

Unbiased Estimation of Structural Parameters in Credibility Models with Dependence Induced by Common Effects

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Abstract. In credibility models, the so-called structural parameters must be estimated before the credibility estimators can be calculated. Several existing methods provide estimators, but these estimators are not necessarily unbiased or simple to use. In this paper, we introduce alternative unbiased estimators for structural parameters in credibility models with dependence induced by common effects. The main advantage of our estimators is their ease of application.

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1. Introduction

One of the basic challenges of developing insurance policies is determining their premiums. If we have observations of past claims for a set of contracts, it might be possible to calculate an appropriate premium for a future period. These premiums must strongly reflect the features the expected insurance risks. Several methodologies of insurance pricing have been developed for this purpose; one of the most important methods is credibility ratemaking. In credibility techniques, the premiums of each contract in a heterogeneous portfolio are separately and adaptively determined by combining the policyholder's claim experiences and the portfolio's particular risk features. According to Klugman *et al.* [1], "Credibility theory is a set of quantitative tools which allows an insurer to perform prospective experience rating (adjust future premiums based on past experience) on a risk or group of risks." Based on the experience and the collective premium, the credibility theory determines the credibility premium by the following credibility form:

$$\text{Credibility premium} = Z \times (\text{experience}) + (1 - Z) \times (\text{collective premium}),$$

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where Z , a value between 0 and 1, is the credibility factor and needs to be chosen. Clearly, the choice of the credibility factor Z is of an immense importance and has attracted a lot of research interest.

In credibility models, there are so-called structural parameters that must be estimated before the credibility estimators can be calculated. The focus of this paper is the estimation of the variance structure parameters in the hierarchical credibility model and their applications to a credibility model with dependence induced by common effects. Several textbooks present unbiased estimators of the structural parameters for the Bühlmann and Bühlmann-Straub models. There are simple estimators for structural parameters for Jewell's hierarchical credibility model that are not necessarily unbiased; see, e.g., [2]. Ohlsson [3] presented an alternative, unbiased estimator, similar to those of the Bühlmann-Straub model. The estimation of structural parameters in higher-level hierarchical credibility models can be found, e.g., in [4]. Belhadj *et al.* [5] focused on the estimation of structural parameters of the hierarchical credibility model. The authors reviewed the estimators of the structural parameters, emphasizing three main sets of variance components estimators: the iterative pseudo-estimators, Ohlsson estimators and Bühlmann-Gisler estimators. Using simulation, the authors then assessed the relative performance of the three sets of estimators and showed that the Bühlmann-Gisler and iterative pseudo-estimators were generally superior to the Ohlsson estimators, but by a minuscule margin.

In this paper, we present alternative, simple estimators of structural parameters of credibility models with dependence induced by common effects. The main advantage of our estimators is their simplicity in calculation and application. In section 2, the construction of the two- and three-level common-effect model, assumptions, and Ohlsson estimators are introduced. In section 3, we derive unbiased estimators of structural parameters for two- and three-level common-effect models for portfolios with the Bühlmann model's structure. The results are extended to the Bühlmann-Straub model in section 4. Section 5 presents two examples. We conclude in section 6.

2. Preliminaries

The purpose of this paper is to study unbiased estimators of structural parameters under a type of dependence structure involving portfolios and individuals and containing common effects. The credibility models with two- and three-level common effects of claim dependence are formulated in a hierarchical way, as described below. Note that all random variables are defined on a common probability space (Ω, F, P) . Also, all claim variables are square-integrable.

2.1. The two-level common-effect model of dependence

A common practice in calculating premiums is to group individual risks to ensure homogeneity and achieve a fair and equitable premium across individuals. Under this approach, the risks within each group are as homogeneous as possible in terms of certain observable risk characteristics. However, not all risks in the group are truly homogeneous. Some unobservable factors will always affect the degree of heterogeneity among the individuals. Thus, the risk level of each individual in the group can be characterized by a risk parameter θ such that all possible values of θ are modeled by a random variable Θ following a probability distribution $\pi(\theta)$.

Consider a portfolio of insurance contracts that consists of I insured individuals, and further suppose that each individual has a total of T time periods of history available. Denote by X_{it} , the claim amount for individual i during period t . To simplify our exposition, the same time periods will be applied to all individuals; we refer to this scenario as the “balanced model.” These assumptions can be modified for the unbalanced model. The claim amounts and individuals can be written as $\mathbf{X}_i = (X_{i1}, X_{i2}, \dots, X_{iT})'$ and $i = 1, 2, \dots, I$, respectively.

In the classical credibility models of Bühlmann [6] and Bühlmann-Straub [7] and studied by many subsequent researchers, a common assumption is that the random vectors (\mathbf{X}_i, Θ_i) , $i = 1, 2, \dots, I$, are independent across individuals (i.e., independence over risks) and that for each i , $X_{i1}, X_{i2}, \dots, X_{iT}$ are conditionally independent given Θ_i (i.e., conditional independence over time). Although such independence assumptions may be at least approximately appropriate in some practical situations, Yeo and Valdez [8] have shown that there exist many important insurance scenarios in which these classical assumptions are certainly violated. In their analysis, claims $X_{1t}, X_{2t}, \dots, X_{It}$ are first assumed to be not independent across individuals for a fixed time t , implying that the claims of one insured individual can directly impact those of other insured individuals. For example, in home insurance, geographic proximity of insureds may result in exposure to a common catastrophe, and in motor insurance, accidents may involve several insureds in the same collision. Second, for a fixed individual $i = 1, 2, \dots, I$, claims $X_{i1}, X_{i2}, \dots, X_{iT}$ are not always assumed to be independent across different time periods. In motor insurance, for example, an individual may suffer from accident proneness. Yeo and Valdez [8] have addressed a simultaneous dependence of claims across individuals for a fixed time period and across time periods for a fixed individual. The authors introduced one common effect affecting all individuals and another common effect affecting a fixed individual over time. They used a random variable Λ to describe the common dependence across the insured individuals, and for a fixed individual i , the random variable Θ_i was used to describe the common dependence across the time periods.

Wen *et al.* [9] studied the Bühlmann and Bühlmann-Straub models with a dependence structure characterized by a stochastic latent risk parameter (referred to as a common effect, following the terminology of Yeo and Valdez [8]). They derived credibility formulas for general credibility models with common effects.

The model of dependence proposed in this subsection allows for both dependence or common effects among individual risks at any point in time and the dependence of a particular individual's risk experience over time. The risk quality of an individual i is characterized by a risk parameter Θ_i , and the common effect is represented by a random variable Λ . Conditional on this common effect Λ , the vectors (\mathbf{X}_i, Θ_i) of individual i 's experience \mathbf{X}_i and random variables Θ_i are independent and identically distributed (*i.i.d.*) over individuals $i = 1, 2, \dots, I$. Furthermore, given the risk structure Θ_i , the claims X_{ij} are assumed to be *i.i.d.* The formal assumptions of the model are stated below.

- (A1). Given Λ , the random vectors (\mathbf{X}_i, Θ_i) , $i = 1, 2, \dots, I$, are mutually independent and identically distributed.
- (A2). For fixed contract i , given Λ and Θ_i , the claims $X_{i1}, X_{i2}, \dots, X_{iT}$ are conditionally *i.i.d.*, with $E(X_{ij}|\Theta_i, \Lambda) = \mu(\Theta_i, \Lambda)$ and $\text{Var}(X_{ij}|\Theta_i, \Lambda) = \sigma^2(\Theta_i, \Lambda)$.

