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On the Logarithmic Integral and Convolutions

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Abstract. The logarithmic integral $\mathrm{li}(x)$ and its associated functions $\mathrm{li}_+(x)$ and $\mathrm{li}_-(x)$ are defined as locally summable functions on the real line. Some commutative neutrix convolutions of these functions and other functions are then found.

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1. Introduction

The *logarithmic integral* $\mathrm{li}(x)$, (see Abramowitz and Stegun [1]) is defined by

$$\mathrm{li}(x) = \begin{cases} \displaystyle\int_0^x \frac{dt}{\ln|t|}, & \text{for } |x| < 1, \\ \displaystyle\mathrm{PV}\int_0^x \frac{dt}{\ln t}, & \text{for } x > 1, \\ \displaystyle\mathrm{PV}\int_0^x \frac{dt}{\ln|t|}, & \text{for } x < -1 \end{cases}$$
$$= \begin{cases} \displaystyle\int_0^x \frac{dt}{\ln|t|}, & \text{for } |x| < 1, \\ \displaystyle\lim_{\varepsilon\to 0^+}\left[\int_0^{1-\varepsilon} \frac{dt}{\ln t} + \int_{1+\varepsilon}^x \frac{dt}{\ln t}\right], & \text{for } x > 1, \\ \displaystyle\lim_{\varepsilon\to 0^+}\left[\int_0^{-1+\varepsilon} \frac{dt}{\ln|t|} + \int_{-1-\varepsilon}^x \frac{dt}{\ln|t|}\right], & \text{for } x < -1 \end{cases}$$

where PV denotes the Cauchy principal value of the integral. We will therefore write

$$\mathrm{li}(x) = \mathrm{PV}\int_0^x \frac{dt}{\ln|t|}$$

for all values of x. The associated functions $\mathrm{li}_+(x)$ and $\mathrm{li}_-(x)$ are now defined by

$$\mathrm{li}_+(x) = H(x)\,\mathrm{li}(x), \quad \mathrm{li}_-(x) = H(-x)\,\mathrm{li}(x),$$

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where $H(x)$ denotes Heaviside's function. The distribution $\ln^{-1}|x|$ is defined by

$$\ln^{-1}|x| = \text{li}'(x)$$

and its associated distributions $\ln^{-1} x_+$ and $\ln^{-1} x_-$ are defined by

$$\ln^{-1} x_+ = H(x)\ln^{-1}|x| = \text{li}'_+(x), \quad \ln^{-1} x_- = H(-x)\ln^{-1}|x| = \text{li}'_-(x).$$

The classical definition of the convolution product of two functions f and g is as follows:

Definition 1.1. *Let f and g be functions. Then the convolution $f * g$ is defined by*

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt$$

for all points x for which the integral exist.

It follows easily from the definition that if $f * g$ exists then $g * f$ exists and $f * g = g * f$. Further, if $(f * g)'$ and $f * g'$ (or $f' * g$) exists, then

$$(f * g)' = f * g' \quad (\text{or } f' * g). \tag{1.1}$$

Definition 1.1 can be extended to define the convolution $f * g$ of two distributions f and g in $\mathscr{D}'$, the space of infinitely differentiable functions with compact support, see Gel'fand and Shilov [6].

Definition 1.2. *Let f and g be distributions in $\mathscr{D}'$. Then the convolution $f * g$ is defined by the equation*

$$\langle (f * g)(x), \varphi \rangle = \langle f(y), \langle g(x), \varphi(x+y) \rangle \rangle$$

for arbitrary φ in $\mathscr{D}$, provided f and g satisfy either of the conditions

(a) *either f or g has bounded support,*
(b) *the supports of f and g are bounded on the same side.*

Note that if f and g are locally summable functions satisfying either of the above conditions and the classical convolution $f * g$ exists, then it is in agreement with Definition 1.1.

