BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **35**(3) (2012), 679–686

An Integral-Type Operator from H^∞ to Zygmund-Type Spaces

XIANGLING ZHU
Department of Mathematics, Jiaying University, Meizhou, Guangdong, 514015, China
xiangling-zhu@163.com; jyuzxl@163.com

Abstract. Let $g \in H(\mathbb{D})$, n be a nonnegative integer and φ be an analytic self-map of $\mathbb{D}$. We study the boundedness and compactness of the integral operator $C^n_{\varphi,g}$ defined by

$$(C^n_{\varphi,g}f)(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)d\xi, \quad z \in \mathbb{D},\ f \in H(\mathbb{D}),$$

from H^∞ to Zygmund-type spaces on the unit disk.

2010 Mathematics Subject Classification: Primary: 47B38; Secondary: 30H05

Keywords and phrases: Integral-type operator, composition operator, Zygmund-type space.

1. Introduction

A positive continuous function ϕ on $[0,1)$ is called normal if there exist positive numbers s and t, $0 < s < t$, and $\delta \in [0,1)$ such that

$$\frac{\phi(r)}{(1-r)^s} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\phi(r)}{(1-r)^s} = 0;$$

$$\frac{\phi(r)}{(1-r)^t} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\phi(r)}{(1-r)^t} = \infty$$

(see, e.g. [11]). From now on we always assume that ω and μ are normal functions and non-negative functions on $[0,1)$ such that $\omega(t_n), \mu(t_n) > 0$ for some sequence $\{t_n\}_0^\infty \subset [0,1)$ with $\lim_{n\to\infty} t_n = 1$.

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, and $H(\mathbb{D})$ be the class of all analytic functions on $\mathbb{D}$. We denote by $H^\infty = H^\infty(\mathbb{D})$ the bounded analytic function space on $\mathbb{D}$. An $f \in H(\mathbb{D})$ is said to belong to the Zygmund-type space, denoted by $\mathscr{Z}_\mu$, if

$$\sup_{z\in\mathbb{D}} \mu(|z|)|f''(z)| < \infty.$$

Under the norm

$$\|f\|_{\mathscr{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \mu(|z|)|f''(z)|,$$

Communicated by Mohammad Sal Moslehian.
Received: August 23, 2010; *Revised:* October 12, 2010.

it is easy to see that $\mathscr{Z}_\mu$ is a Banach space. The little Zygmund-type space $\mathscr{Z}_{\mu,0}$ is defined to be the subspace of $\mathscr{Z}_\mu$ consisting of those $f \in \mathscr{Z}_\mu$ such that

$$\lim_{|z|\to 1} \mu(|z|)|f''(z)| = 0.$$

When $\mu(r) = (1-r^2)$, the induced spaces $\mathscr{Z}_\mu$ and $\mathscr{Z}_{\mu,0}$ become the classical Zygmund space and the little Zygmund space respectively (see [2,4,6]).

Let φ be an analytic self-map of $\mathbb{D}$. The composition operator C_φ is defined by

$$(C_\varphi f)(z) = f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

It will be of interest to provide a function theoretic characterization of when φ induces a bounded or compact composition operator between spaces of analytic functions. The composition operator has been studied by many researchers on various spaces (see, e.g., [1, 18] and the references therein).

Let $g \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. In [6], the authors defined and studied the generalized composition operator as follows

$$(C_\varphi^g f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi, \quad f \in H(\mathbb{D}),\ z \in \mathbb{D}.$$

The boundedness and compactness of the generalized composition operator on Zygmund spaces and Bloch spaces were investigated in [6]. In [3], Li studied another type Volterra composition operator between weighted Bergman spaces and Bloch spaces. In [22], the author of this paper generalized the operator C_φ^g to the unit ball and studied the boundedness and compactness of the corresponding operator on some function spaces. Some related results can be found, e.g., in [5, 7–9, 12–17, 19–22].

Here we generalize the generalized composition operator C_φ^g from another point of view. Let $g \in H(\mathbb{D})$, n be a nonnegative integer and φ be an analytic self-map of $\mathbb{D}$. We define

$$(C_{\varphi,g}^n f)(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)d\xi, \quad z \in \mathbb{D},\ f \in H(\mathbb{D}).$$

When $n = 1$, then $C_{\varphi,g}^1$ is the generalized composition operator defined by Li and Stević in [6]. When $n = 0$, then $C_{\varphi,g}^0$ is the Volterra composition operator defined by Li in [3]. To the best of our knowledge, the operator $C_{\varphi,g}^n$ is studied in the present paper for the first time.

