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On One Parameter Semigroup of Self Mappings Uniformly Satisfying Expansive Kannan Condition

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Abstract. The aim of this paper is to prove existence results on fixed points for asymptotically regular uniformly expansive Kannan semigroup of selfmappings (with constant $\beta < \sqrt{2}$) defined on metric spaces equipped with uniform normal structure which further enjoys a kind of intersection property. As Banach spaces also fall in the class of metric spaces with uniform normal, therefore our results can be viewed as metric versions of some earlier results due to Kannan originally proved in reflexive Banach spaces besides generalizing certain previously known results due to Beg and Azam proved in convex metric spaces.

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1. Introduction

It has been a standard practice these days to utilize known fixed point theorems in Banach spaces as means to prove new results in metric spaces. In 1970, Takahashi [12] introduced the notion of convexity in metric spaces and utilize the same to generalize some known fixed point theorems in Banach spaces by proving corresponding fixed point theorems in convex metric spaces. Subsequently Guay *et al.* [4], Talman [11] and some others have proved similar fixed point theorems in convex metric spaces. In 1973, Kannan [5] proved some interesting results on fixed points for a class of mappings which may also include discontinuous maps is now popularly known as Kannan mappings. Motivated by Takahashi [12], Beg and Azam [1] generalized the results of Kannan [5] using convexity (cf. [12]) in metric spaces. Moreover, several results on fixed point property in metric spaces were established which were essentially patterned after Penot's formulation [10]. The compactness of the convexity structure which appears in Penot's formulation (*cf.* [10]) expresses the weak compactness, or more particularly, the reflexivity in the case of Banach spaces. In 1989, Khamsi [7] defined normal and uniform normal structure for metric spaces and proved that a complete bounded metric space (X,d) equipped with uniform normal structure has

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the fixed point property for nonexpansive mappings and also shares a kind of intersection property which extends a result of Maluta [9] to metric spaces.

In this paper, we prove results on the existence of fixed points for expansive Kannan mappings (with Lipschitz's constant $\beta < \sqrt{2}$) in metric spaces equipped with uniform normal structure which is essentially a notion due to Khamsi [7].

2. Preliminaries

In what follows, we collect relevant definitions and results to make our presentation as self-contained as possible. We begin with a metric space (X,d) wherein F stands for a nonempty family of subsets of X. We say that F defines a convexity structure on X if F is stable under intersection. We also say that F has *the Property (R)* if any decreasing sequence $\{C_n\}$ of closed bounded nonempty subsets of X with $C_n \in F$ has a non void intersection. Also recall that a subset of X is said to be admissible (cf.[3]) if it is an intersection of closed balls centered at the points of X. We denote by $A(X)$, the family of all admissible subsets of X which admits a natural convexity structure on X. In this paper any other convexity structure F on X is always assumed to contain $A(X)$. For a bounded subset A of X, we define the admissible hull of A (denoted by $\mathrm{ad}(A)$) as the intersection of all those admissible subsets of X which contain A i.e.

$$\mathrm{ad}(A) = \bigcap \{B : A \subseteq B \subseteq X \ \ with\ \ B\ \ admissible\}.$$

In respect of foregoing definitions and discussions, the following facts are worth noting:

(i) Every Banach space and all of its possible convex subsets are the natural examples of metric spaces equipped with normal structure.

(ii) Every reflexive Banach space enjoys the *Property (R)*.

(iii) Every weakly compact convex subset of a Banach space enjoys the *Property (R)*.

(iv) There exist many metric spaces equipped with a convexity structure which cannot be embedded in any Banach space.

Let M be a bounded subset of a metric space (X,d) wherein $B(x,r)$ stands for a closed ball centered at x with radius r. Following Lim and Xu [8], we adopt the following notations:

$$r(x,M) = \sup\{d(x,y) : y \in M\} \text{ for } x \in X,$$
$$\delta(M) = \sup\{r(x,M) : x \in M\},$$
$$R(M) = \inf\{r(x,M) : x \in M\}.$$

Proposition 2.1. [8] *For a point $x \in X$ and a bounded subset A of X, we have*

$$r(x,\mathrm{ad}(A)) = r(x,A).$$

Motivated by Kannan [5], we can have the following definition.

