_n of (1.9) such that $F(x_n) > 0$, then x_n is oscillatory where the functional $F(x_n)$ is defined by $F(x_n) := (\Delta x_n)^2 - 2x_{n+1}\Delta^2 x_n - p_n x_{n+2}^2$. However one can easily see that the condition depends on the solution itself whose determination might not be possible.

In [18] the author considered the equation

$$\Delta(\Delta^2 x_n - p_{n+1}x_{n+1}) - q_{n+2}x_{n+2} = 0, \quad n \geq n_0, \tag{1.11}$$

where p_n and q_n are nonnegative real sequences and satisfying (1.10). The author proved that if x_n is a nonoscillatory solution then there exists an integer N for which either $x_n\Delta x_n > 0$ or $x_n\Delta x_n < 0$ for all $n > N$ and proved that the equation (1.11) is oscillatory if and only if the equation

$$\Delta^3 x_n - p_{n+1}\Delta x_{n+1} + q_{n+1}x_{n+1} = 0, \quad n \geq n_0, \tag{1.12}$$

is oscillatory. Further the author gave a connection between the behavior of solutions of (1.12) and (1.11) by proving that if u_n is a solution of (1.12), then the two independent solutions of (1.11) satisfy the self-adjoint second order difference equation

$$\Delta\left(\frac{\Delta x_n}{u_n}\right) + \left(\frac{\Delta^2 u_{n+1} - p_{n+1}u_{n+2}}{u_{n+1}u_{n+2}}\right) x_{n+1} = 0. \tag{1.13}$$

Also in [18] the author proved that if v_n is a nonoscillatory solution of (1.11), then the two independent solutions of (1.12) satisfy the self-adjoint second order equation

$$\Delta\left(\frac{\Delta x_n}{v_n}\right) + \left(\frac{\Delta^2 v_{n-1} - p_n v_n}{v_n v_{n+1}}\right) x_{n+1} = 0. \tag{1.14}$$

Specifically the author proved that the equation (1.11) is oscillatory if and only if (1.13) is oscillatory and also (1.12) is oscillatory if and only if (1.14) is oscillatory. In fact these results can be considered as the discrete analogy of the results that has been given for third order differential equations by Jones [10] where he considered the equation

$$x'''(t) + p(t)x'(t) + q(t)x(t) = 0, \quad t \geq t_0, \tag{1.15}$$

and gave a relationship between oscillation of (1.15) and nonoscillation of its self-adjoint equation

$$x'''(t) + p(t)x'(t) + (p'(t) - q(t))x(t) = 0, \tag{1.16}$$

and proved that if N is a nonoscillatory solution of the adjoint equation (1.16), then there are two independent oscillatory solutions of (1.15) satisfying the equation

$$\left(\frac{x'(t)}{N(t)}\right)' + \left(\frac{N''(t) + p(t)N(t)}{N^2(t)}\right) x(t) = 0. \tag{1.17}$$

In [13] the authors considered the difference equation of the form

$$y_{n+3} + r_n y_{n+2} + q_n y_{n+1} + p_n y_n = 0, \tag{1.18}$$

where r_n, p_n, q_n are sequences of real numbers such that $p_n \neq 0$. The authors proved that if $p_n < 0$, $q_n < 0$ and $r_n < 0$ then equation (1.18) admits two oscillatory solutions and if $p_n < 0$, $q_n > 0$ and $r_n > 0$ then equation (1.18) admits a nonoscillatory solution. In [8] the authors studied the oscillation of the nonlinear difference equation

$$\Delta(c_n\Delta(d_n\Delta(x_n))) + q_n f(x_{n-\sigma+1}) = 0, \quad n \geq n_0, \tag{1.19}$$

where σ is a nonnegative integer, $f : \mathbf{R} \to \mathbf{R}$ is continuous such that $uf(u) > 0$ for $u \neq 0$, and

$$f(u) - f(v) = g(u,v)(u-v), \text{ for } u,v \neq 0 \quad \text{and} \quad g(u,v) \geqslant \mu > 0, \tag{1.20}$$

and c_n, d_n are positive sequences of real numbers such

$$\sum_{n=n_0}^{\infty}\left(\frac{1}{c_n}\right) = \sum_{n=n_0}^{\infty}\left(\frac{1}{d_n}\right) = \infty, \text{ and } \Delta c_n \geqslant 0. \tag{1.21}$$

For the linear case they used the Riccati transformation technique and established some sufficient conditions which ensure that every solution of (1.19) is oscillatory. They proved that if $f(u) = u$ and there exist real valued sequences $h, H : \mathbf{N}\times\mathbf{N} \to \mathbf{R}$ such that $H(n,n) = 0$, $H(n,s) > 0$ for $n > s \geq n_0$, $-\Delta_2 H(n,s) = h(n,s)\sqrt{H(n,s)}$ and

$$\limsup_{n\to\infty} \frac{1}{H(n,n_1)} \sum_{s=n_1} \left[H(n,s)q_s - \frac{c_s d_{s-\sigma} h^2(n,s)}{4(s-\sigma-n_0)}\right] = \infty,$$

$$\sum_{i=n}^{n+m-1} q_i \left[\sum_{j=n}^{i} \frac{1}{d_j}\left(\sum_{k=j}^{i}\frac{1}{c_k}\right)\right] > 1,$$

then every solution of (1.19) is oscillatory. In the nonlinear case some oscillation criteria are given by reducing the oscillation of the equation to the existence of positive solution of a Riccati difference inequality. But one can easily see that the condition (1.20) cannot be tested when $f(u) = u^{\gamma}$ for $\gamma > 0$ and the results are valid only when $\Delta c_n \geqslant 0$. They proved that if (1.20) and (1.21) hold and there exists a positive sequence such that

$$\sum_{s=n_1}^{\infty} \left[\rho_s q_s - \frac{c_s d_{s-\sigma}(\Delta\rho_s)^2}{4\mu(s-\sigma-n_0)}\right] = \infty,$$

and

$$\limsup_{n\to\infty} \sum_{i=n}^{n+m-1} q_i \left[\sum_{j=n}^{i} \frac{1}{d_j}\left(\sum_{k=j}^{i}\frac{1}{c_k}\right)\right] = \infty,$$

then every solution of (1.19) is oscillatory. Note that these results cannot be applied on the equation

$$\Delta^3 x_n + \frac{8n+12}{(n-\sigma+1)(1+(n-\sigma+1)^2))} x_{n-\sigma+1}(1 + x_{n-\sigma+1}^2) = 0, \quad \text{for } n \geq n_0, \tag{1.22}$$

where σ is an odd positive integer, $f(u) = u(1+u^2) \geq u$ satisfies $f(-u) = -f(u)$. Note that this equation has an oscillatory solution $x_n = (-1)^n n$. So one of our aims in this paper is to establish some sufficient conditions bypass these restrictions.