The purpose of this paper is to study the operator $C_{\varphi,g}^n$. The boundedness and compactness of the operator $C_{\varphi,g}^n$ from H^∞ to Zygmund-type spaces are completely characterized. Throughout the paper, C denotes a positive constant which may differ from one occurrence to the other.

2. Main results and proofs

In this section, we give the main results and proofs. Before proving the main results, it is necessary to give some lemmas. By standard arguments (see for e.g. [1, Proposition 3.11]), the following lemma follows.

Lemma 2.1. *Let $g \in H(\mathbb{D})$, n be a nonnegative integer and φ be an analytic self-map of $\mathbb{D}$. Then $C_{\varphi,g}^n : H^\infty \to \mathscr{Z}_\mu$ is compact if and only if $C_{\varphi,g}^n : H^\infty \to \mathscr{Z}_\mu$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in H^∞ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, we have $\|C_{\varphi,g}^n f_k\|_{\mathscr{Z}_\mu} \to 0$ as $k \to \infty$.*

Lemma 2.2. *A closed set K in $\mathscr{Z}_{\mu,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}\mu(|z|)|f''(z)|=0. \tag{2.1}$$

Proof. The proof is similar to that of [10, Lemma 1], and the details are omitted here. ∎

By the Cauch integral formula, we have

Lemma 2.3. *Let $f\in H^\infty$. Then for each $m\in\mathbb{N}$, there is a positive constant C independent of f such that*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^m|f^{(m)}(z)|\le C\|f\|_\infty. \tag{2.2}$$

Now we are in a position to state and prove the main results of this paper.

Theorem 2.1. *Let $g\in H(\mathbb{D})$, n be a nonnegative integer and φ be an analytic self-map of $\mathbb{D}$. Then $C^n_{\varphi,g}:H^\infty\to\mathscr{Z}_\mu$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}}\frac{\mu(|z|)|\varphi'(z)||g(z)|}{(1-|\varphi(z)|^2)^{n+1}}<\infty\quad\text{and}\quad\sup_{z\in\mathbb{D}}\frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^n}<\infty. \tag{2.3}$$

Proof. Suppose that $C^n_{\varphi,g}:H^\infty\to\mathscr{Z}_\mu$ is bounded, i.e., there exists a constant C such that $\|C^n_{\varphi,g}f\|_{\mathscr{Z}_\mu}\le C\|f\|_\infty$ for all $f\in H^\infty$. Taking $f(z)=z^n$ and $f(z)=z^{n+1}$, and using the boundedness of the function $\varphi(z)$, we get

$$\sup_{z\in\mathbb{D}}\mu(|z|)|g'(z)|<\infty, \tag{2.4}$$

and

$$\sup_{z\in\mathbb{D}}\mu(|z|)|g(z)||\varphi'(z)|<\infty. \tag{2.5}$$

For $w\in\mathbb{D}$, set

$$h_w(z)=\frac{1-|w|^2}{1-\overline{w}z}-\frac{1}{n+1}\frac{(1-|w|^2)^2}{(1-\overline{w}z)^2}.$$

It is easy to check that $h_w\in H^\infty$, $\|h_w\|_\infty<(2n+6)/(n+1)$ for every $w\in\mathbb{D}$,

$$h^{(n)}_{\varphi(\lambda)}(\varphi(\lambda))=0\qquad\text{and}\qquad|h^{(n+1)}_{\varphi(\lambda)}(\varphi(\lambda))|=\frac{n!|\varphi(\lambda)|^{n+1}}{(1-|\varphi(\lambda)|^2)^{n+1}}.$$

It follows that

$$\infty>\|C^n_{\varphi,g}h_{\varphi(\lambda)}\|_{\mathscr{Z}_\mu}\ge\frac{n!\mu(|\lambda|)|g(\lambda)||\varphi'(\lambda)||\varphi(\lambda)|^{n+1}}{(1-|\varphi(\lambda)|^2)^{n+1}} \tag{2.6}$$

for every $\lambda\in\mathbb{D}$.