The following convolutions identities were proved in [5] for $r = 0, 1, 2, \ldots$.

$$\text{li}_+(x) * x_+^r = \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{r-i+1} x^i \,\text{li}_+(x^{r-i+2}), \tag{1.2}$$

$$\ln^{-1} x_+ * x_+^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} x^i \,\text{li}_+(x^{r-i+1}), \tag{1.3}$$

$$\text{li}_-(x) * x_-^r = \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{r-i+2} x^i \,\text{li}_-(x^{r-i+2}), \tag{1.4}$$

$$\ln^{-1} x_- * x_-^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i+1} x^i \,\text{li}_-(x^{r-i+1}). \tag{1.5}$$

The convolution product of distributions may be defined without any restriction on the supports. One of the most known is the definition given by Vladimirov. But, several other definitions of the convolution product are equivalent to that of Vladimirov. However, the convolution product in the sense of any of these definitions does not exist for many pairs of distributions.

In [3] the *commutative neutrix convolution product* is defined so that it exists for a considerably large class of pairs of distributions. In that definition, unit-sequences of function in $\mathscr{D}$ are used which allows one to approximate a given distribution by a sequence of distributions of bounded support.

To recall the definition of the commutative neutrix convolution product we first let τ be a function in $\mathscr{D}$, see [7], satisfying the conditions:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \leq \tau(x) \leq 1$,
(iii) $\tau(x) = 1$ for $|x| \leq \frac{1}{2}$,
(iv) $\tau(x) = 0$ for $|x| \geq 1$.

For $n = 1, 2, \ldots.$ the function τ_n is then defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \leq n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n. \end{cases}$$

We have the following definition of the commutative neutrix convolution product.

Definition 1.3. *Let f and g be distributions in $\mathscr{D}'$ and let $f_n = f\tau_n$ and $g_n = g\tau_n$ for $n = 1, 2, \ldots.$ Then the commutative neutrix convolution product $f \boxast g$ is defined as the neutrix limit of the sequence $\{f_n * g_n\}$, provided that the limit h exists in the sense that*

$$\underset{\nu\to\infty}{\text{N}-\lim} \langle f_n * g_n, \varphi \rangle = \langle h, \varphi \rangle$$

for all φ in $\mathscr{D}$, where N is the neutrix (see van der Corput [2]*), having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range N'' the real numbers, with negligible functions finite linear sums of the functions*

$$n^\lambda \ln^{r-1} n,\ \ln^r n, \quad (\lambda \neq 0,\ r = 1, 2, \ldots)$$

and all functions which converge to zero in the usual sense as n tends to infinity.

Note that in this definition the convolution product $f_n * g_n$ is in the sense of Definition 1.1, the distribution f_n and g_n having bounded support since the support of τ_n is contained in the interval $[-n - n^{-n}, n + n^{-n}]$. This neutrix convolution product is also commutative.

It is obvious that any results proved with the original definition hold with the new definition. The following theorem (proved in [3]) therefore hold, the first showing that the commutative neutrix convolution product is a generalization of the convolution product. So the idea of a neutrix lies in neglecting certain numerical sequences diverging to $\pm\infty$, which makes a wider the class of pairs of distributions f and g for which the product exists. It should be noted that, in general, the definition of a commutative neutrix convolution product depends on the choice of the sequence τ_n as well as the set of negligible sequences.

Theorem 1.1. *Let f and g be distributions in $\mathscr{D}'$ satisfying either condition (a) or condition (b) of Gel'fand and Shilov's definition. Then the commutative neutrix convolution product $f \boxast g$ then exists and $f \boxast g = f * g$.*

Note however that $(f \boxast g)'$ is not necessarily equal to $f' \boxast g$ but we do have the following theorem proved in [4].

Theorem 1.2. *Let f and g be distributions in $\mathscr{D}'$ and suppose that the commutative neutrix convolution product $f \boxast g$ exists. If* $\underset{\nu\to\infty}{\mathrm{N-lim}}\langle (f\tau_n') * g_n, \varphi\rangle$ *exists and equals $\langle h, \varphi\rangle$ for all φ in $\mathscr{D}$, then $f' \boxast g$ exists and $(f \boxast g)' = f' \boxast g + h$.*

2. Main result

In the following, we need to extend our set of negligible functions to include finite linear sums of the functions $n^s \operatorname{li}(n^r)$ and $n^s \ln^{-r} n$, $(n > 1)$ for $s = 0, 1, 2, \dots$ and $r = 1, 2, \dots$. Also, we need the following lemma proved in [5].