In the analysis, we use the following notations:

$$E[\mu(\Theta_i, \Lambda)|\Lambda] = \mu(\Lambda), \text{Var}[\mu(\Theta_i, \Lambda)|\Lambda] = \sigma_\theta^2(\Lambda), E[\sigma_\theta^2(\Lambda)] = \sigma_\theta^2,$$

$$\text{Var}(\mu(\Lambda)) = \sigma_\lambda^2, \quad E[\mu(\Lambda)] = \mu, \quad E[\sigma_x^2(\Theta_i, \Lambda)] = \sigma_x^2(\Lambda), \quad E[\sigma_x^2(\Lambda)] = \sigma_x^2.$$

Wen *et al.* [9] have shown that under assumptions (A1) and (A2) and above notations, the optimal linear inhomogeneous unbiased estimator for $X_{i,T+1}$, $i = 1, 2, \dots, I$, is given by $\hat{X}_{i,T+1} = Z_1 \bar{X}_i + Z_2 \bar{X}_{..} + (1 - Z_1 - Z_2)\mu$, $i = 1, 2, \dots, I$ where μ is collective mean or manual premium and

$$Z_1 = \frac{T\sigma_\theta^2}{T\sigma_\theta^2 + \sigma_x^2}, \quad Z_2 = \frac{TI\sigma_\lambda^2\sigma_x^2}{(T\sigma_\theta^2 + \sigma_x^2)(TI\sigma_\lambda^2 + T\sigma_\theta^2 + \sigma_x^2)}, \quad \bar{X}_i = \frac{1}{T} \sum_{t=1}^T X_{it}, \quad \bar{X}_{..} = \frac{1}{I} \sum_{i=1}^I \bar{X}_i.$$

Here and in the future, the dot notation is used to indicate summation.

2.2. The three-level common-effect model of dependence

Consider a set of insurance contracts consisting of K portfolios. For each portfolio, there are I insured individuals. Suppose that each individual has available a history that is T time periods long. Denote by X_{kit} the claim amount in the portfolio k of individual i during period t . We shall use the random matrix $\mathbf{X}_k = (\mathbf{X}_{k1}^I, \mathbf{X}_{k2}^I, \dots, \mathbf{X}_{kI}^I)$ to denote the matrix of claims of a particular portfolio $k = 1, 2, \dots, K$, where \mathbf{X}_{ki} is a random vector with $\mathbf{X}_{ki} = (X_{ki1}, X_{ki2}, \dots, X_{kiT})^I$, the vector of claims in the portfolio k for a particular individual $i = 1, 2, \dots, I$.

In the usual three-level credibility models with hierarchical structure, a common assumption is that the random matrices $(\mathbf{X}_k, \Lambda_k)$, $k = 1, 2, \dots, K$, are independent across portfolios (i.e., independence over portfolios). In addition, for each k , the random vectors $(\mathbf{X}_{ki}, \Theta_{ki})$, $i = 1, 2, \dots, I$, are typically assumed to be independent across individuals (i.e., independence over risks) and for each i , $X_{ki1}, X_{ki2}, \dots, X_{kiT}$ are conditionally independent given Θ_{ki} (i.e., conditional independence over time)(see, e.g., [4,5,10]).

In practice, these classical assumptions are certainly violated in some insurance scenarios with a hierarchical structure. In these scenarios, the random matrices $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_K$ are not independent: the claims of one portfolio can directly impact those of other portfolios. Additionally, for a fixed portfolio $k = 1, 2, \dots, K$, as addressed by Wen *et al.* [9], the random vectors $(\mathbf{X}_{ki}, \Theta_{ki})$, $i = 1, 2, \dots, I$, have a dependent structure across individuals and across different time periods. For example, in motor insurance, different makes of cars are grouped in different portfolios. The claims of one portfolio might affect other portfolios if bad weather conditions lead to accidents are common. For a given make of car, as described by Yeo and Valdez [8], a common economic environment would make dependence across individuals and across different time periods a reasonable assumption.

In this section, the model of dependence allows for the dependence among portfolio risks, dependence of the individual risks and the dependence of experience for a particular individual risk over time. The dependence among portfolio risks is described by a common-effect random variable Γ . Realization of this common effect is denoted by γ . Conditionally on this common effect, the random matrices \mathbf{X}_k , $k = 1, \dots, K$ are mutually independent and identically distributed. Because Γ is a common effect among all portfolios, it will define the dependence structure between portfolios. Thus,

- A1. Given Γ , the random matrices \mathbf{X}_k , $k = 1, \dots, K$ are mutually independent and identically distributed.

For a fixed portfolio k , the dependence among the individual risks is described by another common-effect random variable Λ_k . Realization of this common effect is denoted by

λ_k . Given this common effect and Γ , the random vectors \mathbf{X}_{ki} , $i = 1, \dots, I$, are mutually independent and identically distributed. Similar to Γ , because Λ_k is a common effect among all risks, it will define the dependence structure between risks. Additional assumptions regarding this common effect are:

- A2. The random variables $\Lambda_1, \Lambda_2, \dots, \Lambda_K$ are pairwise independent; that is, Λ_m is independent of Λ_n for all $m \neq n$ where $m, n = 1, \dots, K$.
- A3. For a fixed portfolio k , the random variable Λ_k is independent of Γ .
- A4. For a fixed portfolio k , given Λ_k , the random vectors \mathbf{X}_{ki} , $i = 1, \dots, I$, are mutually independent and identically distributed.

Assumption A2 states that portfolio risk parameters are independent of one another. This assumption is reasonable because the random variable Γ already takes into account all of the common effects across portfolio risks, and thus, Λ_k are truly peculiar common effects within portfolios. Assumption A3 states that the overall risk parameter is independent of all portfolio risk parameters. Assumption A4 merely asserts that given the overall risk parameter and the portfolio's risk parameter, all individuals are independent.

For a fixed portfolio k and fixed individual i , the dependence of claims across time is described by another common-effect random variable, Θ_{ki} . Realization of this common effect is denoted by θ_{ki} . Additional assumptions regarding this common effect are:

- A5. For a fixed portfolio k , the random variables $\Theta_{k1}, \dots, \Theta_{kI}$ are pairwise independent; that is, Θ_{km} is independent of Θ_{kn} for all $m \neq n$ where $m, n = 1, \dots, I$.
- A6. For a fixed portfolio k and a fixed contract i , the random variable Θ_{ki} is independent of Λ_k .
- A7. For a fixed portfolio k and a fixed contract i , given the common effects Γ , Λ_k and Θ_{ki} , the claims $X_{ki1}, X_{ki2}, \dots, X_{kiT}$ are conditionally independent and identically distributed with $E(X_{kit} | \Theta_{ki}, \Lambda_k, \Gamma) = \mu(\Theta_{ki}, \Lambda_k, \Gamma)$ and $\text{Var}(X_{kit} | \Theta_{ki}, \Lambda_k, \Gamma) = \sigma^2(\Theta_{ki}, \Lambda_k, \Gamma)$.

Assumption A5 states that for a fixed portfolio k , individual risk parameters are independent of one another. This assumption is reasonable because the random variable Λ_k already takes into account the common characteristics across individual risks, and thus Θ_{ki} represents truly peculiar individual characteristics. Assumption A6 states that for a fixed portfolio k , the portfolio risk parameter is independent of all individual risk parameters. Finally, assumption A7 merely states that given the overall risk parameter, the portfolio's risk parameter and the individual's risk parameter, the individual risk's experience at a particular time period is independent of that of all other individuals at that point in time as well as the individual risk's experience at other time periods.