In [7] the authors considered the nonlinear delay difference equation

$$\Delta(c_n(\Delta^2 x_n)^{\gamma}) + q_n f(x(g_n)) = 0, \quad n \geq n_0, \tag{1.23}$$

where c_n, g_n, q_n are sequences of nonneagtive real numbers, $g_n < n$, γ is quotient of odd positive integers, f: $\mathbf{R} \to \mathbf{R}$ is continuous such that $uf(u) > 0$ for $u \neq 0$, $f^{'}(x) > 0$, and $-f(-xy) \geq f(xy) \geq f(x)f(y)$ for $xy > 0$ and

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{c_n}\right)^{\gamma} < \infty. \tag{1.24}$$

The main approach of proving the results in [7] is the reduction of the oscillation of (1.23) to the oscillation of first order delay difference equation. They proved that if both of the two difference equations

$$\Delta y_n + cq_n f\left(\sum_{k=n_1}^{g_n-1} \frac{k}{c^{\frac{1}{\gamma}}(n)}\right) f(y^{\frac{1}{\gamma}}(g_n)) = 0,$$

$$\Delta y_n + q_n f(\xi(n)) - g_n f\left(\sum_{k=\xi(n)}^{\eta(n)-1} \frac{k}{c^{\frac{1}{\gamma}}(n)}\right) f(y^{\frac{1}{\gamma}}(\eta(n))) = 0,$$

are oscillatory, and

$$\sum_{l=n_1}^{\infty} \left(\frac{1}{c_l} \sum_{k=n_1}^{l-1} q(k) f\left(\sum_{s=g_k}^{\infty} \frac{1}{c^{\frac{1}{\gamma}}(n)}\right)\right)^{\frac{1}{\gamma}} = \infty,$$

then equation (1.23) is oscillatory. But the results can be applied only in the case when $g_n < n$. Also the restriction $f'(x) > 0$ is required. This condition does not hold and cannot be applied in the case when $f(x) = x\left(1/9 + 1/(1+x^2)\right)$, since $f'(x) = (x^2-2)(x^2-5)/9(1+x^2)^2$ changes sign four times. Note that in this case we have $f(-x) = -x(1/9+1/(1+x^2)) = -f(x)$ which means that condition (h_2) is satisfied.

We note that the equation (1.19) is a special case of (1.1) when $\gamma = 1$ and the equation (1.23) is also a special case of (1.1) when $d_n = 1$. Also the results that has been established for the equation (1.19) in [8] depend on condition (1.21) and the results that in [7] has been established in the special case when $d_n = 1$. Therefore it will be great of interest to establish oscillation criteria for (1.1) when

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{c_n}\right)^{\gamma} = \infty, \quad \sum_{n=n_0}^{\infty} \left(\frac{1}{d_n}\right) = \infty. \tag{1.25}$$

The main aim of this paper is to establish some sufficient conditions which guarantee that the equation (1.1) has oscillatory solutions or the solutions tend to zero as $n \to \infty$. The paper is organized as follows: In Section 2, we state and prove some useful lemmas that will be used in the proofs of the main results. In Section 3, we consider the case when (1.25) holds. In the Subsection 3.1, we consider the advanced case when $g(n) > n$ and in the Subsection 3.2, we consider the delay case when $g(n) < n$. The main investigation of the main oscillation results depends on the Riccati substitution and the analysis of the associated Riccati difference inequality. Our results improve the results improve the results in [8] in the sense that the results do not require the conditions (1.21) and (1.20). Also the results complement the results in [7] in the sense that the results do not require the condition $f^{'}(x) > 0$ and $d_n = 1$ and can be applied on the case when $g(n) \geq n$. Some examples and applications are considered throughout the paper to illustrate the main results.

2. Some preliminary lemmas

In this section, we state and prove some fundamental lemmas that will be used in the proofs of the main results. For the solution x_n of the equation (1.1), we define the quasi differences by

$$x_n^{[0]} = x_n, \quad x_n^{[1]} = d_n \Delta x_n, \quad x_n^{[2]} = c_n \left[\Delta x_n^{[1]}\right]^{\gamma}, \quad \text{and} \quad x_n^{[3]} = \Delta\left(x_n^{[2]}\right). \tag{2.1}$$

We note that if x_n is a solution of (1.1) then $z = -x$ is also solution of (1.1), since from (h_2), $f(-u) = -f(u)$ for $u \neq 0$. Thus, concerning nonoscillatory solutions of (1.1), we can restrict our attention only to the positive ones. We start with the following Lemma which provides the signs of the quasi differences of the solution x_n of (1.1).

Lemma 2.1. *Assume that $(h_1) - (h_2)$ hold. If x_n is a nonoscillatory solution of (1.1), then there exists $N > n_0$ such that $x_n^{[i]} \neq 0$ for $i = 0, 1, 2$ for $n \geq N$.*

Proof. Without loss of generality, we may assume that x_n be an eventually positive solution of (1.1) and there exists a $n_1 \geq n_0$ such that $x_n > 0$ and $x_{g(n)} > 0$ for $n \geq n_1$. Then, since $q_n > 0$, $x_n^{[3]} < 0$, and there exists $n_2 \geq n_1$ such that $x_n^{[2]}$ is either positive or negative for $n \geq n_2$. Thus $x_n^{[1]}$ is either increasing or decreasing for $n \geq n_2$ and so there exists $N \geq n_2$ such that $x_n^{[1]}$ is either positive or negative for $n \geq N$. The proof is complete. ∎

In view of Lemma 2.1, we deduce that all nonoscillatory solutions of (1.1) belong to the following classes:

$$\begin{aligned}
C_0 =& \{x : \exists\, N \text{ such that } x_n x_n^{[1]} < 0,\ x_n x_n^{[2]} > 0 \text{ for } n \geq N\},\\
C_1 =& \{x : \exists\, N \text{ such that } x_n x_n^{[1]} > 0,\ x_n x_n^{[2]} < 0 \text{ for } n \geq N\},\\
C_2 =& \{x : \exists\, N \text{ such that } x_n x_n^{[1]} > 0,\ x_n x_n^{[2]} > 0 \text{ for } n \geq N\},\\
C_3 =& \{x : \exists\, N \text{ such that } x_n x_n^{[1]} < 0, x_n x_n^{[2]} < 0 \text{ for } n > N\}.
\end{aligned}$$

Lemma 2.2. *Assume that $(h_1) - (h_2)$ and (1.25) hold. If x_n be a nonoscillatory solution of (1.1), then $x_n \in C_0 \cup C_2$.*

Proof. Without loss of generality, we may assume that x_n is an eventually positive solution of (1.1). Then there exists $n_1 \geqslant n_0$ such that x_n and $x_{g(n)} > 0$ for $n \geqslant n_1$. Then in view Lemma 2.1, $x_n^{[0]}$, $x_n^{[1]}$ and $x_n^{[2]}$ are monotone and eventually of one sign. So to complete the proof, we prove that the possible cases are the following two cases for $n \geqslant n_1$ sufficiently large:

Case (I): $x_n^{[0]} > 0$, $x_n^{[1]} > 0$, $x_n^{[2]} > 0$,

Case (II): $x_n^{[0]} > 0$, $x_n^{[1]} < 0$, $x_n^{[2]} > 0$.

This means that it is enough to claim that there exists $n_2 \geqslant n_1$ such that $x_n^{[2]} > 0$ for $n \geqslant n_2$,. Suppose to the contrary that $x_n^{[2]} \leq 0$ for $n \geqslant n_2$. From (1.1) and (h_2), we see that $x_n^{[3]} < 0$ for $n \geqslant n_1$ and then $x_n^{[2]}$ is decreasing. Therefore there exist a negative constant C and $n_3 \geqslant n_2$ such that $x_n^{[2]} \leq C$ for $n \geqslant n_3$. So that

$$x_n^{[1]} \leq x_{n_3}^{[1]} + C^{\frac{1}{\gamma}} \sum_{s=n_3}^{n-1} \frac{1}{(c_s)^{\frac{1}{\gamma}}},$$

which implies by (1.25) that $\lim_{n\to\infty} x_n^{[1]} = -\infty$. Thus, there is an integer $n_4 \geqslant n_3$ such that for $n \geqslant n_4$, $d_n\Delta(x_n) \leq d_{n_4}\Delta(x_{n_4}) < 0$. This implies that after summing from n_4 to $n-1$, that

$$x_n - x_{n_4} \leq d_{n_4}\Delta(x_{n_4}) \sum_{s=n_3}^{n-1} \frac{1}{d_s},$$

which implies by (1.25) that $x_n \to -\infty$ as $n \to \infty$. This is a contradiction with $x_n > 0$. Then $x_n^{[2]} > 0$. The proof is complete. ∎

Remark 2.1. We note that most of the results that has been presented in the introduction are obtained under some conditions on the coefficients which ensure that the solutions are of type C_0 and C_2. In the following Lemma, we give a condition which ensure that $C_3 = \emptyset$ and we will consider it in the reminder of the paper. So it would be great of interest to find new conditions for oscillation of (1.1) when (2.2) does not hold and this will be left to the interested reader.