Lemma 2.1.

$$\operatorname{li}(x^r) = PV \int_0^x \frac{t^{r-1}dt}{\ln|t|}. \tag{2.1}$$

The logarithm integral satisfies:

Lemma 2.2.

$$\lim_{n\to\infty} \int_n^{n+n^{-n}} \tau_n(t)\operatorname{li}(t)(x-t)^r\,dt = 0 \quad \textit{for} \quad r = 1, 2, \dots. \tag{2.2}$$

Lemma 2.3.

$$\underset{n\to\infty}{\mathrm{N-lim}} \operatorname{li}[(x+n)^r] = 0, \quad \textit{for} \quad r = 1, 2, \dots. \tag{2.3}$$

$$\underset{n\to\infty}{\mathrm{N-lim}}\, n^r \operatorname{li}[(x+n)] = 0, \quad \textit{for} \quad r = 1, 2, \dots. \tag{2.4}$$

Now we have our main result.

Theorem 2.1. *The commutative neutrix convolution* $\operatorname{li}_+(x) \boxast x^r$ *exists and*

$$\operatorname{li}_+(x) \boxast x^r = 0 \quad \textit{for} \quad r = 0, 1, 2, \dots. \tag{2.5}$$

Proof. We put $[\operatorname{li}_+(x)]_n = \operatorname{li}_+(x)\tau_n(x)$ and $[x^r]_n = x^r\tau_n(x)$ for $n = 1, 2, \dots$. Since this functions has compact support, the classical convolution $[\operatorname{li}_+(x)]_n * [x^r]_n$ exists by Definition 1.1 and we have:

$$[\operatorname{li}_+(x)]_n * [x^r]_n = \int_{-\infty}^{\infty} \operatorname{li}_+(t)(x-t)^r\tau_n(x-t)\tau_n(t)\,dt$$

$$= \int_0^n \operatorname{li}(t)(x-t)^r\tau_n(x-t)\,dt + \int_n^{n+n^{-n}} \operatorname{li}(t)(x-t)^r\tau_n(x-t)\tau_n(t)\,dt$$

$$= I_1 + I_2\,. \tag{2.6}$$

If $0 \le x \le n$, we see that

$$\begin{aligned} I_1 &= \int_0^n \operatorname{li}(t)(x-t)^r\tau_n(x-t)\,dt \\ &= PV \int_0^n (x-t)^r \int_0^t \frac{du}{\ln u}\,dt \\ &= PV \int_0^n \frac{1}{\ln u} \int_u^n (x-t)^r\,dt\,du \\ &= PV \frac{1}{r+1} \sum_{i=0}^{r+1} (-1)^{r-i+1} x^i \binom{r+1}{i} \int_0^n \frac{u^{r-i+1} - n^{r-i+1}}{\ln u}\,du \end{aligned}$$

$$=\frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^{r-i+1}x^i[\mathrm{li}(n^{r-i+2})-n^{r-i+1}\,\mathrm{li}(n)].$$

Thus

$$\mathrm{N-}\lim_{n\to\infty}\int_0^n \mathrm{li}(t)(x-t)^r\tau_n(x-t)\,dt=0. \tag{2.7}$$

Next, if $-n\le x\le 0$, we have

$$\begin{aligned}I_1&=\int_0^n \mathrm{li}(t)(x-t)^r\tau_n(x-t)\,dt\\&=\int_0^{x+n}\mathrm{li}(t)(x-t)^r\,dt+\int_{x+n}^{x+n+n^{-n}}\mathrm{li}(t)(x-t)^r\tau_n(x-t)\,dt,\end{aligned}$$

where

$$\begin{aligned}\int_0^{x+n}\mathrm{li}(t)(x-t)^r\,dt=&PV\int_0^{x+n}(x-t)^r\int_0^t\frac{du}{\ln u}\,dt\\=&PV\int_0^{x+n}\frac{1}{\ln u}\int_u^{x+n}(x-t)^r\,dt\,du\\=&PV\frac{1}{r+1}\sum_{i=0}^{r+1}(-1)^{r-i+1}x^i\binom{r+1}{i}\int_0^{x+n}\frac{u^{r-i+1}}{\ln u}\,du\\&-PV\frac{(-n)^{r+1}}{r+1}\int_0^{x+n}\frac{du}{\ln u}\\=&\frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^{r-i+1}x^i\,\mathrm{li}[(x+n)^{r-i+2}]-\frac{(-n)^{r+1}}{r+1}\,\mathrm{li}(x+n).\end{aligned}$$