The following notations will be used in the analysis:

$$\begin{aligned}
 E[\mu(\Theta_{ki}, \Lambda_k, \Gamma) | \Lambda_k, \Gamma] &= \mu(\Lambda_k, \Gamma), & \text{Var}[\mu(\Theta_{ki}, \Lambda_k, \Gamma) | \Lambda_k, \Gamma] &= \sigma_\theta^2(\Lambda_k, \Gamma), \\
 \text{Var}[\mu(\Lambda_k, \Gamma) | \Gamma] &= \sigma_\lambda^2(\Gamma), & E[\sigma_\lambda^2(\Gamma)] &= \sigma_\lambda^2, & E[\sigma_\theta^2(\Lambda_k, \Gamma) | \Gamma] &= \sigma_\theta^2(\Gamma), \\
 E[\sigma_\theta^2(\Gamma)] &= \sigma_\theta^2, & E[\sigma^2(\Theta_{ki}, \Lambda_k, \Gamma) | \Lambda_k, \Gamma] &= \sigma_x^2(\Lambda_k, \Gamma), & E[\sigma_x^2(\Lambda_k, \Gamma) | \Gamma] &= \sigma_x^2(\Gamma), \\
 E[\sigma_x^2(\Gamma)] &= \sigma_x^2, & E[\mu(\Lambda_k, \Gamma) | \Gamma] &= \mu(\Gamma), & \text{Var}[\mu(\Gamma)] &= \sigma_\gamma^2, & E[\mu(\Gamma)] &= \mu.
 \end{aligned}$$

Ebrahinzadeh *et al.* [11] have shown that under assumptions A1 to A7 and above notations, the optimal linear inhomogeneous unbiased estimator for $X_{k,i,T+1}$, $i = 1, 2, \dots, I$, is given by $\hat{X}_{k,i,T+1} = Z_1 \bar{X}_{ki} + Z_2 \bar{X}_{k..} + Z_3 \bar{X}_{...} + (1 - Z_1 - Z_2 - Z_3)\mu$, $k = 1, 2, \dots, K$ and

$i = 1, 2, \dots, I$ where μ is collective mean or manual premium and

$$Z_1 = \frac{T\sigma_\theta^2}{T\sigma_\theta^2 + \sigma_x^2}, \quad Z_2 = \frac{TI\sigma_\lambda^2\sigma_x^2}{(T\sigma_\theta^2 + \sigma_x^2)(TI\sigma_\lambda^2 + T\sigma_\theta^2 + \sigma_x^2)},$$

$$Z_3 = \frac{TIK\sigma_\gamma^2\sigma_x^2}{(TI\sigma_\lambda^2 + T\sigma_\theta^2 + \sigma_x^2)(TIK\sigma_\gamma^2 + TI\sigma_\lambda^2 + T\sigma_\theta^2 + \sigma_x^2)}, \quad \bar{X}_{ki.} = \frac{1}{T} \sum_{t=1}^T X_{kit},$$

$$\bar{X}_{k..} = \frac{1}{I} \sum_{i=1}^I \bar{X}_{ki.} \quad \text{and} \quad \bar{X}_{...} = \frac{1}{K} \sum_{k=1}^K \bar{X}_{k..}.$$

3. New estimators in Bühlmann model's structure

Here and in the next section, we do not discuss about estimator for the collective mean, μ . This is well established in the actuarial literature (see, e.g., [1,3,4,5]) and is not subject to much controversy. In this section, we consider the unbiased estimation of structural parameters, σ_x^2 , σ_θ^2 , σ_λ^2 and σ_γ^2 in two- and three-level common-effect situations, assuming that each portfolio has the Bühlmann model's structure. The following lemma will be used later.

Lemma 3.1. *Consider a portfolio consisting of I individuals; suppose that each individual has available a history of length T time periods; and $E(X_{it}|\Theta_i) = \mu(\Theta_i)$, $\text{Var}(X_{it}|\Theta_i) = v(\Theta_i)$, and $\text{Var}[\mu(\Theta_i)] = a$. Furthermore, $X_{i1}, X_{i2}, \dots, X_{iT}$ are conditionally independent. Also assume that different individuals' past data are independent. In this case $E[v(\Theta_i)] = v$, and unbiased estimators for $v(\Theta_i)$, v and a are given by*

$$\hat{v}_i = \widehat{v(\Theta_i)} = \frac{1}{T-1} \sum_{t=1}^T (X_{it} - \bar{X}_i)^2, \quad \hat{v} = \frac{1}{I} \sum_{i=1}^I \hat{v}_i, \quad \hat{a} = \frac{1}{I-1} \sum_{i=1}^I (\bar{X}_i - \bar{X}_{..})^2 - \frac{\hat{v}}{T}$$

Proof. See pages 592–593 of [1]. ■

3.1. The estimators of the two-level common-effect model

Theorem 3.1. *Consider R portfolios of the two-level common-effect model satisfying assumptions (A1) and (A2) of subsection 2.1. Using the notations in subsection 2.1, unbiased estimators for the structural parameters, σ_x^2 , σ_θ^2 and σ_λ^2 , are given by*

$$(1) \quad \widehat{\sigma_x^2} = \frac{1}{RI} \sum_{r=1}^R \sum_{i=1}^I \widehat{\text{Var}}(X_{rit}),$$

$$(2) \quad \widehat{\sigma_\theta^2} = \frac{1}{R} \sum_{r=1}^R \widehat{\text{Var}}(\bar{X}_{ri.}) - \frac{\widehat{\sigma_x^2}}{T}, \quad \text{and}$$

$$(3) \quad \widehat{\sigma_\lambda^2} = \widehat{\text{Var}}(\bar{X}_{r..}) - \frac{\widehat{\sigma_\theta^2}}{T},$$

where $\bar{X}_{ri.} = \frac{1}{T} \sum_{t=1}^T X_{rit}$, $\bar{X}_{r..} = \frac{1}{I} \sum_{i=1}^I \bar{X}_{ri.}$, $\bar{X}_{...} = \frac{1}{R} \sum_{r=1}^R \bar{X}_{r..}$, $\widehat{\text{Var}}(X_{rit}) = \frac{1}{T-1} \sum_{t=1}^T (X_{rit} - \bar{X}_{ri.})^2$, $\widehat{\text{Var}}(\bar{X}_{ri.}) = \frac{1}{I-1} \sum_{i=1}^I (\bar{X}_{ri.} - \bar{X}_{r..})^2$ and $\widehat{\text{Var}}(\bar{X}_{r..}) = \frac{1}{R-1} \sum_{r=1}^R (\bar{X}_{r..} - \bar{X}_{...})^2$.

Proof. (1) and (2) are straightforward; see, e.g., [1] and [4].

Here, we only prove (3). We have

$$E(\bar{X}_{r..}|\Lambda) = \frac{1}{I} \sum_{i=1}^I E(\bar{X}_{r. i}|\Lambda) = \frac{1}{I} \sum_{i=1}^I \mu(\Lambda) = \mu(\Lambda).$$

Thus,

$$E(\bar{X}_{r..}) = E[E(\bar{X}_{r..}|\Lambda)] = E[\mu(\Lambda)] = \mu$$

and

$$\text{Var}(\bar{X}_{r..}) = \text{Var}[E(\bar{X}_{r..}|\Lambda)] + E[\text{Var}(\bar{X}_{r..}|\Lambda)] = \text{Var}[\mu(\Lambda)] + E\left[\frac{\sigma_{\theta}^2(\Lambda)}{I}\right] = \sigma_{\lambda}^2 + \frac{\sigma_{\theta}^2}{I}.$$

Therefore, $\bar{X}_{1..}, \bar{X}_{2..}, \dots, \bar{X}_{R..}$ are independent with common mean μ and common variance $\sigma_{\lambda}^2 + \frac{\sigma_{\theta}^2}{I}$. Consequently, an unbiased estimator of $\sigma_{\lambda}^2 + \frac{\sigma_{\theta}^2}{I}$ is $\widehat{\text{Var}}(\bar{X}_{r..}) = \frac{1}{R-1} \sum_{r=1}^R (\bar{X}_{r..} - \bar{X}_{....})^2$, and an unbiased estimator of σ_{λ}^2 is given by

$$\widehat{\sigma}_{\lambda}^2 = \widehat{\text{Var}}(\bar{X}_{r..}) - \frac{\widehat{\sigma}_{\theta}^2}{I}. \quad \blacksquare$$

3.2. The estimators of the three-level common-effect model

Theorem 3.2. Consider R sets of portfolios of the three-level common-effect model satisfying assumptions A1 to A7 of subsection 2.2. Under the notations in subsection 2.2, unbiased estimators for structural parameters, $\sigma_x^2, \sigma_{\theta}^2, \sigma_{\lambda}^2$ and σ_{γ}^2 are given by