Lemma 2.3. *Assume that* $(h_1)-(h_2)$ *hold. If*

$$\sum_{n=n_1}^{\infty} \frac{1}{d_n} \sum_{s=n_1}^{n-1} \frac{1}{(c_s)^{\frac{1}{\gamma}}} = \infty, \tag{2.2}$$

then C_3 *is empty.*

Proof. To prove that C_3 is empty, we prove that if there is a positive solution x_n of (1.1), then

$$x_n x_n^{[1]} < 0,\ x_n x_n^{[2]} < 0,\ \ for \ \ n \geq N > n_0,$$

is impossible. Assume for the sake of contradiction that there exists $n_1 > n_0$ such that x_n and $x_{g(n)} > 0$, $x_n^{[2]} < 0$ and $x_n^{[1]} < 0$ for $n \geq n_1$. Denote $a_0 = x_{n_1}^{[2]} < 0$. Then, since $x_n^{[2]}$ is decreasing we have $c_n(\Delta x_n^{[1]})^\gamma < a_0$ for $n \geq n_1$ and thus by summation from n_1 to $n-1$, we have

$$x_n^{[1]} < x_{n_1}^{[1]} + a_0^{\frac{1}{\gamma}} \sum_{s=n_1}^{n-1} \frac{1}{(c_s)^{\frac{1}{\gamma}}}.$$

Now, since $x_{n_1}^{[1]} < 0$, we see after summation from n_1 to $n-1$, that

$$x_n < x_{n_1} + a_0^{\frac{1}{\gamma}} \sum_{n=n_1}^{n-1} \frac{1}{d_n} \sum_{s=n_1}^{n-1} \frac{1}{(c_s)^{\frac{1}{\gamma}}}.$$

Letting $n \to \infty$, we get by (2.2) that $\lim_{n\to\infty} x_n = -\infty$, which contradicts the positivity. ∎

Remark 2.2. In the proof [7, Theorem 2.1] the authors assumed that the case $x_n > 0$, $\Delta x_n < 0$ and $\Delta^2 x_n < 0$ cannot hold (this equivalent to the case $C_3 = \emptyset$). In fact this is not the case, since to prove this we should assume that $\Delta^2 x_n$ is decreasing. From the equation (1.23), we see that the term which is decreasing is $c_n(\Delta^2 x_n)^\gamma$ (not $\Delta^2 x_n$) and then $c_n(\Delta^2 x_n)^\gamma < c_{n_1}(\Delta^2 x_{n_1})^\gamma$ for $n > n_1$. Then, we get $\Delta^2 x_n < A c_n^{-1/\gamma}$, where $A = c_{n_1}(\Delta^2 x_{n_1})^\gamma < 0$. This implies that $\Delta x_n - \Delta x_{n_1} < A \sum_{n_1}^{n-1} c_s^{-1/\gamma}$, and since $\Delta x_{n_1} < 0$, we get $\Delta x_n < A C_n$ where $C_n = \sum_{n_1}^{n-1} c_s^{-1/\gamma}$. After summing, we get $x_n < x_{n_1} - A \sum_{n_1}^{n-1} C_n$. So to get a contradiction with the positivity of x_n, we have to assume that $\sum_{n_1}^{\infty} C_n = \sum_{n=n_1}^{\infty} \sum_{s=n_1}^{n-1} c_s^{-1/\gamma} = \infty$.

Lemma 2.4. *Assume that $(h_1)-(h_2)$ hold. Let x_n is a nonoscillatory solution of (1.1) such that $x_n \in C_0$. If*

(h_3). $\sum_{n=n_0}^{\infty} \frac{1}{d_n} \sum_{t=n_0}^{n-1} \left(\frac{1}{c_t} \sum_{s=n_0}^{t-1} q_s \right)^{\frac{1}{\gamma}} = \infty.$

Then $\lim_{n\to\infty} x_n = 0$.

Proof. Without loss of generality, we may assume that $x_n > 0$, $x_{g(n)} > 0$ for $n \geq n_1$ where $n_1 > n_0$ is chosen sufficiently large. Since $x_n \in C_0$, then there exists $n_1 > n_0$ such that $x_n^{[1]} < 0$, $x_n^{[2]} > 0$ for $n \geq n_1$. From (1.1), we obtain

$$\Delta(c_n [\Delta(d_n \Delta x_n)]^\gamma) + K q_n x_{g(n)}^\gamma \leq 0, \quad n \geq n_2. \tag{2.3}$$

Since x_n is positive and decreasing it follows that $\lim_{n\to\infty} x_n = b \geq 0$. Now we claim that $b = 0$. If not then $x_{g(n)}^\gamma \to b^\gamma > 0$ as $n \to \infty$, and hence there exists $n_2 \geq n_1$ such that $x_{g(n)}^\gamma \geq b^\gamma$. Therefore from (2.3), we have

$$\Delta(c_n [\Delta(d_n \Delta x_n)]^\gamma) + K q_n b^\gamma \leq 0, \quad n \geq n_2.$$

Define the sequence $u_n = c_n [\Delta(d_n \Delta x_n)]^\gamma$ for $n \geq n_2$. Then $\Delta u_n \leq -A q_n$, where $A = K b^\gamma > 0$. Summing the last inequality from n_2 to $n-1$, we get $u_n \leq u_{n_2} - A \sum_{s=n_2}^{n-1} q_s$. In view of (h_3), it is possible to choose an integer n_3 sufficiently large such that $u_n \leq -(A/2) \sum_{s=n_2}^{n-1} q_s$ for all $n \geq n_3$. Hence

$$[\Delta(d_n \Delta x_n)]^\gamma \leq -\frac{A}{2} \frac{1}{c_n} \sum_{s=n_2}^{n-1} q_s.$$

Summing the last inequality from n_3 to $n-1$, we obtain

$$d_n \Delta x_n \leq d_{n_3} \Delta x_{n_3} - \left(\frac{A}{2}\right)^{\frac{1}{\gamma}} \sum_{t=n_3}^{n-1} \left(\frac{1}{c_t} \sum_{s=n_2}^{t-1} q_s \right)^{\frac{1}{\gamma}}.$$

Since $\Delta x_n < 0$ for $n \geq n_0$, the last inequality implies that

$$\Delta x_n \leq -\left(\frac{A}{2}\right)^{\frac{1}{\gamma}} \frac{1}{d_n} \sum_{t=n_3}^{n-1} \left(\frac{1}{c_t} \sum_{s=n_2}^{t-1} q_s \right)^{\frac{1}{\gamma}}.$$

Summing from n_3 to $n-1$, we have

$$x_n \leq x_{n_3} - \left(\frac{A}{2}\right)^{\frac{1}{\gamma}} \sum_{l=n_3}^{n-1} \frac{1}{d_l} \sum_{t=n_3}^{l-1} \left(\frac{1}{c_t} \sum_{s=n_2}^{t-1} q_s \right)^{\frac{1}{\gamma}}.$$