For any fixed $r\in(0,1)$, from (2.6) we have

$$\begin{aligned}\sup_{|\varphi(\lambda)|>r}\frac{\mu(|\lambda|)|g(\lambda)||\varphi'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{n+1}}&\le\sup_{|\varphi(\lambda)|>r}\frac{1}{r^{n+1}}\frac{\mu(|\lambda|)|g(\lambda)||\varphi'(\lambda)||\varphi(\lambda)|^{n+1}}{(1-|\varphi(\lambda)|^2)^{n+1}}\\&\le C\|C^n_{\varphi,g}\|_{H^\infty\to\mathscr{Z}_\mu}<\infty.\end{aligned} \tag{2.7}$$

By (2.5),

$$\sup_{|\varphi(\lambda)|\le r}\frac{\mu(|\lambda|)|g(\lambda)||\varphi'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{n+1}}\le\frac{1}{(1-r^2)^{n+1}}\sup_{|\varphi(\lambda)|\le r}\mu(|\lambda|)|g(\lambda)||\varphi'(\lambda)|<\infty. \tag{2.8}$$

Therefore, (2.7) and (2.8) yield the first inequality of (2.3).

Next, set $f_w(z) = (1-|w|^2)/(1-\overline{w}z)$. Then $f_w \in H^\infty$ and $\sup_{w\in\mathbb{D}} \|f_w\|_\infty \le 2$. Hence,

$$\begin{aligned}
\infty > 2\|C^n_{\varphi,g}\|_{H^\infty\to\mathscr{Z}_\mu} &\ge \|C^n_{\varphi,g} f_{\varphi(\lambda)}\|_{\mathscr{Z}_\mu} \\
&\ge \sup_{z\in\mathbb{D}} \mu(|z|)|(C^n_{\varphi,g} f_{\varphi(\lambda)})''(z)| \\
&= \sup_{z\in\mathbb{D}} \mu(|z|)\left| f^{(n+1)}_{\varphi(\lambda)}(\varphi(z))g(z)\varphi'(z) + f^{(n)}_{\varphi(\lambda)}(\varphi(z))g'(z)\right| \\
&\ge \mu(|\lambda|)\left| \frac{n!g'(\lambda)(\overline{\varphi(\lambda)})^n}{(1-|\varphi(\lambda)|^2)^n} + \frac{(n+1)!g(\lambda)\varphi'(\lambda)(\overline{\varphi(\lambda)})^{n+1}}{(1-|\varphi(\lambda)|^2)^{n+1}}\right| \\
&\ge \mu(|\lambda|)\left| \frac{n!g'(\lambda)(\overline{\varphi(\lambda)})^n}{(1-|\varphi(\lambda)|^2)^n}\right| - \mu(|\lambda|)\left| \frac{(n+1)!g(\lambda)\varphi'(\lambda)(\overline{\varphi(\lambda)})^{n+1}}{(1-|\varphi(\lambda)|^2)^{n+1}}\right| \\
&= \frac{n!\mu(|\lambda|)|g'(\lambda)||\varphi(\lambda)|^n}{(1-|\varphi(\lambda)|^2)^n} - \frac{(n+1)!\mu(|\lambda|)|g(\lambda)||\varphi'(\lambda)||\varphi(\lambda)|^{n+1}}{(1-|\varphi(\lambda)|^2)^{n+1}}
\end{aligned}$$

for every $\lambda \in \mathbb{D}$. Therefore

$$(2.9)\quad \frac{\mu(|\lambda|)|g'(\lambda)||\varphi(\lambda)|^n}{(1-|\varphi(\lambda)|^2)^n} \le \frac{2}{n!}\|C^n_{\varphi,g}\|_{H^\infty\to\mathscr{Z}_\mu} + \frac{(n+1)\mu(|\lambda|)|g(\lambda)||\varphi'(\lambda)||\varphi(\lambda)|^{n+1}}{(1-|\varphi(\lambda)|^2)^{n+1}}.$$

From (6) and (9), we get

$$(2.10)\quad \sup_{\lambda\in\mathbb{D}} \frac{\mu(|\lambda|)|g'(\lambda)||\varphi(\lambda)|^n}{(1-|\varphi(\lambda)|^2)^n} < \infty.$$

Combining (2.10) with (2.4), similar to the former proof, we get the second inequality of (2.3).