- (1) $\widehat{\sigma}_x^2 = \frac{1}{RKI} \sum_{r=1}^R \sum_{k=1}^K \sum_{i=1}^I \widehat{\text{Var}}(X_{rkit}),$
- (2) $\widehat{\sigma}_{\theta}^2 = \frac{1}{RK} \sum_{r=1}^R \sum_{k=1}^K \widehat{\text{Var}}(\bar{X}_{rki.}) - \frac{\widehat{\sigma}_x^2}{I},$ and
- (3) $\widehat{\sigma}_{\lambda}^2 = \frac{1}{R} \sum_{r=1}^R \widehat{\text{Var}}(\bar{X}_{rk..}) - \frac{\widehat{\sigma}_{\theta}^2}{I},$
- (4) $\widehat{\sigma}_{\gamma}^2 = \widehat{\text{Var}}(\bar{X}_{r...}) - \frac{\widehat{\sigma}_{\lambda}^2}{K},$

where

$$\begin{aligned} \bar{X}_{rki.} &= \frac{1}{T} \sum_{t=1}^T X_{rkit}, & \bar{X}_{rk..} &= \frac{1}{I} \sum_{i=1}^I \bar{X}_{rki.}, & \bar{X}_{r...} &= \frac{1}{K} \sum_{k=1}^K \bar{X}_{rk..}, & \bar{X}_{....} &= \frac{1}{R} \sum_{r=1}^R \bar{X}_{r...}, \\ \widehat{\text{Var}}(\widehat{X}_{rkit}) &= \frac{1}{T-1} \sum_{t=1}^T (X_{rkit} - \bar{X}_{rki.})^2, & \widehat{\text{Var}}(\widehat{\bar{X}}_{rki.}) &= \frac{1}{I-1} \sum_{i=1}^I (\bar{X}_{rki.} - \bar{X}_{rk..})^2, \\ \widehat{\text{Var}}(\widehat{\bar{X}}_{rk..}) &= \frac{1}{K-1} \sum_{k=1}^K (\bar{X}_{rk..} - \bar{X}_{r...})^2 & \text{and} & \widehat{\text{Var}}(\widehat{\bar{X}}_{r...}) &= \frac{1}{R-1} \sum_{r=1}^R (\bar{X}_{r...} - \bar{X}_{....})^2. \end{aligned}$$

Proof. As above, theorems (1) and (2) are straightforward; see, e.g., [1] and [4].

To prove (3), for the r th set, $r = 1, 2, \dots, R$, we can use Lemma 3.1 to replace a, v and T by $b_r = \text{Var}[\mu(\Lambda_k)], a_r = E[a_{rk}(\Lambda_k)]$ and I , respectively. An unbiased estimator of b_r is given by

$$\widehat{b}_r = \widehat{\text{Var}}(\widehat{\bar{X}}_{rk..}) - \frac{\widehat{a}_r}{I} = \widehat{\text{Var}}(\widehat{\bar{X}}_{rk..}) - \frac{1}{KI} \sum_{k=1}^K \left[\widehat{\text{Var}}(\widehat{\bar{X}}_{rki.}) - \frac{1}{IT} \sum_{i=1}^I \widehat{\text{Var}}(\widehat{X}_{rkit}) \right].$$

Similarly, an unbiased estimator of a_r is

$$\widehat{a}_r = \frac{1}{K} \sum_{k=1}^K \widehat{a}_{rk} = \frac{1}{K} \sum_{k=1}^K \left[\widehat{\text{Var}}(\overline{X}_{rki.}) - \frac{1}{IT} \sum_{i=1}^I \widehat{\text{Var}}(X_{rkit}) \right].$$

Furthermore,

$$E(\widehat{b}_r) = E[E(\widehat{b}_r|\Gamma)] = E[b_r(\Gamma)] = E\{\text{Var}[\mu(\Lambda_k, \Gamma)|\Gamma]\} = E[\sigma_\lambda^2(\Gamma)] = \sigma_\lambda^2,$$

and \widehat{b}_r is unbiased for σ_λ^2 . Hence, an unbiased estimator of σ_λ^2 based on all data is

$$\widehat{\sigma}_\lambda^2 = \frac{1}{R} \sum_{r=1}^R \widehat{b}_r = \frac{1}{R} \sum_{r=1}^R \widehat{\text{Var}}(\overline{X}_{rk..}) - \frac{\widehat{\sigma}_\theta^2}{I}.$$

To prove (4), we have

$$E(\overline{X}_{r...}|\Gamma) = \frac{1}{K} \sum_{k=1}^K E(\overline{X}_{rk..}|\Gamma) = \frac{1}{K} \sum_{k=1}^K \mu(\Gamma) = \mu(\Gamma).$$

Thus,

$$E(\overline{X}_{r...}) = E[E(\overline{X}_{r...}|\Gamma)] = E[\mu(\Gamma)] = \mu$$

and

$$\text{Var}(\overline{X}_{r...}) = \text{Var}[E(\overline{X}_{r...}|\Gamma)] + E[\text{Var}(\overline{X}_{r...}|\Gamma)] = \sigma_\gamma^2 + \frac{\sigma_\lambda^2}{K}.$$

Therefore, as before, we can show that an unbiased estimator of $\sigma_\gamma^2 + \frac{\sigma_\lambda^2}{K}$ is

$\widehat{\text{Var}}(\overline{X}_{r...}) = \frac{1}{R-1} \sum_{r=1}^R (\overline{X}_{r...} - \overline{X}_{....})^2$. Consequently, an unbiased estimator of σ_γ^2 is given by

$$\widehat{\sigma}_\gamma^2 = \widehat{\text{Var}}(\overline{X}_{r...}) - \frac{\widehat{\sigma}_\lambda^2}{K}. \quad \blacksquare$$

4. New estimators under the structure of the Bühlmann-Straub model

The credibility model with weights was developed by Bühlmann and Straub [7] and hence is known as the Bühlmann-Straub model, see also, e.g. [4]. This model has been broadly applied in the practice of insurance, and it has thus been one of the building blocks of credibility theory. In this section, we consider unbiased estimation of structural parameters σ_x^2 , σ_θ^2 , σ_λ^2 and σ_γ^2 in two- and three-level common-effect scenarios, assuming that each portfolio has the structure of the Bühlmann-Straub model. The lemmas below will be used later.

Lemma 4.1. *Consider a portfolio consisting of I individuals; suppose that each individual has available a history of length T time periods; and $E(X_{it}|\Theta_i) = \mu(\Theta_i)$, $\text{Var}(X_{it}|\Theta_i) = v(\Theta_i)/m_{it}$, and $\text{Var}[\mu(\Theta_i)] = a$, where m_{it} is a known constant measuring exposure. Furthermore, $X_{i1}, X_{i2}, \dots, X_{iT}$ are independent, conditional on Θ_i , and $\overline{X} = m_{..}^{-1} \sum_{i=1}^I m_i \overline{X}_i$, where $m_i = \sum_{t=1}^T m_{it}$ and $m_{..} = \sum_{i=1}^I m_i$. In addition, different individuals' past data are independent. In this case, $E[v(\Theta_i)] = v$, and unbiased estimators of $v(\Theta_i)$, v and a are given by*

$$\widehat{v}_i = \widehat{v}(\Theta_i) = \frac{1}{T-1} \sum_{t=1}^T m_{it} (X_{it} - \overline{X}_i)^2, \quad \widehat{v} = \frac{1}{I} \sum_{i=1}^I \widehat{v}_i, \quad \text{and}$$

$$\widehat{\alpha} = \left(m_{..} - m_{..}^{-1} \sum_{i=1}^I m_i^2 \right)^{-1} \left[\sum_{i=1}^I m_i (\bar{X}_i - \bar{X}_{..})^2 - \widehat{v}(I - 1) \right].$$