Condition (h_3) implies that $x_n \to -\infty$ as $n \to \infty$ which is a contradiction with the fact that x_n is positive. Then $b = 0$ and this completes the proof. ∎

To prove the next lemma we will use the functions $h_k(n,s)$ which are define by

$$h_k(n,s) := \frac{(n-s)^{(k)}}{k!}, \quad k = 0, 1, 2, ..., \tag{2.4}$$

where $t^{(k)} = t(t-1)\cdots(t-k+1)$ is the so-called falling function (cf. Kelley and Peterson [11]). The summation and difference of the functions in (2.4) are defined by

$$h_{k+1}(n,s) = \sum_{\tau=s}^{n-1} h_k(\tau,s), \; \Delta_1 h_k(n,s) = h_{k-1}(n,s) \quad \textit{and} \quad \Delta_2 h_k(n,s) = -h_{k-1}(n,s),$$

where Δ_1 denotes the difference with respect to n and Δ_2 denotes the difference with respect to s. As a special case when $n = 2$, we see that $n^{(2)} = n(n-1)$ and we can prove easily that $\Delta n^{(2)} = 2n$. Also one can easily see that $\Delta(1/n^{(2)}) = -2/(n+1)^{(3)}$ and then deduce that $\sum_{s=n}^{\infty}\left(-2/(n+1)^{(3)}\right) = 1/n^{(2)}$.

This lemma will be used in the proof of delay case.

Lemma 2.5. *Assume that $g(n) \le n$, and*

$$x_n > 0,\ \Delta x_n > 0,\ \Delta^2 x_n > 0, \quad and \quad \Delta^3 x_n < 0, \quad for \quad n \ge n_0. \tag{2.5}$$

Then

$$\liminf_{n\to\infty} \frac{n x_n}{h_2(n,n_0)\Delta x_n} \ge 1, \tag{2.6}$$

and there exists $N > n_0$ such that

$$\frac{\Delta x_{g(n)}}{\Delta x_{n+1}} \ge \frac{g(n)-N}{n+1-N}. \tag{2.7}$$

Proof. First, we prove that (2.6) holds. To do this we define G_n by

$$G_n := (n-N)x_n - \frac{(n-N)^{(2)}}{2}\Delta x_n.$$

Then $G_N = 0$, and

$$\begin{aligned}
\Delta G_n =&(n+1-N)\Delta x_n + x_n - \frac{(n+1-N)^{(2)}}{2}\Delta^2 x_n - (n-N)\Delta x_n \\
=&\Delta x_n + x_n - \frac{(n+1-N)^{(2)}}{2}\Delta^2 x_n \\
=&x_{n+1} - \frac{(n+1-N)^{(2)}}{2}\Delta^2 x_n \\
=&x_{n+1} - \sum_{\tau=N}^{n}(\tau - N)\Delta^2 x_n.
\end{aligned}$$

By the discrete Taylor's Theorem [1, Theorem 1.113] of the sequence f_n,

$$f_n := \sum_{k=0}^{m-1} h_k(n,\alpha)\Delta^k f(\alpha) + \frac{1}{(m-1)!}\sum_{\tau=\alpha}^{n-m} h_{m-1}(n,\tau+1)\Delta^m f(\tau),$$

where $h_n(t,s)$ be as defined by (2.4). Putting $f_n = x_{n+1}$ and $m = 2$, we have

$$\begin{aligned}
x_{n+1} &= \sum_{k=0}^{2-1} h_k(n+1,N)\Delta^k x_N + \frac{1}{(2-1)!}\sum_{\tau=N}^{n+1-2} h_{2-1}(n+1,\tau+1)\Delta^2 x_\tau \\
&= x_N + (n+1-N)\Delta x_N + \sum_{\tau=N}^{n-1} h_1(n+1,\tau+1)\Delta^2 x_\tau \\
&\ge x_N + (n+1-N)\Delta x_N + \Delta^2 x_n \sum_{\tau=N}^{n-1} h_1(n+1,\tau+1),
\end{aligned}$$

since from (2.5) $\Delta^2 x_n$ is decreasing. It would follow that $\Delta G_n > 0$ on $[N,\infty)$ provided, we can prove that

$$\sum_{\tau=N}^{n-1} h_1(n+1,\tau+1) = \sum_{\tau=N}^{n} (\tau - N).$$

To see this, we use the summation by parts formula [1, Theorem 1.77],

$$\sum_{\tau=a}^{b} f(\tau+1)\Delta g(\tau) = f(\tau)g(\tau)]_a^{b+1} - \sum_{\tau=a}^{b} \Delta f(\tau)g(\tau),$$

to get

$$\sum_{\tau=N}^{n} h_1(n+1,\tau+1) = h_1(n+1,\tau)(\tau-N)_{\tau=N}^{\tau=n+1} - \sum_{\tau=N}^{n}(-1)(\tau-N) = \sum_{\tau=N}^{n}(\tau-N),$$

which is the desired result. Hence $\Delta G_n > 0$ for $n \geq N$. Since $G_N = 0$, we get that $G_n > 0$ for $n \geq N$. This implies that

$$\frac{(n-N)x_n}{h_2(n,N)\Delta x_n} \geq 1, \quad \text{for } n \geq N. \tag{2.8}$$

Therefore, since

$$\frac{nx_n}{h_2(n,n_0)\Delta x_n} = \frac{(n-N)x_n}{h_2(n,N)\Delta x_n}\frac{n}{n-N}\frac{h_2(n,N)}{h_2(n,n_0)},$$

and since

$$\lim_{n\to\infty}\frac{n}{n-N} = 1 = \lim_{n\to\infty}\frac{h_2(n,N)}{h_2(n,n_0)},$$

we get that

$$\liminf_{n\to\infty}\frac{nx(n)}{h_2(n,n_0)\Delta x_n} \geq 1.$$

which proves (2.6). Next, we prove that (2.7) holds. From (2.5), since $\Delta^2 x_n$ is decreasing, we have

$$\Delta x_n - \Delta x_N \geq \Delta^2 x_n (n-N).$$

Dividing by $\Delta x_n \Delta x_{n+1}$, we get

$$\frac{\Delta x_n - \Delta x_N - \Delta^2 x_n(n-N)}{\Delta x_n \Delta x_{n+1}} \geq 0.$$

This implies that

$$\Delta\left(\frac{n-N}{\Delta x_n}\right) \geq 0,$$

which proves that $((n-N)/\Delta x_n)$ is a nondecreasing function. Then, since $g(n) \leq n < n+1$, we have

$$\frac{(n+1-N)}{\Delta x_{n+1}} \geq \frac{(g(n)-N)}{\Delta x_{g(n)}}.$$

Hence

$$(\Delta x_{g(n)})^\gamma \geq \left(\frac{(g(n)-N)}{(n+1-N)}\right)^\gamma (\Delta x_{n+1})^\gamma,$$

which proves (2.7). The proof is complete. ∎

3. Main oscillation results

In this section, we consider that case when (1.25) holds and establish some sufficient conditions which guarantee that the solution x_n of (1.1) oscillates or satisfies $\lim_{n\to\infty} x_n = 0$. In view of Lemma 2.2, it is clear that if x_n is a solution of (1.1), then the solution $x_n \in C_0 \cup C_2$.