Definition 2.1. *A map $T : X \to X$ is said to be uniformly expansive Kannan mapping if for each integer $n \geq 1$, there exists a constant $\beta_n > 0$ such that*

$$d(T^n x, T^n y) \leq \beta_n [d(x,T^n x) + d(y,T^n y)] \ \ \forall\ x,y \in X. \tag{2.1}$$

If $\beta_n < \frac{1}{2}$ $\forall\, n \geq 1$, then the map T is called uniformly Kannan.

Definition 2.2. [8] *A metric space (X,d) is said to have the property (P) if for any two bounded sequences $\{x_n\}$ and $\{z_n\}$ in X, one can find some $z \in \bigcap_{n=1}^{\infty} ad\{z_j : j \geq n\}$ such that*

$$\limsup_{n\to\infty} d(z,x_n) \leq \limsup_{j\to\infty} \limsup_{n\to\infty} d(z_j,x_n).$$

Definition 2.3. [6] *A mapping $T : X \to X$ is said to asymptotically regular, if*

$$\lim_{n\to\infty} d(T^{n+1}x, T^n x) = 0 \ \ \forall \ x \in X.$$

In 1973, Kannan proved the following fixed point theorem in reflexive Banach spaces.

Theorem 2.1. [5] *Let T be a self mapping of a nonempty bounded, closed and convex subset K of a reflexive Banach space X wherein T satisfies a condition corresponding to (2.1) with $n = 1$. If $\sup_{y\in F} \|y - Ty\| < \delta(F)$ for every nonempty bounded closed convex subset F of K comprising of more than one points and are mapped back into itself by T, then T has a unique fixed point in K.*

Let G be subsemigroup of $[0,\infty)$ with respect to addition and $T = \{T(t) : t \in G\}$ be a family of self mappings on X. Then T is called a (one-parameter) semigroup on X if the following conditions are satisfied:

(i) $T(0)x = x \ \forall \ x \in X$;
(ii) $T(s+t)x = T(s)(T(t)x) \ \forall \ s,t \in G$ and $x \in X$;
(iii) for all $x \in X$, a mapping $t \to T(t)x$ from G into X is continuous when G is equipped with the relative topology of $[0,\infty)$;
(iv) for each $t \in G$, $T(t) : X \to X$ is continuous.

A semigroup $T = \{T(t) : t \in G\}$ on X is said to be asymptotically regular at a point $x \in X$ if

$$\lim_{t\to\infty} d(T(t+h)x, T(t)x) = 0 \ \ \forall \ h \in \ G.$$

If T is asymptotically regular at each point $x \in X$, then T is called an asymptotically regular semigroup of selfmappings defined on X.

Definition 2.4. *A semigroup $\Im = \{T(t) : t \in G\}$ on X is called a uniformly expansive Kannan semigroup if*

$$\sup\{\beta(t) : t \in G\} = \beta < \infty,$$

where

$$\beta(t) = \sup\{\frac{d(T(t)x, T(t)y)}{(d(x, T(t)x) + d(y, T(t)y)) \neq 0} : \ x, y \in \ X\}. \tag{2.2}$$

Also, if $\sup\{\beta(t) : t \in G\} = \frac{1}{2}$, then $\Im$ is called uniformly Kannan semigroup.