For the converse, suppose that (2.3) holds. For any $f \in H^\infty$, by Lemma 2.3, we have

$$\begin{aligned}
\mu(|z|)|(C^n_{\varphi,g} f)''(z)| &= \mu(|z|)|(f^{(n)}(\varphi)g)'(z)| \\
&\le \mu(|z|)|g(z)||\varphi'(z)||f^{(n+1)}(\varphi(z))| + \mu(|z|)|g'(z)||f^{(n)}(\varphi(z))| \\
&\le C\frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}}\|f\|_\infty + C\frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^n}\|f\|_\infty.
\end{aligned}$$

Moreover, $|(C^n_{\varphi,g} f)(0)| = 0$ and

$$|(C^n_{\varphi,g} f)'(0)| = |f^{(n)}(\varphi(0))g(0)| \le \frac{|g(0)|}{(1-|\varphi(0)|^2)^n}\|f\|_\infty.$$

From (2.3), we see that

$$\|C^n_{\varphi,g} f\|_{\mathscr{Z}_\mu} = |(C^n_{\varphi,g} f)(0)| + |(C^n_{\varphi,g} f)'(0)| + \sup_{z\in\mathbb{D}} \mu(|z|)|(C^n_{\varphi,g} f)'(z)| < \infty.$$

Therefore $C^n_{\varphi,g} : H^\infty \to \mathscr{Z}_\mu$ is bounded. The proof of the theorem is complete. ∎

Theorem 2.2. *Let $g \in H(\mathbb{D})$, n be a nonnegative integer and φ be an analytic self-map of $\mathbb{D}$. Then $C^n_{\varphi,g} : H^\infty \to \mathscr{Z}_\mu$ is compact if and only if $C^n_{\varphi,g} : H^\infty \to \mathscr{Z}_\mu$ is bounded,*

$$(2.11)\quad \lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}} = 0 \quad \textit{and} \quad \lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^n} = 0.$$

Proof. Suppose that $C^n_{\varphi,g}: H^\infty \to \mathscr{Z}_\mu$ is compact. Then $C^n_{\varphi,g}: H^\infty \to \mathscr{Z}_\mu$ is bounded. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Set

$$h_k(z) = \frac{1-|\varphi(z_k)|^2}{1-\overline{\varphi(z_k)}z} - \frac{1}{n+1}\frac{(1-|\varphi(z_k)|^2)^2}{(1-\overline{\varphi(z_k)}z)^2}, \quad k \in \mathbb{N}.$$

Notice that h_k is a sequence in H^∞ and converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$,

$$h_k^{(n)}(\varphi(z_k)) = 0 \quad \text{and} \quad |h_k^{(n+1)}(\varphi(z_k))| = \frac{n!|\varphi(z_k)|^{n+1}}{(1-|\varphi(z_k)|^2)^{n+1}}.$$

The compactness of $C^n_{\varphi,g}: H^\infty \to \mathscr{Z}_\mu$ implies $\lim_{k\to\infty} \|C^n_{\varphi,g}h_k\|_{\mathscr{Z}_\mu} = 0$. On the other hand, similar to the proof of Theorem 2.1, we have

$$n!\frac{\mu(|z_k|)|g(z_k)||\varphi'(z_k)||\varphi(z_k)|^{n+1}}{(1-|\varphi(z_k)|^2)^{n+1}} \le \|C^n_{\varphi,g}h_k\|_{\mathscr{Z}_\mu},$$

i.e. we get

$$\lim_{k\to\infty}\frac{\mu(|z_k|)|g(z_k)||\varphi'(z_k)||\varphi(z_k)|^{n+1}}{(1-|\varphi(z_k)|^2)^{n+1}} = \lim_{k\to\infty}\|C^n_{\varphi,g}h_k\|_{\mathscr{Z}_\mu} = 0.$$

Therefore

$$\begin{aligned}\text{(2.12)} \qquad \lim_{|\varphi(z_k)|\to 1}\frac{\mu(|z_k|)|g(z_k)||\varphi'(z_k)|}{(1-|\varphi(z_k)|^2)^{n+1}} &= \lim_{|\varphi(z_k)|\to 1}\frac{\mu(|z_k|)|g(z_k)||\varphi'(z_k)||\varphi(z_k)|^{n+1}}{(1-|\varphi(z_k)|^2)^{1+n}} \\ &= \lim_{k\to\infty}\frac{\mu(|z_k|)|g(z_k)||\varphi'(z_k)||\varphi(z_k)|^{n+1}}{(1-|\varphi(z_k)|^2)^{n+1}} = 0.\end{aligned}$$