3.1. The case when $g(n) > n$

To simplify the presentation of the results, we introduce the following notations:

$$Q_n = Kq_n \left(\frac{D_n \quad (c_n)^{\frac{1}{\gamma}} C_n}{(c_n)^{\frac{1}{\gamma}} C_n + 1} \right)^{\gamma}, \qquad C_n := \sum_{s=N}^{n-1} c_s^{-\frac{1}{\gamma}}, \qquad D_n = \sum_{s=n}^{g(n)-1} \frac{1}{d_s}.$$

$$r := \liminf_{n\to\infty} \frac{n^{\gamma} w_{n+1}}{c_n}, \qquad R := \limsup_{n\to\infty} \frac{n^{\gamma} w_{n+1}}{c_n},$$

$$q_* := \liminf_{n\to\infty} \frac{1}{n} \sum_{s=N}^{n-1} \frac{s^{\gamma+1}}{c_s} Q_s, \qquad p_* := \liminf_{n\to\infty} \frac{n^{\gamma}}{c_n} \sum_{s=n+1}^{\infty} Q_s,$$

Theorem 3.1. *Assume that (h_1), (h_2) and (1.25) hold. Furthermore assume that $g(n) > n$. Let x_n be a solution of (1.1) such that $x_n \in C_2$ and make the Riccati substitution*

$$w_n := \frac{x_n^{[2]}}{\left(x_n^{[1]}\right)^{\gamma}}. \tag{3.1}$$

Then $w_n > 0$, and

$$\Delta w_n + Q_n + \frac{\gamma}{(c_n)^{\frac{1}{\gamma}}} (w_{n+1})^{1+\frac{1}{\gamma}} \le 0, \text{ for } \; n \in [N, \infty). \tag{3.2}$$

Proof. Let x_n be as in the statement of this Theorem and without loss of generality, we may assume that there is $n_1 > n_0$ such that $x_n > 0$ and $x_{g(n)} > 0$. Since $x_n \in C_2$, then there exists $N > n_1$ such that $x_n^{[0]} > 0$, $x_n^{[1]} > 0$, $x_n^{[2]} > 0$, $x_n^{[3]} \le 0$. By the difference quotient rule, we have

$$\begin{aligned}
\Delta w_n =& \Delta \frac{x^{[2]}}{\left(x^{[1]}\right)^{\gamma}} = \frac{\left(x_n^{[1]}\right)^{\gamma} x_n^{[3]} - \Delta \left(x_n^{[1]}\right)^{\gamma} x_n^{[2]}}{(x_n^{[1]})^{\gamma} (x_{n+1}^{[1]})^{\gamma}} \\
=& \frac{x_n^{[3]} \left(x_{g(n)}^{[0]}\right)^{\gamma}}{\left(x_{g(n)}^{[0]}\right)^{\gamma} \left(x_{n+1}^{[1]}\right)^{\gamma}} - \frac{\Delta \left(x_n^{[1]}\right)^{\gamma} x_n^{[2]}}{(x_n^{[1]})^{\gamma} (x_{n+1}^{[1]})^{\gamma}} \\
\le& -Kq_n \frac{\left(x_{g(n)}^{[0]}\right)^{\gamma}}{\left(x_{n+1}^{[1]}\right)^{\gamma}} - \frac{\Delta \left(x_n^{[1]}\right)^{\gamma} x_n^{[2]}}{(x_n^{[1]})^{\gamma} (x_{n+1}^{[1]})^{\gamma}}.
\end{aligned} \tag{3.3}$$

Using the inequality ([9, p. 39]),

$$\gamma x^{\gamma-1}(x-y) \ge x^{\gamma} - y^{\gamma} \ge \gamma y^{\gamma-1}(x-y) \;\text{ for all x} \ne y \text{ and } \gamma \ge 1, \tag{3.4}$$

we have

$$\Delta \left(x_n^{[1]}\right)^{\gamma} = \left(x_{n+1}^{[1]}\right)^{\gamma} - \left(x_n^{[1]}\right)^{\gamma} \ge \gamma \left(x_n^{[1]}\right)^{\gamma-1} \left(\Delta x_n^{[1]}\right), \text{ when } \gamma \ge 1.$$

From the definition of $x_n^{[2]} = c_n\left(\Delta x_n^{[1]}\right)^{\gamma}$, we see that $\Delta x_n^{[1]} = \left(\frac{x_n^{[2]}}{c_n}\right)^{\frac{1}{\gamma}}$. So that

$$\Delta\left(x_n^{[1]}\right)^{\gamma} \geq \gamma\left(x_n^{[1]}\right)^{\gamma-1}\left(\frac{x_n^{[2]}}{c_n}\right)^{\frac{1}{\gamma}}. \tag{3.5}$$

Using the inequality ([9, p. 39]),

$$\gamma y^{\gamma-1}(x-y) \geq x^{\gamma} - y^{\gamma} \geq \gamma x^{\gamma-1}(x-y) \ \text{ for all x} \neq y \text{ and } 0<\gamma\leq 1, \tag{3.6}$$

we have, when $0<\gamma\leq 1$, that

$$\Delta\left(x_n^{[1]}\right)^{\gamma} = \left(x_{n+1}^{[1]}\right)^{\gamma} - \left(x_n^{[1]}\right)^{\gamma} \geq \gamma\left(x_{n+1}^{[1]}\right)^{\gamma-1}\left(\Delta x_n^{[1]}\right) \geq \gamma\left(x_{n+1}^{[1]}\right)^{\gamma-1}\left(\frac{x_n^{[2]}}{c_n}\right)^{\frac{1}{\gamma}}. \tag{3.7}$$

From (3.5) and (3.7), since $x_n^{[1]}$ is increasing and $x^{[2]}$ is decreasing, we get

$$\begin{aligned}
\frac{\Delta\left(x_n^{[1]}\right)^{\gamma} x_n^{[2]}}{\left(x_n^{[1]}\right)^{\gamma}\left(x_{n+1}^{[1]}\right)^{\gamma}} &\geq \frac{\gamma x_n^{[2]}(x_n^{[2]})^{\frac{1}{\gamma}}}{(c_n)^{\frac{1}{\gamma}}\left(x_n^{[1]}\right)\left(x_{n+1}^{[1]}\right)^{\gamma}} \\
&\geq \frac{\gamma\left(x_{n+1}^{[2]}\right)\left(x_{n+1}^{[2]}\right)^{\frac{1}{\gamma}}}{(c_n)^{\frac{1}{\gamma}}\left(x_{n+1}^{[1]}\right)\left(x_{n+1}^{[1]}\right)^{\gamma}} \\
&= \frac{\gamma}{(c_n)^{\frac{1}{\gamma}}}(w_{n+1})^{\frac{1}{\gamma}+1}, \text{ for } \gamma>0.
\end{aligned}$$

Substituting in (3.3), we have

$$\Delta w_n \leq -Kq_n\left(\frac{x_{g(n)}}{x_{n+1}^{[1]}}\right)^{\gamma} - \frac{\gamma}{(c_n)^{\frac{1}{\gamma}}}(w_{n+1})^{1+\frac{1}{\gamma}}. \tag{3.8}$$

Next, we consider the coefficient of q_n in (3.8). Since $x_{n+1}^{[1]} = x_n^{[1]} + \Delta(x_n^{[1]})$, we have

$$x_{n+1}^{[1]}/x_n^{[1]} = 1 + \Delta(x^{[1]})/x_n^{[1]} = 1 + c_n^{-\frac{1}{\gamma}}\left(x_n^{[2]}\right)^{\frac{1}{\gamma}}/x_n^{[1]}.$$