Proof. See pages 594–596 of [1]. ■

Lemma 4.2. *Suppose X_1, X_2, \dots, X_T are independent with common mean $\mu = E(X_j)$ and variance $\text{Var}(X_j) = \beta + \alpha/m_j$; $\alpha, \beta > 0$ and all $m_j \geq 1$. Let $m = \sum_{j=1}^T m_j$. Then*

$$E \left[\sum_{j=1}^T m_j (X_j - \bar{X})^2 \right] = \beta \left(m - m^{-1} \sum_{j=1}^T m_j^2 \right) + \alpha(T - 1).$$

Proof. See pages 525–528 of [1]. ■

4.1. The estimators of the two-level common-effect model

Consider we are given a portfolio of I risks or “individuals” under assumptions (A1) of subsection 2.1 and the following (A2)′:

(A2)′. For a fixed contract i , given Λ and Θ_i , $X_{i1}, X_{i2}, \dots, X_{iT}$ are conditionally independent with $E(X_{ij}|\Theta_i, \Lambda) = \mu(\Theta_i, \Lambda)$ and $\text{Var}(X_{ij}|\Theta_i, \Lambda) = \sigma_x^2(\Theta_i, \Lambda)/m_{ij}$, where m_{ij} are known weights.

Theorem 4.1. *Consider R portfolios of the two-level common-effect model satisfying assumptions (A1) and (A2)′. Under the notations in subsection 2.1, unbiased estimators of structural parameters, σ_x^2 , σ_θ^2 and σ_λ^2 are given by*

- (1) $\widehat{\sigma}_x^2 = \frac{1}{RI} \sum_{r=1}^R \sum_{i=1}^I \widehat{v}_{ri}$,
- (2) $\widehat{\sigma}_\theta^2 = \frac{1}{R} \sum_{r=1}^R \left\{ \left(m_{r..} - m_{r..}^{-1} \sum_{i=1}^I m_{ri}^2 \right)^{-1} \left[\sum_{i=1}^I m_{ri} (\bar{X}_{ri} - \bar{X}_{r..})^2 - \widehat{v}_r(I - 1) \right] \right\}$, and
- (3) $\widehat{\sigma}_\lambda^2 = \widehat{\text{Var}}(\bar{X}_{r..}) - \frac{\widehat{\sigma}_\theta^2}{I}$,

where $m_{ri} = \sum_{t=1}^T m_{rit}$, $m_{r..} = \sum_{i=1}^I m_{ri}$, $\bar{X}_{ri} = \frac{1}{m_{ri}} \sum_{t=1}^T m_{rit} X_{rit}$, $\bar{X}_{r..} = \frac{1}{I} \sum_{i=1}^I \bar{X}_{ri}$, $\bar{X}_{...} = \frac{1}{R} \sum_{r=1}^R \bar{X}_{r..}$,

$\widehat{v}_{ri} = \frac{1}{T-1} \sum_{t=1}^T m_{rit} (X_{rit} - \bar{X}_{ri})^2$, $\widehat{v}_r = \frac{1}{I} \sum_{i=1}^I \widehat{v}_{ri}$ and $\widehat{\text{Var}}(\bar{X}_{r..}) = \frac{1}{R-1} \sum_{r=1}^R (\bar{X}_{r..} - \bar{X}_{...})^2$.

Proof. (1) Consider

$$\widehat{v}_{ri} = \frac{1}{T-1} \sum_{t=1}^T m_{rit} (X_{rit} - \bar{X}_{ri})^2.$$

Recall that for fixed $r = 1, 2, \dots, R$ and fixed $i = 1, 2, \dots, I$, the random variables $X_{ri1}, X_{ri2}, \dots, X_{riT}$ are independent with common mean μ and variances $\text{Var}(X_{rit}|\Theta_i, \Lambda) = \sigma_x^2(\Theta_i, \Lambda)/m_{rit}$, conditional on Θ_i and Λ . Consequently, by using Lemma 4.2 with $\beta = 0$ and $\alpha = \sigma_x^2(\Theta_i, \Lambda)$, we have

$$E(\widehat{v}_{ri}|\Theta_i, \Lambda) = \frac{1}{T-1} E \left[\left(\sum_{t=1}^T m_{rit} (X_{rit} - \bar{X}_{ri})^2 \right) | \Theta_i, \Lambda \right] = \sigma_x^2(\Theta_i, \Lambda).$$

Thus,

$$E(\widehat{v}_{ri}) = E[E(\widehat{v}_{ri}|\Theta_i, \Lambda)] = E \{ E[\sigma_x^2(\Theta_i, \Lambda)|\Lambda] \} = E[\sigma_x^2(\Lambda)] = \sigma_x^2$$

and \widehat{v}_{ri} is unbiased for σ_x^2 . Hence, an unbiased estimator of σ_x^2 based on all data is

$$\widehat{\sigma}_x^2 = \frac{1}{RI} \sum_{r=1}^R \sum_{i=1}^I \widehat{v}_{ri}.$$

(2) For the r th portfolio, $r = 1, 2, \dots, R$, by using Lemma 4.1 to replace a, v and m_{it} by a_r, v_r and m_{rit} , respectively, an unbiased estimator of a_r is given by

$$\widehat{a}_r = \left(m_{r..} - m_{r..}^{-1} \sum_{i=1}^I m_{ri.}^2 \right)^{-1} \left[\sum_{i=1}^I m_{ri.} (\overline{X}_{ri.} - \overline{X}_{r..})^2 - \widehat{v}_r (I - 1) \right].$$

Furthermore,

$$E(\widehat{a}_r) = E[E(\widehat{a}_r | \Lambda)] = E[a_r(\Lambda)] = E\{\text{Var}[\mu(\Theta_i, \Lambda) | \Lambda]\} = E[\sigma_\theta^2(\Lambda)] = \sigma_\theta^2$$

and \widehat{a}_r is unbiased for σ_θ^2 . Hence, an unbiased estimator of σ_θ^2 based on all data is

$$\widehat{\sigma}_\theta^2 = \frac{1}{R} \sum_{r=1}^R \left\{ \left(m_{r..} - m_{r..}^{-1} \sum_{i=1}^I m_{ri.}^2 \right)^{-1} \left[\sum_{i=1}^I m_{ri.} (\overline{X}_{ri.} - \overline{X}_{r..})^2 - \widehat{v}_r (I - 1) \right] \right\}.$$

(3) As for (3) of Theorem 3.1. ■

Remark 4.1. For the model in this subsection, Ohlsson [3] presented the following estimators of σ_λ^2 and σ_θ^2 :

$$\widehat{\sigma}_\theta^2 = \left[m_{...} - \sum_{r=1}^R \left(\sum_{i=1}^I m_{ri.}^2 \right) / m_{r..} \right]^{-1} \left[\sum_{r=1}^R \sum_{i=1}^I m_{ri.} \left(\overline{X}_{ri.}^\omega - \overline{X}_{r..}^{\omega\omega} \right)^2 - \widehat{\sigma}_x^2 R (I - 1) \right] \text{ and}$$

$$\widehat{\sigma}_\lambda^2 = \left(z_{..} - \sum_{r=1}^R z_{r.}^2 / z_{..} \right)^{-1} \left[\sum_{r=1}^R z_{r.} \left(\overline{X}_{r..}^{z\omega} - \overline{X}_{...}^{zz\omega} \right)^2 - \widehat{\sigma}_\theta^2 (R - 1) \right],$$

where

$$\overline{X}_{ri.}^\omega = \frac{\sum_{t=1}^T m_{rit} X_{rit}}{\sum_{t=1}^T m_{rit}}, \quad \overline{X}_{r..}^{\omega\omega} = \frac{\sum_{i=1}^I m_{ri.} \overline{X}_{ri.}^\omega}{\sum_{i=1}^I m_{ri.}}, \quad z_{ri} = \frac{m_{ri.}}{m_{ri.} + \widehat{\sigma}_x^2 / \widehat{\sigma}_\theta^2},$$

$$z_{r.} = \frac{1}{I} \sum_{i=1}^I z_{ri}, \quad z_{..} = \frac{1}{R} \sum_{r=1}^R z_{r.}, \quad \overline{X}_{r..}^{z\omega} = \frac{\sum_{i=1}^I z_{ri} \overline{X}_{ri.}^\omega}{\sum_{i=1}^I z_{ri}} \text{ and } \overline{X}_{...}^{zz\omega} = \frac{\sum_{r=1}^R z_{r.} \overline{X}_{r..}^{z\omega}}{\sum_{r=1}^R z_{r.}}.$$