Definition 2.5. *The simplest uniformly Kannan semigroup is a semigroup of iterates of a mapping $T : X \to X$ with*

$$\sup\{\beta_n : n \in N\} = \beta < \infty,$$

$$\beta_n = \sup\{\frac{d(T^n x, T^n y)}{(d(x, T^n x) + d(y, T^n y)) \neq 0} : \ x, y \in \ X\}.$$

Definition 2.6. [7] *A metric space (X,d) is said to have normal (resp. uniform normal) structure if there exists a convexity structure F on X such that $R(A) < \delta(A)$ (resp. $R(A) \leq c.\delta(A)$ for some constant $c \in (0,1)$) for all $A \in F$ which is bounded and always consists of more than one points.*

We define the normal structure coefficient $N(X)$ of X (with respect to a given convexity structure F) as the number

$$\sup\left\{\frac{R(A)}{\delta(A)}\right\}$$

where the supremum is taken over all bounded subsets $A \in F$ with $\delta(A) > 0$. A metric space X is said to have a uniform normal structure if and only if $N(X) < 1$.

Definition 2.7. [13] *Let (X,d) be a metric space and $T = \{T(t) : t \in G\}$ be a semigroup on X. Let us write the set*

$$w(\infty) = \{\{t_n\} : \{t_n\} \subset G \ \text{and} \ t_n \to \infty\}.$$

Definition 2.8. [13] *Let (X,d) be a complete bounded metric space and $T = \{T(t) : t \in G\}$ be a semigroup on X. Then T is said to have the property $(*)$ if for each $x \in X$ and each $\{t_n\} \in w(\infty)$, the following conditions are satisfied:*

(a) *the sequence $\{T(t_n)x\}$ is bounded;*
(b) *for any sequence $\{z_n\}$ in $ad\{T(t_n)x : n \geq 1\}$ there exists some $z \in \bigcap_{n=1}^{\infty} ad\{z_j : j \geq n\}$ such that*

$$\limsup_{n\to\infty} d(z, T(t_n)x) \leq \limsup_{j\to\infty} \limsup_{n\to\infty} d(z_j, T(t_n)x).$$

Yao and Zeng [13] proved the following lemma which will be instrumental in the proof of our main result.

Lemma 2.1. *Let (X,d) be a complete bounded metric space equipped with uniform normal structure and $T = \{T(t) : t \in G\}$ be a semigroup of selfmappings defined on X which satisfy the property $(*)$. Then for each $x \in X$, each $\{t_n\} \in w(\infty)$ and for any constant $N(X) < c$, (wherein $N(X)$ stands for the normal structure coefficient with respect to a given convexity structure F) there exists some $z \in \bigcap_{n=1}^{\infty} ad\{z_j : j \geq n\}$ satisfying the properties:*

(I) $\limsup_{n\to\infty} d(z, T(t_n)x) \leq c.A(\{T(t_n)x\})$, *where*

$$A(\{T(t_n)x\}) = \limsup_{n\to\infty} d(T(t_i)x, T(t_j)x) : i, j \geq n;$$

(II) $d(z,y) \leq \limsup_{n\to\infty} d(T(t_n)x, y)$ *for all $y \in X$.*

3. Main results

Now, we are equipped to prove our main result as follows:

Theorem 3.1. *Let (X,d) be a complete bounded metric space equipped with uniform normal structure. Let $\Im = \{T(t) : t \in G\}$ be an asymptotically regular as well as uniformly expansive Kannan semigroup of continuous selfmappings defined on X (with constant $\beta < \sqrt{2}$) enjoying the property $(*)$ and the condition (2.2) (with $\beta < 1/\sqrt{N(X)}$). If*

$$\sup_{y\in F} d(y, T(t)y) < \delta(F),$$

for every nonempty bounded closed convex subset F of X with nonzero diameter which are mapped back into itself by the members of $\Im$, then there exists some $z \in X$ such that $T(t)z = z$ for all $t \in G$.

Proof. Choose a constant c such that $N(X) < c < 1$ and $\beta < 1/\sqrt{c}$. We can pick a sequence $\{t_n\} \in w(\infty)$ such that $\{t_{n+1} - t_n\} \in w(\infty)$ and $\lim_{n\to\infty} \beta(t_n) = \beta$, where $\beta > 0$.