Also since $x_n^{[2]}$ is decreasing, we get

$$x_n^{[1]} = x_N^{[1]} + \sum_{s=N}^{n-1}\left(x_s^{[2]}\right)^{\frac{1}{\gamma}}\frac{1}{(c_s)^{\frac{1}{\gamma}}} \geq x_N^{[1]} + \left(x_n^{[2]}\right)^{\frac{1}{\gamma}}\sum_{s=N}^{n-1}\frac{1}{(c_s)^{\frac{1}{\gamma}}} > \left(x_n^{[2]}\right)^{\frac{1}{\gamma}}\sum_{s=N}^{n-1}\frac{1}{(c_s)^{\frac{1}{\gamma}}}. \tag{3.9}$$

It follows that

$$x_n^{[1]}/\left(x_n^{[2]}\right)^{\frac{1}{\gamma}} \geq \sum_{s=N}^{n-1}\frac{1}{(c_s)^{\frac{1}{\gamma}}} = C_n. \tag{3.10}$$

Hence

$$(x_{n+1}^{[1]})/x_n^{[1]} = 1 + \left(\Delta(x^{[1]})/(x_n^{[1]})\right) = 1 + \left(\frac{1}{(c_n)^{\frac{1}{\gamma}}}\left(x_n^{[2]}\right)^{\frac{1}{\gamma}}/x_n^{[1]}\right) \leq \frac{(c_n)^{\frac{1}{\gamma}}C_n + 1}{(c_n)^{\frac{1}{\gamma}}C_n}.$$

Hence, we have

$$\frac{x_n^{[1]}}{(x_{n+1}^{[1]})} \geq \frac{(c_n)^{\frac{1}{\gamma}} C_n}{(c_n)^{\frac{1}{\gamma}} C_n + 1}.$$

So that

$$\frac{x_{g(n)}}{x_{n+1}^{[1]}} = \left(\frac{x_{g(n)}}{x_n^{[1]}}\right)\left(\frac{x_n^{[1]}}{x_{n+1}^{[1]}}\right) \geq \left(\frac{x_{g(n)}}{x_n^{[1]}}\right)\frac{(c_n)^{\frac{1}{\gamma}} C_n}{(c_n)^{\frac{1}{\gamma}} C_n + 1}. \tag{3.11}$$

Now, since $g(n) > n$ and $x_n^{[1]}$ is increasing, we have

$$x_{g(n)} > x_{g(n)} - x_n = \sum_{s=n}^{g(n)-1} \Delta x_s = \sum_{s=n}^{g(n)-1} \frac{x_s^{[1]}}{d_s} \geq x_n^{[1]} \sum_{s=n}^{g(n)-1} \frac{1}{d_s} = x_n^{[1]} D_n.$$

This and (3.11) show that

$$\frac{x_{g(n)}}{x_{n+1}^{[1]}} \geq \frac{D_n (c_n)^{\frac{1}{\gamma}} C_n}{(c_n)^{\frac{1}{\gamma}} C_n + 1}. \tag{3.12}$$

Substituting from (3.12) into (3.8), we have the inequality (3.2) and this completes the proof. ∎

In order for the definition of p_* to make sense, we assume that

$$\sum_{s=n_0}^{\infty} Q_s < \infty, \tag{3.13}$$

which is different from the assumption that has been posed in all the above mentioned results in the introduction.

Now, we are ready to state and prove the main oscillation theorem in the advanced case.

Theorem 3.2. *Assume that $(h_1) - (h_3)$, and (1.25) hold. Furthermore assume that $g(n) > n$, and $\Delta c_n \geq 0$. Let x_n be a solution of (1.1). If*

$$p_* > \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}, \tag{3.14}$$

or

$$p_* + q_* > 2^{\gamma(\gamma+1)}. \tag{3.15}$$

Then either x_n oscillates or $\lim_{n\to\infty} x_n = 0$.

Proof. Suppose the contrary and assume that x_n is a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that $x_n > 0$, $x_{g(n)} > 0$ for $n \geq n_1$ where n_1 is chosen so large. We consider only this case, because the proof when $x_n < 0$ is similar, since $f(-u) = -f(u)$. From Lemma 2.2, since (1.25) holds, we see that $x_n \in C_0 \cup C_2$. If $x_n \in C_0$, then since (h_3) holds, we are back to the proof of Lemma 2.4 to show that $\lim_{n\to\infty} x_n = 0$. Next, we consider the case when $x_n \in C_2$ and define the sequence w_n be as given by (3.1) in Theorem 3.1. Then from Theorem 3.1, there exists $n_2 > n_1$ such that $w_n > 0$ and satisfies the difference inequality

$$\Delta w_n \leq -Q_n - \frac{\gamma}{(c_n)^{\frac{1}{\gamma}}} (w_{n+1})^{1+\frac{1}{\gamma}} \text{ for } n \geq n_2. \tag{3.16}$$

Also from Theorem 3.1, since

$$x_n^{[1]} > \left(x_n^{[2]}\right)^{\frac{1}{\gamma}} \sum_{s=N}^{n-1} \frac{1}{(c_s)^{\frac{1}{\gamma}}},$$

we see that

$$w_n := \frac{x_n^{[2]}}{\left(x_n^{[1]}\right)^{\gamma}} < \left(\sum_{s=N}^{n-1} \frac{1}{(c_s)^{\frac{1}{\gamma}}}\right)^{-\gamma}.$$

Then from (1.25), we have $\lim_{n\to\infty} w_n = 0$. First, we assume that (3.14) holds. Summing (3.16) from $n+1$ to ∞ and using that $\lim_{n\to\infty} w_n = 0$, we get

$$w_{n+1} \geq \sum_{n+1}^{\infty} Q_s + \gamma \sum_{n+1}^{\infty} c_s^{-\frac{1}{\gamma}} (w_{s+1})^{\frac{1}{\gamma}} w_{s+1}. \tag{3.17}$$

It follows from (3.17) that

$$\frac{n^{\gamma} w_{n+1}}{c_n} \geq \frac{n^{\gamma}}{c_n} \sum_{n+1}^{\infty} Q_s + \gamma \frac{n^{\gamma}}{c_n} \sum_{n+1}^{\infty} \frac{1}{(c_s)^{\frac{1}{\gamma}}} (w_{s+1})^{\frac{1}{\gamma}} w_{s+1}. \tag{3.18}$$

Let $\varepsilon > 0$, then by the definition of p_* and r we can pick $N \geq n_2$, sufficiently large, so that

$$\frac{n^{\gamma}}{c_n} \sum_{n+1}^{\infty} Q_s \geq p_* - \varepsilon, \quad \text{and} \quad \frac{n^{\gamma} w_{n+1}}{c_n} \geq r - \varepsilon, \text{ for } n \geq N. \tag{3.19}$$

From (3.18) and (3.19) and using the fact $\Delta c_n \geq 0$, we get that

$$\begin{aligned} \frac{n^{\gamma} w_{n+1}}{c_n} &\geq (p_* - \varepsilon) + \gamma \frac{n^{\gamma}}{c_n} \sum_{n+1}^{\infty} \frac{c_s}{s^{\gamma+1}} \frac{s(w_{s+1})^{\frac{1}{\gamma}}}{(c_s)^{\frac{1}{\gamma}}} \frac{s^{\gamma} w_{s+1}}{c_s} \\ &\geq (p_* - \varepsilon) + (r - \varepsilon)^{1+\frac{1}{\gamma}} \frac{n^{\gamma}}{c_n} \sum_{n+1}^{\infty} \frac{\gamma c_s}{s^{\gamma+1}} \\ &\geq (p_* - \varepsilon) + (r - \varepsilon)^{1+\frac{1}{\gamma}} n^{\gamma} \sum_{n+1}^{\infty} \frac{\gamma}{s^{\gamma+1}}. \end{aligned} \tag{3.20}$$