4.2. The estimators of the three-level common-effect model

Assume that we are given a set of insurance contracts consisting of K portfolios. For each portfolio, there are I insured individuals. In addition, the model satisfies assumptions A1 to A6 of subsection 2.2, and the following A7':

A7' For a fixed portfolio k and a fixed contract i , given the common effects Γ, Λ_k and Θ_{ki} , the claims $X_{ki1}, X_{ki2}, \dots, X_{kiT}$ are conditionally independent and identically distributed with $E(X_{kit} | \Theta_{ki}, \Lambda_k, \Gamma) = \mu(\Theta_{ki}, \Lambda_k, \Gamma)$ and variance $\text{Var}(X_{kit} | \Theta_{ki}, \Lambda_k, \Gamma) = \sigma^2(\Theta_{ki}, \Lambda_k, \Gamma) / m_{kit}$, where m_{kit} are known weights.

Theorem 4.2. Consider R sets of portfolios of the three-level common-effect model satisfying assumptions A1 to A6 of subsection 2.2 and A7'. Under the notations in subsection 2.2, unbiased estimators of the structural parameters, $\sigma_x^2, \sigma_\theta^2, \sigma_\lambda^2$ and σ_γ^2 , are given by

- (1) $\widehat{\sigma}_x^2 = \frac{1}{RKL} \sum_{r=1}^R \sum_{k=1}^K \sum_{i=1}^I \widehat{v}_{rki}$,
- (2) $\widehat{\sigma}_\theta^2 = \frac{1}{RK} \sum_{r=1}^R \sum_{k=1}^K \left\{ \left(m_{rk..} - m_{rk..}^{-1} \sum_{i=1}^I m_{rki}^2 \right)^{-1} \left[\sum_{i=1}^I m_{rki} (\bar{X}_{rki} - \bar{X}_{rk..})^2 - \widehat{v}_{rk} (I - 1) \right] \right\}$,
- (3) $\widehat{\sigma}_\lambda^2 = \frac{1}{R} \sum_{r=1}^R \left[\widehat{\text{Var}}(\bar{X}_{rk..}) - \widehat{v}_r \right]$, and
- (4) $\widehat{\sigma}_\gamma^2 = \widehat{\text{Var}}(\bar{X}_{r...}) - \frac{\widehat{\sigma}_\lambda^2}{K}$,

where

$$\begin{aligned}
 m_{rki} &= \sum_{t=1}^T m_{rkit}, & m_{rk..} &= \sum_{i=1}^I m_{rki}, & \bar{X}_{rki} &= \frac{1}{m_{rki}} \sum_{t=1}^T m_{rkit} X_{rkit}, \\
 \bar{X}_{rk..} &= \frac{1}{I} \sum_{i=1}^I \bar{X}_{rki}, & \bar{X}_{r...} &= \frac{1}{K} \sum_{k=1}^K \bar{X}_{rk..}, & \bar{X}_{....} &= \frac{1}{R} \sum_{r=1}^R \bar{X}_{r...}, \\
 \widehat{v}_{rki} &= \frac{1}{T-1} \sum_{t=1}^T m_{rkit} (X_{rkit} - \bar{X}_{rki})^2, & \widehat{v}_{rk} &= \frac{1}{I} \sum_{i=1}^I \widehat{v}_{rki}, & \widehat{v}_r &= \frac{1}{K} \sum_{k=1}^K \widehat{v}_{rk}, \\
 \widehat{\text{Var}}(\bar{X}_{rk..}) &= \frac{1}{K-1} \sum_{k=1}^K (\bar{X}_{rk..} - \bar{X}_{r...})^2 \text{ and } \widehat{\text{Var}}(\bar{X}_{r...}) &= \frac{1}{R-1} \sum_{r=1}^R (\bar{X}_{r...} - \bar{X}_{....})^2.
 \end{aligned}$$

Proof. (1) Consider

$$\widehat{v}_{rki} = \frac{1}{T-1} \sum_{t=1}^T m_{rkit} (X_{rkit} - \bar{X}_{rki})^2.$$

Recall that for fixed $r = 1, 2, \dots, R$, fixed $k = 1, 2, \dots, K$ and fixed $i = 1, 2, \dots, I$, the random variables $X_{rki1}, X_{rki2}, \dots, X_{rkiT}$ are independent with common mean μ and variances $\text{Var}(X_{rkit} | \Theta_{ki}, \Lambda_k, \Gamma) = \sigma_x^2(\Theta_{ki}, \Lambda_k, \Gamma) / m_{rkit}$, conditional on Θ_{ki}, Λ_k and Γ . Consequently, by using Lemma 4.2 with $\beta = 0$ and $\alpha = \sigma_x^2(\Theta_{ki}, \Lambda_k, \Gamma)$, we have

$$E(\widehat{v}_{rki} | \Theta_{ki}, \Lambda_k, \Gamma) = \frac{1}{T-1} E \left[\left(\sum_{t=1}^T m_{rkit} (X_{rkit} - \bar{X}_{rki})^2 \right) | \Theta_{ki}, \Lambda_k, \Gamma \right] = \sigma_x^2(\Theta_{ki}, \Lambda_k, \Gamma).$$

Thus,

$$E(\widehat{v}_{rki}) = E \{ E \{ E(\widehat{v}_{rki} | \Theta_{ki}, \Lambda_k, \Gamma) | \Lambda_k, \Gamma \} | \Gamma \} = E \{ E \{ \sigma_x^2(\Theta_{ki}, \Lambda_k, \Gamma) | \Lambda_k, \Gamma \} | \Gamma \} = \sigma_x^2$$

and \widehat{v}_{rki} is unbiased for σ_x^2 . Hence, an unbiased estimator of σ_x^2 based on all data is

$$\widehat{\sigma}_x^2 = \frac{1}{RKL} \sum_{r=1}^R \sum_{k=1}^K \sum_{i=1}^I \widehat{v}_{rki}.$$

(2) For the k th portfolio, $k = 1, 2, \dots, K$, from the r th set, $r = 1, 2, \dots, R$, by using Lemma 4.1 to replace a, v and m_{it} by a_{rk}, v_{rk} and m_{rkit} , respectively, an unbiased estimator of a_{rk} is given by

$$\widehat{a}_{rk} = \left(m_{rk..} - m_{rk..}^{-1} \sum_{i=1}^I m_{rki}^2 \right)^{-1} \left[\sum_{i=1}^I m_{rki} (\bar{X}_{rki} - \bar{X}_{rk..})^2 - \widehat{v}_{rk} (I - 1) \right].$$

Furthermore,

$$E(\widehat{a}_{rk}) = E [E(\widehat{a}_{rk} | \Lambda_k, \Gamma)] = E \{ E [E(\widehat{a}_{rk} | \Lambda_k, \Gamma) | \Gamma] \} = E \{ E [a_{rk}(\Lambda_k, \Gamma) | \Gamma] \} = \sigma_\theta^2,$$

and \widehat{a}_{rk} is unbiased for σ_θ^2 . Hence an unbiased estimator of σ_θ^2 based on all data is

$$\widehat{\sigma}_\theta^2 = \frac{1}{RK} \sum_{r=1}^R \sum_{k=1}^K \left\{ \left(m_{rk..} - m_{rk..}^{-1} \sum_{i=1}^I m_{rki}^2 \right)^{-1} \left[\sum_{i=1}^I m_{rki} (\bar{X}_{rki} - \bar{X}_{rk..})^2 - \widehat{v}_{rk}(I-1) \right] \right\}.$$

(3) As for (3) of *Theorem 3.2*.