Now fix an $x_0 \in X$. Then, in view of Lemma 2.1, we can inductively construct a sequence $\{x_l\}_{l=1}^{\infty} \subset X$ such that $x_{l+1} \in \bigcap_{n=1}^{\infty} ad\{T(t_i)x_l : i \geq n\}$; for each integer $l \geq 0$,

(III) $\limsup_{n\to\infty} d(T(t_n)x_l, x_{l+1}) \le c.A(\{T(t_n)x_l\})$, where
$A(\{T(t_n)x_l\}) = \limsup_{n\to\infty} d(T(t_i)x_l, T(t_j)x_l) : i,j \ge n;$

(IV) $d(x_{l+1}, y) \le \limsup_{n\to\infty} d(T(t_n)x_l, y)$ for all $y \in X$.

Let

$$D_l = \limsup_{n\to\infty} d(x_{l+1}, T(t_n)x_l) \ \ \forall \ \ l \ge 0.$$

From (IV), we have

$$\begin{aligned} d(x_l, T(t_j)x_l) &\le \limsup_{n\to\infty} d(x_l, T(t_n+t_j)x_{l-1}) \le \limsup_{n\to\infty} d(x_l, T(t_n)x_{l-1}) \\ &+ \limsup_{n\to\infty} d(T(t_n)x_{l-1}, T(t_n+t_j)x_{l-1}) \le D_{l-1}. \end{aligned} \tag{3.1}$$

Observe that for each $j > i \ge 1$, using (2.2) and (3.1), we can write

$$\begin{aligned} d(T(t_i)x_l, T(t_j)x_l) &\le \beta(t_j - t_i)[d(T(t_i)x_l, x_l) + d(T(t_j - t_i)x_l, T(t_j)x_l)] \\ &\le \beta(t_j - t_i) \sup_{x_l \in X} d(T(t_i)x_l, x_l) \le \beta D_{l-1} \end{aligned}$$

which implies that for each $n \ge 1$,

$$\begin{aligned} \sup\{d(T(t_i)x_l, T(t_j)x_l) : i,j \ge n\} &= \sup\{d(T(t_i)x_l, T(t_j)x_l) : i > j \ge n\} \\ &\le \sup_{i > j \ge n} \beta(t_j - t_i) D_{l-1}. \end{aligned} \tag{3.2}$$

Hence using (III) together with (3.2), we have

$$\begin{aligned} D_l &= \limsup_{n\to\infty} d(x_{l+1}, T(t_n)x_l) \le c.A(\{T(t_n)x_l\}) \\ &\le c. \limsup_{n\to\infty} \{d(T(t_i)x_l, T(t_j)x_l) : i,j \ge n\} \\ &\le c. \limsup_{n\to\infty} \beta(t_n) D_{l-1} \\ &\le (\beta c) D_{l-1} \le (\beta c)^2 D_{l-2} \le \dots \le (\beta c)^l D_0. \end{aligned} \tag{3.3}$$

Now using the asymptotic regularity of T on X, we have (for each integer $n \ge 1$)

$$\begin{aligned} d(x_{l+1}, x_l) &\le \limsup_{n\to\infty} d(T(t_n)x_l, x_l) \le \limsup_{n\to\infty} \limsup_{m\to\infty} d(x_l, T(t_m+t_n)x_{l-1}) \\ &\le \limsup_{m\to\infty} d(x_l, T(t_m)x_{l-1}) + \limsup_{n\to\infty} \limsup_{m\to\infty} d(T(t_m)x_{l-1}, T(t_m+t_n)x_{l-1}) \ \le D_{l-1}, \end{aligned}$$

which together with (3.3) gives rise that

$$d(x_{l+1}, x_l) \le D_{l-1} \le (\beta c)^{l-1} D_0.$$

Hence $\lim_{l\to\infty} d(x_{l+1}, x_l) = 0$. Consequently $\{x_l\}$ is Cauchy and hence convergent as X is complete. Let $z = \lim_{l\to\infty} x_l$. Then, we can have