Using the inequality (3.4), we have

$$\Delta\left(\frac{-1}{s^{\gamma}}\right) = \frac{(s+1)^{\gamma} - s^{\gamma}}{s^{\gamma}(s+1)^{\gamma}} \leq \frac{\gamma(s+1)^{\gamma-1}}{s^{\gamma}(s+1)^{\gamma}} = \frac{\gamma}{s^{\gamma}(s+1)} < \frac{\gamma}{s^{\gamma+1}}, \gamma \geq 1. \tag{3.21}$$

Using the inequality (3.6), we have

$$\Delta\left(\frac{-1}{s^{\gamma}}\right) = \frac{(s+1)^{\gamma} - s^{\gamma}}{s^{\gamma}(s+1)^{\gamma}} \leq \frac{\gamma(s)^{\gamma-1}}{s^{\gamma}(s+1)^{\gamma}} = \frac{\gamma}{s(s+1)^{\gamma}} < \frac{\gamma}{s^{\gamma+1}}, 0 < \gamma < 1.$$

So that for $\gamma > 0$, we have

$$\sum_{n+1}^{\infty} \frac{\gamma}{s^{\gamma+1}} > \sum_{n+1}^{\infty} \Delta\left(\frac{-1}{s^{\gamma}}\right) = \frac{1}{(n+1)^{\gamma}}. \tag{3.22}$$

Then from (3.20), (3.21) and (3.22), we obtain

$$\frac{n^{\gamma} w_{n+1}}{c_n} \geq (p_* - \varepsilon) + (r - \varepsilon)^{1+\frac{1}{\gamma}} \left(\frac{n}{n+1}\right)^{\gamma}, \text{ for } \gamma > 0.$$

Taking the lim inf of both sides as $n \to \infty$, we get that

$$r \geq p - \varepsilon + (r - \varepsilon)^{1+\frac{1}{\gamma}}.$$

Since $\varepsilon > 0$ is arbitrary, we get

$$p_* \leq r - r^{1+\frac{1}{\gamma}}. \tag{3.23}$$

Using the fact that

$$u - u^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}},$$

we have

$$p_* \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}},$$

which contradicts (3.14). Next, we assume (3.15) holds. Multiplying both sides of (3.16) by $\frac{n^{\gamma+1}}{c_n}$, and summing from N to $n-1$ $(n-1 \geq N)$, we get

$$\sum_{s=N}^{n-1} \frac{s^{\gamma+1}}{c_s}\Delta w_s \leq -\sum_{s=N}^{n-1} \frac{s^{\gamma+1}}{c_s} Q_s - \gamma \sum_{s=N}^{n-1} \left(\frac{s^\gamma w_{s+1}}{c_s}\right)^{\frac{\gamma+1}{\gamma}}.$$

Using summation by parts, we obtain

$$\frac{n^{\gamma+1} w_n}{c_n} \leq \frac{N^{\gamma+1} w_N}{c_N} + \sum_{s=N}^{n-1} \Delta\left(\frac{s^{\gamma+1}}{c_s}\right) w_{s+1} - \sum_{s=N}^{n} \frac{s^{\gamma+1}}{c_s} Q_s - \gamma \sum_{s=N}^{n-1} \left(\frac{s^\gamma w_{s+1}}{c_s}\right)^{\frac{\gamma+1}{\gamma}}.$$

By the quotient rule, we have

$$\Delta\left(\frac{s^{\gamma+1}}{c_s}\right) = \frac{\Delta(s^{\gamma+1})}{c_{s+1}} - \frac{s^{\gamma+1}\Delta c_s}{c_s c_{s+1}} \leq \frac{(\gamma+1)(s+1)^\gamma}{c_{s+1}} \leq \frac{(\gamma+1)(s+1)^\gamma}{c_s}. \tag{3.24}$$

Hence

$$\frac{n^{\gamma+1} w_n}{c_n} \leq \frac{N^{\gamma+1} w_N}{c_N} - \sum_{s=N}^{n-1} \frac{s^{\gamma+1}}{c_s} Q_s + \sum_{s=N}^{n-1} (\gamma+1)\left(\frac{(s+1)^\gamma w_{s+1}}{c_s}\right)$$
$$-\gamma \sum_{s=N}^{n-1} \left(\frac{s^\gamma w_{s+1}}{c_s}\right)^{\frac{\gamma+1}{\gamma}}.$$

Now, since $s > n_0 > 0$ we can assume for s sufficiently large that $(s+1) \leq Ls < 2s$. Using this and the last inequality, we obtain

$$\frac{n^{\gamma+1} w_n}{c_n} \leq \frac{N^{\gamma+1} w_N}{c_N} - \sum_{s=N}^{n-1} \frac{s^{\gamma+1}}{c_s} Q_s + \sum_{s=N}^{n-1} \left\{(\gamma+1)L^\gamma W_{s+1} - \gamma W_{s+1}^{\frac{\gamma+1}{\gamma}}\right\}.$$

where $W_{s+1} := \frac{s^\gamma w_{s+1}}{c_s}$. Using the inequality

$$Bu - Au^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma},$$

we have

$$\frac{n^{\gamma+1} w_n}{c_n} \leq \frac{N^{\gamma+1} w_N}{c_N} - \sum_{s=N}^{n-1} \frac{s^{\gamma+1}}{c_s} Q_s + \sum_{s=N}^{n-1} \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{[(\gamma+1)L^\gamma]^{\gamma+1}}{\gamma^\gamma}$$

$$=\frac{N^{\gamma+1}w_N}{c_N}-\sum_{s=N}^{n-1}\frac{s^{\gamma+1}}{c_s}Q_s+L^{\gamma(\gamma+1)}(n-N).$$

It follows from this that

$$\frac{n^{\gamma}w_n}{c_n}\le\frac{N^{\gamma+1}w_N}{nc_N}-\frac{1}{n}\sum_{s=N}^{n-1}\frac{s^{\gamma+1}}{c_s}Q_s+L^{\gamma(\gamma+1)}\left(1-\frac{N}{n}\right).$$

Since $w_{n+1}\le w_n$, we get

$$\frac{n^{\gamma}w_{n+1}}{c_n}\le\frac{N^{\gamma+1}w_N}{nc_N}-\frac{1}{n}\sum_{s=N}^{n-1}\frac{s^{\gamma+1}}{c_s}Q_s+L^{\gamma(\gamma+1)}\left(1-\frac{N}{n}\right).$$

Taking the lim sup of both sides as $n\to\infty$, we obtain

$$R\le -q_*+L^{\gamma(\gamma+1)}=-q_*+L^{\gamma(\gamma+1)},$$

which implies that

$$R\le -q_*+2^{\gamma(\gamma+1)}.$$

Using this and the inequality (3.23), we get

$$p_*\le r-r^{1+\frac{1}{\gamma}}\le r\le R\le -q_*+2^{\gamma(\gamma+1)}.$$

Therefore

$$p_*+q_*\le 2^{\gamma(\gamma+1)},$$

which contradicts (3.15). The proof is complete. ∎

From Theorem 3.2, we have the following results immediately.

Corollary 3.1. *Assume that $(h_1)-(h_3)$ and (1.25) hold. Furthermore assume that $g(n)>n$, and $\Delta c_n\ge 0$. Let x_n be a solution of (1.1). If*

$$\liminf_{n\to\infty}\frac{n^{\gamma}}{c_n}\sum_{s=n+1}^{\infty}Q_s>2^{\gamma(\gamma+1)}, \tag{3.25}$$

then either x_n oscillates or $\lim_{n\to\infty}x_n=0$.