Next, set

$$f_k(z) = \frac{1-|\varphi(z_k)|^2}{1-\overline{\varphi(z_k)}z}, \quad k \in \mathbb{N}.$$

Then $f_k \in H^\infty$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Since $C^n_{\varphi,g}: H^\infty \to \mathscr{Z}_\mu$ is compact, we have $\lim_{k\to\infty}\|C^n_{\varphi,g}f_k\|_{\mathscr{Z}_\mu} = 0$. On the other hand, we have

$$\|C^n_{\varphi,g}f_k\|_{\mathscr{Z}_\mu} \ge \frac{n!\mu(|z_k|)|g'(z_k)||\varphi(z_k)|^n}{(1-|\varphi(z_k)|^2)^n} - \frac{(n+1)!\mu(|z_k|)|g(z_k)||\varphi'(z_k)||\varphi(z_k)|^{n+1}}{(1-|\varphi(z_k)|^2)^{n+1}},$$

which implies that

$$\lim_{|\varphi(z_k)|\to 1}\frac{(n+1)\mu(|z_k|)|g(z_k)||\varphi'(z_k)||\varphi(z_k)|^{n+1}}{(1-|\varphi(z_k)|^2)^{1+n}} = \lim_{|\varphi(z_k)|\to 1}\frac{\mu(|z_k|)|g'(z_k)||\varphi(z_k)|^n}{(1-|\varphi(z_k)|^2)^n},$$

if one of these two limits exists. From the last equality and (2.12), we have

$$\text{(2.13)} \qquad \lim_{|\varphi(z_k)|\to 1}\frac{\mu(|z_k|)|g'(z_k)|}{(1-|\varphi(z_k)|^2)^n} = \lim_{|\varphi(z_k)|\to 1}\frac{\mu(|z_k|)|g'(z_k)||\varphi(z_k)|^n}{(1-|\varphi(z_k)|^2)^n} = 0.$$

From (2.12) and (2.13) we obtain the desired results.

Conversely, suppose that $C^n_{\varphi,g}: H^\infty \to \mathscr{Z}_\mu$ is bounded and (2.11) holds. Assume $(f_k)_{k\in\mathbb{N}}$ is a sequence in H^∞ such that f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. By the assumption, for any $\varepsilon > 0$, there exists a $\delta \in (0,1)$,

$$\text{(2.14)} \qquad \frac{\mu(|z|)|\varphi'(z)||g(z)|}{(1-|\varphi(z)|^2)^{n+1}} < \varepsilon \quad \text{and} \quad \frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^n} < \varepsilon,$$

when $\delta < |\varphi(z)| < 1$. By the boundedness of $C^n_{\varphi,g} : H^\infty \to \mathscr{Z}_\mu$ and the proof of Theorem 2.1,

$$C_1 = \sup_{z\in\mathbb{D}} \mu(|z|)|g'(z)| < \infty \quad \text{and} \quad C_2 = \sup_{z\in\mathbb{D}} \mu(|z|)|g(z)||\varphi'(z)| < \infty. \tag{2.15}$$

Let $K = \{z \in \mathbb{D} : |\varphi(z)| \le \delta\}$. Then by (2.14) and (2.15), we have that

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} \mu(|z|)|(C^n_{\varphi,g} f_k)''(z)| \\
&\le \sup_{z\in K} \mu(|z|)|g(z)||\varphi'(z)||f_k^{(n+1)}(\varphi(z))| + \sup_{z\in K} \mu(|z|)|g'(z)||f_k^{(n)}(\varphi(z))| \\
&\quad + C \sup_{z\in\mathbb{D}\setminus K} \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}} \|f_k\|_\infty + C \sup_{z\in\mathbb{D}\setminus K} \frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^n} \|f_k\|_\infty \\
&\le C_2 \sup_{z\in K} |f_k^{(n+1)}(\varphi(z))| + C_1 \sup_{z\in K} |f_k^{(n)}(\varphi(z))| + C\varepsilon \|f_k\|_\infty,
\end{aligned}$$

i.e. we get

$$\begin{aligned}
\|C^n_{\varphi,g} f_k\|_{\mathscr{Z}_\mu} &\le C_2 \sup_{|w|\le\delta} |f_k^{(n+1)}(w)| + C_1 \sup_{|w|\le\delta} |f_k^{(n)}(w)| \\
&\quad + C\varepsilon \|f_k\|_\infty + |g(0)||f_k^{(n)}(\varphi(0))|.
\end{aligned} \tag{2.16}$$