$$\limsup_{n\to\infty} d(z, T(t_n)z) = \lim_{l\to\infty} \limsup_{n\to\infty} d(x_l, T(t_n)x_l) \le \lim_{l\to\infty} D_{l-1} \le \lim_{l\to\infty} (\beta c)^{l-1} D_0 = 0$$

i.e., $\lim_{n\to\infty} d(z, T(t_n)z) = 0$. Hence for each $s \in G$, we deduce

$$\begin{aligned} d(z, T(s)z) &= \lim_{l\to\infty} d(x_l, T(s)x_l) \le \lim_{l\to\infty} \limsup_{n\to\infty} d(x_l, T(t_n+s)x_{l-l}) \\ &\le \lim_{l\to\infty} D_{l-1} \le \lim_{l\to\infty} (\beta c)^{l-1} D_0 = 0, \end{aligned}$$

implying thereby $d(z,T(s)z)=0$, i.e. $T(s)z=z$ for each $s\in G$. ∎

In case $\beta<1/2$ in (2.2), we obtain the following theorem which is a metric version of Kannan fixed point theorem (cf. [5]).

Theorem 3.2. *Let (X,d) be a complete bounded metric space equipped with uniform normal structure. If $\Im=\{T(t):t\in G\}$ is an asymptotically regular as well as uniformly Kannan semigroup of continuous self mappings on X (with constant $\beta<1/2$) enjoying the property $(*)$, which also satisfy condition (2.2). If $N(X)<2/3$, then there exists some $z\in X$ such that $T(t)z=z$ for all $t\in G$.*

Proof. Choose a constant c such that $\tilde{N}(X)<c<2/3$. We can pick a sequence $\{t_n\}\in w(\infty)$ such that $\{t_{n+1}-t_n\}\in w(\infty)$.

Now fix an $x_0\in X$. Then , in view of Lemma 2.1, we can inductively construct a sequence $\{x_l\}_{l=1}^{\infty}\subset X$ such that $x_{l+1}\in\bigcap_{n=1}^{\infty}\operatorname{ad}\{T(t_i)x_l:i\geq n\}$; for each integer $l\geq 0$, and $\{x_l\}_{l=1}^{\infty}$ satisfy (III) and (IV) utilized in the proof of Theorem 3.1.
Let

$$D_l=\limsup_{n\to\infty}d(x_{l+1},T(t_n)x_l)\ \ \forall\ \ l\geq 0.$$

Owing to (IV), we have

$$\begin{aligned}d(x_l,T(t_j)x_l)&\leq\limsup_{n\to\infty}d(x_l,T(t_n+t_j)x_{l-1})\leq\limsup_{n\to\infty}d(x_l,T(t_n)x_{l-1})\\&+\limsup_{n\to\infty}d(T(t_n)x_{l-1},T(t_n+t_j)x_{l-1})\leq D_{l-1}.\end{aligned}\tag{3.4}$$

Observe that for each $j>i\geq 1$, we write (using (2.2))

$$\begin{aligned}d(T(t_i)x_l,T(t_j)x_l)&\leq\frac{1}{2}[d(T(t_i)x_l,x_l)+d(T(t_j-t_i)x_l,T(t_j)x_l)]\\&\leq\frac{1}{2}[d(T(t_i)x_l,x_l)+d(T(t_j-t_i)x_l,x_l)+d(x_l,T(t_j)x_l)]\leq\frac{3}{2}D_{l-1}\end{aligned}$$

which implies that (for each $n\geq 1$),

$$\sup\{d(T(t_i)x_l,T(t_j)x_l):i,j\geq n\}=\sup\{d(T(t_i)x_l,T(t_j)x_l):i>j\geq n\}\leq\frac{3}{2}D_{l-1}.\tag{3.5}$$