Corollary 3.2. *Assume that $(h_1)-(h_3)$ and (1.25) hold. Furthermore assume that $g(n)>n$, and $\Delta c_n\ge 0$. Let x_n be a solution of (1.1). If*

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{s=N}^{n}\frac{s^{\gamma+1}}{c_s}Q_s>2^{\gamma(\gamma+1)}, \tag{3.26}$$

then either x_n oscillates or $\lim_{n\to\infty}x_n=0$.

3.2. The case when $g(n)\le n$ and $d_n=1$

For the delay case we introduce the following notations:

$$A_*:=\liminf_{n\to\infty}\frac{n^{\gamma}}{c_n}\sum_{s=n+1}^{\infty}A_s,\quad B_*:=\liminf_{n\to\infty}\frac{1}{n}\sum_{s=N}^{n-1}\frac{s^{\gamma+1}}{c_s}A_s,\quad A_n=\quad Kq_n\left(\frac{h_2(g(n),n_0)}{n+1}\right)^{\gamma}.$$

If x_n is a solution of (1.1) such that $x_n\in C_2$, $d_n=1$ and $\Delta c_n\ge 0$, then we can deduce that if $x_n>0$, then

$$\Delta x_n>0,\quad \Delta^2x_n>0,\quad \text{and } \Delta^3x_n<0, \tag{3.27}$$

and the quasi differences in this case defined by

$$y_n^{[0]} = x_n > 0,\, y_n^{[1]} = \Delta x_n,\, y_n^{[2]} = c_n\left[\Delta^2 x_n\right]^{\gamma},\;\; y_n^{[3]} = \Delta(y_n^{[2]}).$$

In order for the definition of A_* to make sense, we assume that

$$\sum_{s=n+1}^{\infty} A_s < \infty.$$

Theorem 3.3. *Assume that* $(h_1)-(h_2)$ *and (1.25) hold. Furthermore assume that* $d_n = 1$, $\Delta c_n \geq 0$, *and* $g(n) \leq n$. *Let* x_n *be a solution of (1.1) such that* $x_n \in C_2$ *and make the Riccati substitution*

$$u_n := \frac{y_n^{[2]}}{\left(y_n^{[1]}\right)^{\gamma}}.$$

Then $u_n > 0$, *and*

$$\Delta u_n + A_n + \frac{\gamma}{(c_n)^{\frac{1}{\gamma}}}(u_{n+1})^{1+\frac{1}{\gamma}} \leq 0, \;\textit{for}\;\; n \geq N. \tag{3.28}$$

Proof. Let x_n be as in the statement of this Theorem and without loss of generality, we may assume that there is $n_1 > n_0$ such that $x_n > 0$ and $x_{g(n)} > 0$. Now, since $x_n \in C_2$ then there exists $N > n_1$ such that $x_n > 0$, $y^{[1]} = \Delta x_n > 0$, $y_n^{[2]} = c_n\left[\Delta^2 x_n\right]^{\gamma} > 0$, $y_n^{[3]} \leq 0$. Since, $\Delta c_n \geq 0$ then (3.27) is satisfied. From the definition of u_n, by quotient rule and continue as in the proof of Theorem 3.1, we get

$$\Delta u_n \leq -Kq_n\left(\frac{x_{g(n)}}{y_{n+1}^{[1]}}\right)^{\gamma} - \frac{\gamma}{(c_n)^{\frac{1}{\gamma}}}(u_{n+1})^{1+\frac{1}{\gamma}}. \tag{3.29}$$

Now we consider the coefficient of q_n in (3.29). This coefficient can be written in the form

$$\frac{x_{g(n)}}{y_{n+1}^{[1]}} = \frac{x_{g(n)}}{y_{g(n)}^{[1]}}\frac{y_{g(n)}^{[1]}}{y_{n+1}^{[1]}}. \tag{3.30}$$

From Lemma 2.5, since $\lim_{t\to\infty} g(n) = \infty$, we can choose $N_k \geq N$ such that

$$\frac{x_{g(n)}}{y_{g(n)}^{[1]}} = \frac{g(n)x_{g(n)}}{\Delta x_{g(n)}} \geq \sqrt{k}\frac{h_2(g(n),n_0)}{g(n)},\text{ for } n > N_k, \tag{3.31}$$

and

$$\frac{y_{g(n)}^{[1]}}{y_{n+1}^{[1]}} = \frac{\Delta x_{g(n)}}{\Delta x_{n+1}} \geq \frac{1}{\sqrt{k}}\frac{g(n)}{(n+1)},\text{ for } 0 < k < 1. \tag{3.32}$$

Then from (3.30)–(3.32), we have

$$\frac{x_{g(n)}}{\Delta x_{n+1}} \geq \frac{h_2(g(n),n_0)}{g(n)}\frac{g(n)}{n+1} = \frac{h_2(g(n),n_0)}{(n+1)}. \tag{3.33}$$

Substituting from (3.33) into (3.29), we have the inequality (3.28) and this completes the proof. ∎

The following theorem gives sufficient conditions for oscillation of (1.1) in the delay case.

Theorem 3.4. *Assume that* $(h_1)-(h_3)$ *and (1.25) hold. Furthermore assume that* $d_n = 1$, $\Delta c_n \geq 0$, $f(u)/u^{\gamma} \geqslant K > 0$ *and* $g(n) \leq n$. *Let* x_n *be a solution of (1.1). If*

$$A_* > \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}, \tag{3.34}$$

or

$$A_* + B_* > 2^{\gamma(\gamma+1)}. \tag{3.35}$$

then x_n *is oscillatory or* $\lim_{n\to\infty} x_n = 0$.

Proof. The proof is similar to the proof of Theorem 3.2, by replacing w_n by u_n, and Q_n by A_n and hence is omitted. ∎

From Theorem 3.4, we have the following results.

Corollary 3.3. *Assume that* $(h_1)-(h_3)$ *and (1.25) hold. Furthermore assume that* $d_n = 1$, $\Delta c_n \geq 0$, *and* $g(n) \leq n$. *Let* x_n *be a solution of (1.1). If*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{s=N}^{n-1} \frac{s^{\gamma+1}}{c_s} A_s > 2^{\gamma(\gamma+1)}. \tag{3.36}$$

Then x_n *is oscillatory or* $\lim_{n\to\infty} x_n = 0$.

Corollary 3.4. *Assume that* $(h_1)-(h_3)$ *and (1.25) hold. Furthermore assume that* $d_n = 1$, $\Delta c_n \geq 0$, *and* $g(n) \leq n$. *Let* x_n *be a solution of (1.1). If*

$$\liminf_{n\to\infty} \frac{n^{\gamma}}{c_n} \sum_{s=n+1}^{\infty} A_s > 2^{\gamma(\gamma+1)}. \tag{3.37}$$

Then x_n *is oscillatory or* $\lim_{n\to\infty} x_n = 0$.

For more illustration, we consider the following example with explicit values of the roots of the characteristic equation.

Example 3.1. Consider the difference equation

$$\Delta^3 x_n + \frac{8n+12}{n(1+n^2)} x_n(1+x_n^2) = 0, \quad \text{where } g(n) = n \text{ for } n \geq 1. \tag{3.38}$$

It is clear that $(h_1)-(h_3)$ and (3.36) hold. Then the conditions of Corollary 3.3 are satisfied and then the solution x_n of (3.38) is oscillatory or converges to zero. In fact $x_n = (-1)^n n$ is such a solution.

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