(4) As for (4) of *Theorem 3.2*. ■

Remark 4.2. We can obtain a similar proof for (1) and (2) for theorems 3.1, 3.2, 4.1 and 4.2 in standard textbooks on credibility models; see, e.g., [2] and [4].

Remark 4.3. Note that due to the subtraction in estimators $\widehat{\sigma}_\theta^2$, $\widehat{\sigma}_\lambda^2$ and $\widehat{\sigma}_\gamma^2$, it is possible for each of them to be negative. When the estimator is negative, it is customary to set it equals to zero.

Remark 4.4. Considering the two- and three-level common-effect formulas in this paper, one can easily conjecture the multi-level model formulas if all assumptions are maintained. The multi-level model as follows.

For the multi-level formula of the Bühlmann model, consider that $X_{n_h, n_{h-1}, \dots, n_1, n_0}$ gives the claim amount for individual n_1 during the period n_0 in the h -level common-effect model where $n_j = 1, 2, \dots, N_j$ and $j = 0, 1, 2, \dots, h$. Unbiased estimators of the structural parameters are given by

$$(1) \quad \widehat{\sigma}_0^2 = \frac{1}{\prod_{j=1}^h N_j} \sum_{n_h=1}^{N_h} \sum_{n_{h-1}=1}^{N_{h-1}} \dots \sum_{n_1=1}^{N_1} \widehat{\text{Var}}(X_{n_h, n_{h-1}, \dots, n_1, n_0}),$$

$$(2) \quad \widehat{\sigma}_g^2 = \frac{1}{\prod_{j=g+1}^h N_j} \sum_{n_h=1}^{N_h} \sum_{n_{h-1}=1}^{N_{h-1}} \dots \sum_{n_{g+1}=1}^{N_{g+1}} \widehat{\text{Var}}(\bar{X}_{n_h, n_{h-1}, \dots, n_g, \dots}) - \frac{\widehat{\sigma}_{g-1}^2}{N_{g-1}}, \quad \text{for } 1 \leq g < h, \text{ and}$$

$$(3) \quad \widehat{\sigma}_h^2 = \widehat{\text{Var}}(\bar{X}_{h, \dots}) - \frac{\widehat{\sigma}_{h-1}^2}{N_{h-1}},$$

where

$$\bar{X}_{n_h, n_{h-1}, \dots, n_g, \dots} = \frac{1}{N_{g-1}} \sum_{n_{g-1}=1}^{N_{g-1}} \bar{X}_{n_h, n_{h-1}, \dots, n_{g-1}, \dots}, \quad \bar{X}_{n_h, n_{h-1}, \dots, n_1, n_0} = X_{n_h, n_{h-1}, \dots, n_1, n_0}$$

and

$$\widehat{\text{Var}}(\bar{X}_{n_h, n_{h-1}, \dots, n_g, \dots}) = \frac{1}{N_g - 1} \sum_{n_g=1}^{N_g} (\bar{X}_{n_h, n_{h-1}, \dots, n_g, \dots} - \bar{X}_{n_h, n_{h-1}, \dots, n_{g+1}, \dots})^2$$

for $0 \leq g < h$.

For the multi-level formulas of the Bühlmann-Straub model, consider $X_{n_h, n_{h-1}, \dots, n_1, n_0}$ the claim amount for individual n_1 during period n_0 in the h -level common-effect model where

$n_j = 1, 2, \dots, N_j$ and $j = 0, 1, 2, \dots, h$. Furthermore, let $m_{n_h, n_{h-1}, \dots, n_1, n_0}$ denote the known weights. The unbiased nonparametric estimators of the structural parameters are given by

$$(1) \widehat{\sigma}_0^2 = \frac{1}{\prod_{j=1}^h N_j} \sum_{n_h=1}^{N_h} \sum_{n_{h-1}=1}^{N_{h-1}} \dots \sum_{n_1=1}^{N_1} \widehat{v}_{n_h, n_{h-1}, \dots, n_1},$$

$$(2) \widehat{\sigma}_1^2 = \frac{1}{\prod_{j=1}^h N_j} \sum_{n_h=1}^{N_h} \sum_{n_{h-1}=1}^{N_{h-1}} \dots \sum_{n_2=1}^{N_2} \left\{ \left(m_{n_h, n_{h-1}, \dots, n_2, \dots} - m_{n_h, n_{h-1}, \dots, n_2, \dots}^{-1} \sum_{n_1=1}^{N_1} m_{n_h, n_{h-1}, \dots, n_1, \dots}^2 \right)^{-1} \right. \\ \left. \times \left[\sum_{n_1=1}^{N_1} m_{n_h, n_{h-1}, \dots, n_1, \dots} (\overline{X}_{n_h, n_{h-1}, \dots, n_1, \dots} - \overline{X}_{n_h, n_{h-1}, \dots, n_2, \dots})^2 \right] \right\},$$

$$- \widehat{v}_{n_h, n_{h-1}, \dots, n_2} (N_1 - 1)$$

$$(3) \widehat{\sigma}_2^2 = \frac{1}{\prod_{j=2}^h N_j} \sum_{n_h=1}^{N_h} \sum_{n_{h-1}=1}^{N_{h-1}} \dots \sum_{n_3=1}^{N_3} \left[\widehat{\text{Var}}(\overline{X}_{n_h, n_{h-1}, \dots, n_2, \dots}) - \frac{\widehat{v}_{n_h, n_{h-1}, \dots, n_3}}{N_1} \right],$$

$$(4) \widehat{\sigma}_g^2 = \frac{1}{\prod_{j=g+1}^h N_j} \sum_{n_h=1}^{N_h} \sum_{n_{h-1}=1}^{N_{h-1}} \dots \sum_{n_{g+1}=1}^{N_{g+1}} \widehat{\text{Var}}(\overline{X}_{n_h, n_{h-1}, \dots, n_g, \dots, \dots}) - \frac{\widehat{\sigma}_{g-1}^2}{N_{g-1}}, \quad \text{for } 3 \leq g < h, \text{ and}$$

$$(5) \widehat{\sigma}_h^2 = \widehat{\text{Var}}(\overline{X}_{h, \dots, \dots}) - \frac{\widehat{\sigma}_{h-1}^2}{N_{h-1}},$$

where, for $2 \leq g < h$,

$$m_{n_h, n_{h-1}, \dots, n_1, \dots} = \sum_{n_0=1}^{N_0} m_{n_h, n_{h-1}, \dots, n_1, n_0}, \quad m_{n_h, n_{h-1}, \dots, n_2, \dots} = \sum_{n_1=1}^{N_1} m_{n_h, n_{h-1}, \dots, n_1, \dots},$$

$$\overline{X}_{n_h, n_{h-1}, \dots, n_1, \dots} = \frac{1}{m_{n_h, n_{h-1}, \dots, n_1, \dots}} \sum_{n_0=1}^{N_0} m_{n_h, n_{h-1}, \dots, n_1, n_0} X_{n_h, n_{h-1}, \dots, n_1, n_0},$$

$$\overline{X}_{n_h, n_{h-1}, \dots, n_g, \dots, \dots} = \frac{1}{N_{g-1}} \sum_{n_{g-1}=1}^{N_{g-1}} \overline{X}_{n_h, n_{h-1}, \dots, n_{g-1}, \dots, \dots},$$

$$\widehat{v}_{n_h, n_{h-1}, \dots, n_1, \dots} = \frac{1}{N_0 - 1} \sum_{n_0=1}^{N_0} m_{n_h, n_{h-1}, \dots, n_1, n_0} (X_{n_h, n_{h-1}, \dots, n_1, n_0} - \overline{X}_{n_h, n_{h-1}, \dots, n_1, \dots})^2$$

$$\widehat{v}_{n_h, n_{h-1}, \dots, n_g, \dots, \dots} = \frac{1}{N_{g-1}} \sum_{n_{g-1}=1}^{N_{g-1}} \widehat{v}_{n_h, n_{h-1}, \dots, n_{g-1}, \dots},$$

$$\widehat{\text{Var}}(\overline{X}_{n_h, n_{h-1}, \dots, n_g, \dots, \dots}) = \frac{1}{N_g - 1} \sum_{n_g=1}^{N_g} (\overline{X}_{n_h, n_{h-1}, \dots, n_g, \dots, \dots} - \overline{X}_{n_h, n_{h-1}, \dots, n_{g+1}, \dots, \dots})^2.$$