Thus, on using lemma 2.3, we have

$$\mathrm{N-}\lim_{n\to\infty}\int_0^{x+n}\mathrm{li}(t)(x-t)^r\,dt=0. \tag{2.8}$$

Further, using lemma 2.2, it is easily seen that

$$\lim_{n\to\infty}\int_{x+n}^{x+n+n^{-n}}\tau_n(x-t)\,\mathrm{li}(t)(x-t)^r\,dt=0 \tag{2.9}$$

and we have from equations (2.7), (2.8) and (2.9) that

$$\mathrm{N-}\lim_{n\to\infty}I_1=0. \tag{2.10}$$

Further, it is easily seen that for each fixed x

$$\lim_{n\to\infty}I_2=\lim_{n\to\infty}\int_n^{n+n^{-n}}\mathrm{li}(t)(x-t)^r\tau_n(x-t)\tau_n(t)\,dt=0 \tag{2.11}$$

and now equation (2.5) follows immediately equations from (2.6), (2.10) and (2.11), proving the theorem. ∎

Corollary 2.1. *The neutrix convolution* $\mathrm{li}_-(x)\boxed{*}x^r$ *exists and*

$$\mathrm{li}_-(x)\boxed{*}x^r=0,\quad \textit{for}\quad r=0,1,2,\ldots. \tag{2.12}$$

Proof. Equation (2.12) follow immediately on replacing x by $-x$ in equation (2.5). ∎

Corollary 2.2. *The neutrix convolution* $\mathrm{li}(x) \boxast x^r$ *exists and*

$$\mathrm{li}(x) \boxast x^r = 0, \quad \textit{for} \quad r = 0,1,2,\ldots. \tag{2.13}$$

Proof. Equation (2.13) follow on adding equations (2.12) and (2.5). ∎

Corollary 2.3. *The neutrix convolutions* $\mathrm{li}_+(x) \boxast x_-^r$ *and* $\mathrm{li}_-(x) \boxast x_+^r$ *exist and*

$$\mathrm{li}_+(x) \boxast x_-^r = \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{i+1} x^i \,\mathrm{li}_+(x^{r-i+2}), \quad \textit{for} \quad r = 0,1,2,\ldots. \tag{2.14}$$

$$\mathrm{li}_-(x) \boxast x_+^r = \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{i} x^i \,\mathrm{li}_-(x^{r-i+2}), \quad \textit{for} \quad r = 0,1,2,\ldots. \tag{2.15}$$

Proof. Equation (2.14) follows from equations (1.2) and (2.5) on noting that

$$\mathrm{li}_+(x) \boxast x^r = \mathrm{li}_+(x) \boxast x_+^r + (-1)^r \,\mathrm{li}_+(x) \boxast x_-^r$$

and equation (2.15) follows on replacing x by $-x$ in equation (2.14). ∎

Theorem 2.2. *The commutative neutrix convolution* $\ln^{-1} x_+ \boxast x^r$ *exists and*

$$\ln^{-1} x_+ \boxast x^r = 0, \quad \textit{for} \quad r = 0,1,2,\ldots. \tag{2.16}$$

Proof. Differentiating equation (2.5) and using Theorem 1.2 we get

$$\ln^{-1} x_+ \boxast x^r = \mathop{\mathrm{N-lim}}_{n\to\infty} [\mathrm{li}_+(x)\tau_n'(x)] * (x^r)_n \tag{2.17}$$

where, on integration by parts we have

$$\begin{aligned}
[\mathrm{li}_+(x)\tau_n'(x)] * (x^r)_n &= \int_n^{n+n^{-n}} \mathrm{li}(t)(x-t)^r \tau_n(x-t)\, d\tau_n(t) \\
&= -\,\mathrm{li}(n)(x-n)^r \tau_n(x-t) - \int_n^{n+n^{-n}} \ln^{-1}(t)(x-t)^r \tau_n(x-t)\tau_n(t)\, dt \\
&\quad + r\int_n^{n+n^{-n}} \mathrm{li}(t)(x-t)^{r-1} \tau_n(t)\tau_n(x-t)\, dt \\
&\quad + \int_n^{n+n^{-n}} \mathrm{li}(t)(x-t)^r \tau_n(t)\tau_n'(x-t)\, dt.
\end{aligned} \tag{2.18}$$