On using (III) together with (3.5), we have

$$\begin{aligned}D_l&=\limsup_{n\to\infty}d(x_{l+1},T(t_n)x_l)\leq c.A(\{T(t_n)x_l\})\\&\leq c.\limsup_{n\to\infty}\{d(T(t_i)x_l,T(t_j)x_l):i,j\geq n\}\\&\leq\left(\frac{3}{2}c\right)D_{l-1}\leq\left(\frac{3}{2}c\right)^2D_{l-2}\leq\ldots\leq\left(\frac{3}{2}c\right)^lD_0.\end{aligned}\tag{3.6}$$

Now using the asymptotic regularity of T on X, (for each integer $n\geq 1$) we have

$$\begin{aligned}d(x_{l+1},x_l)&\leq\limsup_{n\to\infty}d(T(t_n)x_l,x_l)\leq\limsup_{n\to\infty}\limsup_{m\to\infty}d(x_l,T(t_m+t_n)x_{l-1})\\&\leq\limsup_{m\to\infty}d(x_l,T(t_m)x_{l-1})+\limsup_{n\to\infty}\limsup_{m\to\infty}d(T(t_m)x_{l-1},T(t_m+t_n)x_{l-1})\leq D_{l-1},\end{aligned}$$

which together with (3.6) gives rise that

$$d(x_{l+1},x_l)\leq D_{l-1}\leq\left(\frac{3}{2}c\right)^{l-1}D_0.$$

Hence $\lim_{l\to\infty} d(x_{l+1}, x_l) = 0$. Consequently $\{x_l\}$ is Cauchy and hence convergent as X is complete. Let $z = \lim_{l\to\infty} x_l$. Then, we can have

$$\limsup_{n\to\infty} d(z, T(t_n)z) = \lim_{l\to\infty} \limsup_{n\to\infty} d(x_l, T(t_n)x_l) \le \lim_{l\to\infty} D_{l-1} \le \lim_{l\to\infty} \left(\frac{3}{2}c\right)^{l-1} D_0 = 0$$

i.e., $\lim_{n\to\infty} d(z, T(t_n)z) = 0$. Hence for each $s \in G$, we can write

$$\begin{aligned} d(z, T(s)z) &= \lim_{l\to\infty} d(x_l, T(s)x_l) \le \lim_{l\to\infty} \limsup_{n\to\infty} d(x_l, T(t_n + s)x_{l-l}) \\ &\le \lim_{l\to\infty} D_{l-1} \le \lim_{l\to\infty} \left(\frac{3}{2}c\right)^{l-1} D_0 = 0, \end{aligned}$$

implying thereby $d(z, T(s)z) = 0$, i.e. $T(s)z = z$ for each $s \in G$. This concludes the proof. ∎

If in our foregoing theorems, we replace the one parameter semigroup of Kannan mappings with corresponding semigroup of iterates of Kannan mappings, then one can immediately derive the following two corollaries.

Corollary 3.1. *Let (X, d) be a complete bounded metric space equipped with uniform normal structure enjoying the property (P) whereas $T : X \to X$ be an asymptotically regular semigroup of $n-$iterates of Kannan mappings defined on X which satisfy condition (2.1). Then there exists some $z \in X$ such that $Tz = z$.*

The following definition is to be utilized in our next corollary.

Definition 3.1. *A convex metric space K (defined by Takahashi* [12]*) is said to have property (C) if every decreasing net of nonempty closed and convex subsets of K has nonempty intersection.*

Since the definition of convexity structure introduced by Khamsi [7] is more general than the one introduced by Takahashi [12], therefore the following corollary due to Beg and Azam [2] is deduced:

Corollary 3.2. [2] *Let T be a Kannan selfmapping of a nonempty bounded closed convex subset Z of a convex metric space K enjoying the property (C). If $\inf_{y\in H} d(y, Ty) < \delta(H)$ for every nonempty bounded closed convex T-invariant subset H of K with non-zero diameter, then T has a unique fixed point in Z.*

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