Table 1. Comparison of the estimators using the data in Table 3.1 of [2]

	True value	Ohlsson's estimators	Our estimators
σ_θ^2	1.000	1.209 (1.093)	1.256 (1.093)
σ_λ^2	25.000	25.300 (25.259)	25.302 (25.318)

5. Numerical examples

5.1. Numerical example with the two-level common-effect models

For the two-level common-effect model, we can see that the differences between the Ohlsson's unbiased estimators and our unbiased estimators are usually rather small. For confidentiality reasons, we use the artificial population in section 3.3 of [2] and the results from Table 4.1 in [3]. We take the stated value $\widehat{\sigma}_x^2 = 15.89$ as given and only estimate σ_θ^2 and σ_λ^2 . The results are given in Table 1. In parenthesis we give the values when all the weights m_{rit} are set equal to one. The difference between the estimators is negligible for σ_λ^2 and less than 4 for σ_θ^2 . In this case, we know the true values because the population was artificially generated. The present estimators have the advantage due to the simplicity in extension and application for hierarchical case.

5.2. Numerical example with the three-level common-effect model

To illustrate numerically the three-level common-effect model, we generate claims data and use simulation to examine what effect there might be from assuming some level of dependence between portfolios, between individuals and across time. These simulations use the three-level common-effect framework introduced in section 2.2, and we compare its estimators with those of the two-level model and the ordinary Bayesian normal model. The Bayesian normal model assumes dependence only across individuals; this assumption is for simplicity and mathematical tractability and always made without regard as to whether it is violated or not. First, we simulate observations satisfying the assumptions of the three-level common-effect model and then compare the results to those three different models: the two and three-level common-effect models described in this paper, assuming normal distributions for the common effects, and the conventional Bayesian normal model, which allows for only one level of common effect. We are interested in the numerical comparisons of the resulting unbiased estimators of the structural parameters under each model.

The specifications, descriptions, and parameter values used in the simulation are given in Table 2. To allow for meaningful comparison between these three models, we have chosen the parameters in the three models to be consistent with each other. More precisely, for example, we set $\mu = \mu_\theta + \mu_\lambda + \mu_\gamma$ and $\sigma^2 = \sigma_\theta^2 + \sigma_\lambda^2 + \sigma_\gamma^2$. The variances are additive in the Bayesian normal model because it is assumed that θ_{ki} , λ_k and γ are independent of each other. We generated $R = 10$ different sets of insurance contracts consisting of 10-year paths of claims for 10 different individuals for 10 different portfolios assuming that the three-level common-effect model is the true model for each of the portfolios. Hence, we are assuming that in reality, there are three common effects as described in this paper that are inducing the claims. Recall that we assumed that for each portfolio and individual, the claims amount for each time period conditional on θ_{ki} , λ_k and γ , i.e., $X_{kit} | \theta_{ki}, \lambda_k, \gamma$, is normally distributed with

Table 2. Summary of model assumptions and the parameters used in the simulation

Specification	Description
Model III :Three level normal common-effect model Conditional density 'Individual' common effect 'Portfolio' common effect 'Overall' common effect Assumption Parameter values	For $k = 1, 2, \dots, K, i = 1, 2, \dots, I$ and $t = 1, 2, \dots, T$ $X_{kit} \theta_{ki}, \lambda_k, \gamma \sim N(\theta_{ki} + \lambda_k + \gamma, \sigma_x^2)$ $\theta_{ki} \sim N(\mu_\theta, \sigma_\theta^2)$ $\lambda_k \sim N(\mu_\lambda, \sigma_\lambda^2)$ $\gamma \sim N(\mu_\gamma, \sigma_\gamma^2)$ $K = 10, I = 10, T = 10$ $\sigma_x^2 = 22000$ $\mu_\theta = 100, \sigma_\theta^2 = 1000$ $\mu_\lambda = 200, \sigma_\lambda^2 = 4000$ $\mu_\gamma = 300, \sigma_\gamma^2 = 16000$
Model II :Two level normal common-effect model Conditional density 'Individual' common effect 'Overall' common effect Assumption Parameter values	For $i = 1, 2, \dots, I$ and $t = 1, 2, \dots, T$ $X_{it} \theta_i, \lambda \sim N(\theta_i + \lambda, \sigma_x^2)$ $\theta_i \sim N(\mu_\theta, \sigma_\theta^2)$ $\lambda \sim N(\mu_\lambda, \sigma_\lambda^2)$ $I = 100$ individuals, $T = 10$ years $\sigma_x^2 = 22000$ $\mu_\theta = 100, \sigma_\theta^2 = 1000$ $\mu_\lambda = 500, \sigma_\lambda^2 = 20000$
Model I :Bayesian normal model Conditional density Single common effect Assumption Parameter values	For $i = 1, 2, \dots, I$ and $t = 1, 2, \dots, T$ $X_{it} \theta \sim N(\theta, \sigma_x^2)$ $\theta \sim N(\mu_\theta, \sigma_\theta^2)$ $I = 100$ individuals, $T = 10$ years $\sigma_x^2 = 22000$ $\mu_\theta = 600, \sigma_\theta^2 = 21000$

mean $\theta_{ki} + \lambda_k + \gamma$ and variance σ_x^2 . Parameters θ_{ki} , λ_k and γ are also normally distributed with means μ_θ , μ_λ and μ_γ and variances σ_θ^2 , σ_λ^2 and σ_γ^2 , respectively.

We refer to the three-level normal common-effect model as Model III, the two-level normal common-effect model as Model II and the Bayesian normal model as Model I. For each sample set, we compute the estimators of the structural parameters for the three models. We repeat the above simulation $n = 1000$ times and then compute the average of 1000 estimator values for each of the structural parameters of the three models. The differences between the true values and our unbiased estimators are generally rather small. The results are given in Table 3. For Model I, the differences between the averages of our estimators and the true values is negligible for σ_x^2 and less than 6% for σ_θ^2 . For Model II, the difference is less than 4% for σ_λ^2 . Finally, for Model III, the difference is less than 5% for σ_λ^2 and less than 4% for σ_γ^2 . These results appear reasonable because we estimate the structural parameters in each level and use them in the next level to estimate the other structural parameters. Thus, the sources of errors increase with higher-level models.

Table 3. Comparison of the estimators of the three models in Table 2

Model	Parameter	True value	Average of our estimation
III	σ_x^2	22000.00	22010.57
	σ_θ^2	1000.00	995.52
	σ_λ^2	4000.00	4199.34
	σ_γ^2	16000.00	16531.52
II	σ_x^2	22000.00	22010.57
	σ_θ^2	1000.00	995.52
	σ_λ^2	20000.00	19219.12
I	σ_x^2	22000.00	22010.57
	σ_θ^2	21000.00	19823.37

6. Conclusion

One of the primary challenges in using credibility models in practice is the estimation of structural parameters. Several methods found in published literature provide estimators that are not necessarily unbiased or simple. In this paper, we introduced a new method to estimate structural parameters of credibility models with dependence induced by common effects. We find that the difference between the true values and our unbiased estimators is generally rather small. The new estimators have the advantage of being easy to use.

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