Now $\tau_n(x-n)$ is either 0 or 1 for large enough n and so

$$\mathop{\mathrm{N-lim}}_{n\to\infty} \mathrm{li}(n)(x-n)^r \tau_n(x-n) = 0. \tag{2.19}$$

∎

Next, it is clear that

$$\lim_{n\to\infty} \int_n^{n+n^{-n}} \ln^{-1}(t)(x-t)^r \tau_n(t)\tau_n(x-t)\, dt = 0, \tag{2.20}$$

$$\lim_{n\to\infty} \int_n^{n+n^{-n}} \mathrm{li}(t)(x-t)^{r-1} \tau_n(t)\tau_n(x-t)\, dt = 0. \tag{2.21}$$

Noting that $\tau_n'(x-t) = 0$ for large enough n and $x \neq 0$, it follows that

$$\lim_{n\to\infty} \int_n^{n+n^{-n}} \mathrm{li}(t)(x-t)^r \tau_n(t)\tau_n'(x-t)\, dt = 0.$$

If $x = 0$, then

$$\begin{aligned}\int_n^{n+n^{-n}} \mathrm{li}(t)(x-t)^r \tau_n(t)\tau_n'(-t)\,dt &= \frac{1}{2}\int_n^{n+n^{-n}} \mathrm{li}(t)(x-t)^r d\tau_n^2(t)\,dt \\ &= \frac{1}{2}\,\mathrm{li}(n)(x-n)^r \\ &\quad + \int_n^{n+n^{-n}} \left[\ln^{-1}(t)(x-t) - r\,\mathrm{li}(t)\right](x-t)^{r-1}\tau_n^2(t)\,dt\end{aligned}$$

and it follows that

$$\mathrm{N}\!-\!\lim_{n\to\infty} \int_n^{n+n^{-n}} \mathrm{li}(t)(x-t)^r \tau_n(t)\tau_n'(-t)\,dt = 0. \tag{2.22}$$

It now follows from the equation (2.18) to (2.22) that

$$\mathrm{N}\!-\!\lim_{n\to\infty} [\mathrm{li}_+(x)\tau_n'(x)] * [x^r]_n = 0,$$

proving the theorem.

Corollary 2.4. *The neutrix convolution* $\ln^{-1} x_- \boxed{*} x^r$ *exists and*

$$\ln^{-1} x_- \boxed{*} x^r = 0 \quad \text{for} \quad r = 0,1,2,\ldots. \tag{2.23}$$

Proof. Equation (2.23)follows immediately on replacing x by $-x$ in equation (2.16). ∎

Corollary 2.5. *The neutrix convolution* $\ln^{-1}(x) \boxed{*} x^r$ *exists and*

$$\ln^{-1}|x| \boxed{*} x^r = 0. \quad r = 0,1,2,\ldots. \tag{2.24}$$

Proof. Equation (2.24) follows on adding equations (2.23) and (2.16). ∎

Corollary 2.6. *The neutrix convolutions* $\ln_+^{-1}(x) \boxed{*} x_-^r$ *and* $\ln_-^{-1}(x) \boxed{*} x_+^r$ *exist and*

$$\ln^{-1} x_+ \boxed{*} x_-^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} x^i\,\mathrm{li}_+(x^{r-i+1}), \quad \text{for} \quad r = 0,1,2,\ldots. \tag{2.25}$$

$$\ln^{-1} x_- \boxed{*} x_+^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} x^i\,\mathrm{li}_-(x^{r-i+1}) \quad \text{for} \quad r = 0,1,2,\ldots, \tag{2.26}$$

Proof. Equation (2.25) follows from equations (1.3) and (2.16) on noting that

$$\ln^{-1} x_+ \boxed{*} x^r = \ln^{-1} x_+ \boxed{*} x_+^r + (-1)^r \ln^{-1} x_+ \boxed{*} x_-^r$$

and equation (2.26) follows on replacing x by $-x$ in equation (2.25). ∎

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