Since f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, Cauchy's estimate gives that $f_k^{(n)} \to 0$ as $k \to \infty$ on compact subsets of $\mathbb{D}$. Hence, letting $k \to \infty$ in (2.16), and using the fact that ε is an arbitrary positive number, we obtain $\lim_{k\to\infty} \|C^n_{\varphi,g} f_k\|_{\mathscr{Z}_\mu} = 0$. Applying Lemma 2.1 the result follows. ∎

Theorem 2.3. *Let $g \in H(\mathbb{D})$, n be a nonnegative integer and φ be an analytic self-map of $\mathbb{D}$. Then $C^n_{\varphi,g} : H^\infty \to \mathscr{Z}_{\mu,0}$ is compact if and only if*

$$\lim_{|z|\to1} \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}} = 0 \quad \textit{and} \quad \lim_{|z|\to1} \frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^n} = 0. \tag{2.17}$$

Proof. Assume that $C^n_{\varphi,g} : H^\infty \to \mathscr{Z}_{\mu,0}$ is compact. Then $C^n_{\varphi,g} : H^\infty \to \mathscr{Z}_\mu$ is compact and $C^n_{\varphi,g} : H^\infty \to \mathscr{Z}_{\mu,0}$ is bounded. Taking $f(z) = z^n$ and $f(z) = z^{n+1}$, and using the boundedness of $C^n_{\varphi,g} : H^\infty \to \mathscr{Z}_{\mu,0}$ and the function $\varphi(z)$, we get

$$\lim_{|z|\to1} \mu(|z|)|g'(z)| = 0 \tag{2.18}$$

and

$$\lim_{|z|\to1} \mu(|z|)|g(z)||\varphi'(z)| = 0. \tag{2.19}$$

If $\|\varphi\|_\infty < 1$, from (2.18) and (2.19) we get

$$\lim_{|z|\to1} \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}} \le \frac{1}{(1-\|\varphi\|_\infty^2)^{n+1}} \lim_{|z|\to1} \mu(|z|)|g(z)||\varphi'(z)| = 0$$

and

$$\lim_{|z|\to1} \frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^n} \le \frac{1}{(1-\|\varphi\|_\infty^2)^n} \lim_{|z|\to1} \mu(|z|)|g'(z)| = 0.$$

The result follows.

Now we assume that $\|\varphi\|_\infty = 1$. By the compactness of $C^n_{\varphi,g} : H^\infty \to \mathscr{Z}_\mu$ and Theorem 2.2 we have

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}} = 0 \tag{2.20}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^{n}} = 0. \tag{2.21}$$

From (2.18) and (2.21), for any $\varepsilon > 0$, there exists an $r \in (0,1)$ such that

$$\frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^{n}} < \varepsilon$$

when $r < |\varphi(z)| < 1$ and there exists a $\sigma \in (0,1)$ such that

$$\mu(|z|)|g'(z)| \le \varepsilon(1-r^2)^n$$

when $\sigma < |z| < 1$. Therefore, when $\sigma < |z| < 1$ and $r < |\varphi(z)| < 1$, we have

$$\frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^{n}} < \varepsilon. \tag{2.22}$$

On the other hand, when $\sigma < |z| < 1$ and $|\varphi(z)| \le r$, we have

$$\frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^{n}} < \frac{1}{(1-r^2)^n}\mu(|z|)|g'(z)| < \varepsilon. \tag{2.23}$$

Combining (2.22) and (2.23), we obtain the second equality of (2.17). Similar to the above proof we get the first equality of (2.17).

Conversely suppose that (2.17) holds. Let $f \in H^\infty$. We have

$$\mu(|z|)|(C^n_{\varphi,g}f)''(z)| \le C\left(\frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}} + \frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^{n}}\right)\|f\|_\infty.$$

Taking the supremum in this inequality over all $f \in H^\infty$ such that $\|f\|_\infty \le 1$, apply (2.17) we obtain

$$\lim_{|z|\to 1} \sup_{\|f\|_\infty \le 1} \mu(|z|)|(C^n_{\varphi,g}f)''(z)| = 0.$$

The result follows from Lemma 2.2. ∎

References

[1] C. C. Cowen and B. D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC, Boca Raton, FL, 1995.

[2] P. L. Duren, *Theory of H^p spaces*, Pure and Applied Mathematics, Vol. 38 Academic Press, New York, 1970.

[3] S. Li, Volterra composition operators between weighted Bergman spaces and Bloch type spaces, *J. Korean Math. Soc.* **45** (2008), no. 1, 229–248.

[4] S. Li and S. Stević, Volterra-type operators on Zygmund spaces, *J. Inequal. Appl.* **2007**, Art. ID 32124, 10 pp.

[5] S. Li and S. Stević, Products of Volterra type operator and composition operator from H^∞ and Bloch spaces to Zygmund spaces, *J. Math. Anal. Appl.* **345** (2008), no. 1, 40–52.

[6] S. Li and S. Stević, Generalized composition operators on Zygmund spaces and Bloch type spaces, *J. Math. Anal. Appl.* **338** (2008), no. 2, 1282–1295.

[7] S. Li and S. Stević, Weighted composition operators from Zygmund spaces into Bloch spaces, *Appl. Math. Comput.* **206** (2008), no. 2, 825–831.

[8] S. Li and S. Stević, Products of integral-type operators and composition operators between Bloch-type spaces, *J. Math. Anal. Appl.* **349** (2009), no. 2, 596–610.

[9] S. Li and S. Stević, On an integral-type operator from iterated logarithmic Bloch spaces into Bloch-type spaces, *Appl. Math. Comput.* **215** (2009), no. 8, 3106–3115.

[10] K. Madigan and A. Matheson, Compact composition operators on the Bloch space, *Trans. Amer. Math. Soc.* **347** (1995), no. 7, 2679–2687.

[11] A. L. Shields and D. L. Williams, Bounded projections, duality, and multipliers in spaces of analytic functions, *Trans. Amer. Math. Soc.* **162** (1971), 287–302.

[12] S. Stević, On a new integral-type operator from the weighted Bergman space to the Bloch-type space on the unit ball, *Discrete Dyn. Nat. Soc.* **2008**, Art. ID 154263, 14 pp.

[13] S. Stević, On a new operator from the logarithmic Bloch space to the Bloch-type space on the unit ball, *Appl. Math. Comput.* **206** (2008), no. 1, 313–320.

[14] S. Stević, On a new integral-type operator from the Bloch space to Bloch-type spaces on the unit ball, *J. Math. Anal. Appl.* **354** (2009), no. 2, 426–434.

[15] S. Stević, On an integral operator from the Zygmund space to the Bloch-type space on the unit ball, *Glasg. Math. J.* **51** (2009), no. 2, 275–287.

[16] W. Yang, On an integral-type operator between Bloch-type spaces, *Appl. Math. Comput.* **215** (2009), no. 3, 954–960.

[17] W. Yang, Volterra composition operators from $F(p,q,s)$ spaces to Bloch-type spaces, *Bull. Malays. Math. Sci. Soc. (2)* **34** (2011), no. 2, 267–277.

[18] K. H. Zhu, *Operator theory in function spaces*, Monographs and Textbooks in Pure and Applied Mathematics, 139, Dekker, New York, 1990.

[19] X. Zhu, Generalized composition operators and Volterra composition operators on Bloch spaces in the unit ball, *Complex Var. Elliptic Equ.* **54** (2009), no. 2, 95–102.

[20] X. Zhu, Volterra composition operators on logarithmic Bloch spaces, *Banach J. Math. Anal.* **3** (2009), no. 1, 122–130.

[21] X. Zhu, Volterra composition operators from generalized weighted Bergman spaces to μ-Bloch spaces, *J. Funct. Spaces Appl.* **7** (2009), no. 3, 225–240.

[22] X. Zhu, Generalized composition operators from generalized weighted Bergman spaces to Bloch type spaces, *J. Korean Math. Soc.* **46** (2009), no. 6, 1219–1232.