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# On the Cozero-Divisor Graphs of Commutative Rings and Their Complements

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**Abstract.** Let $R$ be a commutative ring with non-zero identity. The cozero-divisor graph of $R$, denoted by $\Gamma'(R)$, is a graph with vertices in $W^*(R)$, which is the set of all non-zero and non-unit elements of $R$, and two distinct vertices $a$ and $b$ in $W^*(R)$ are adjacent if and only if $a \notin bR$ and $b \notin aR$. In this paper, we characterize all commutative rings whose cozero-divisor graphs are forest, star, double-star or unicyclic.

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## 1. Introduction

Let $R$ be a commutative ring with non-zero identity and let $Z(R)$ be the set of all zero-divisors of $R$. Set $Z^*(R) := Z(R) \setminus \{0\}$. The zero-divisor graph of $R$, denoted by $\Gamma(R)$, is an undirected graph whose vertices are elements of $Z^*(R)$ with two distinct vertices $a$ and $b$ are adjacent if and only if $ab = 0$.

The concept of the zero-divisor graph of a commutative ring was introduced by Beck [4], but this work was mostly concerned with coloring of rings. The above definition first appeared in Anderson and Livingston [3], which contained several fundamental results concerning $\Gamma(R)$. The zero-divisor graph of commutative rings has been studied extensively by Anderson, Frazier, Lauve and Livingston (cf. [2] and [3]).

Let $W(R)$ be the set of all non-unit elements of $R$ and $W^*(R) := W(R) \setminus \{0\}$. For an arbitrary commutative ring $R$, the cozero-divisor graph $\Gamma'(R)$ of $R$ was introduced in [1], which is a dual of the zero-divisor graph $\Gamma(R)$ "in some sense". The vertex-set of $\Gamma'(R)$ is $W^*(R)$ and for two distinct vertices $a$ and $b$ in $W^*(R)$, $a$ is adjacent to $b$ if and only if $a \notin bR$ and $b \notin aR$, where $cR$ is the ideal generated by the element $c$ in $R$. Some basic results on the structure of this graph and the relations between the graphs $\Gamma(R)$ and $\Gamma'(R)$ were studied in [1].

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In this paper, we study some more properties of the cozero-divisor graph $\Gamma'(R)$, where $R$ is a commutative ring. In section two, we characterize all commutative rings whose cozero-divisor graphs are double-star, unicyclic, star or forest. On the other hand, for a semigroup $H$ and a subset $S$ of $H$, the Cayley graph $\mathrm{Cay}(H,S)$ of $H$ relative to $S$ is defined as the graph with vertex-set $H$ and edge-set $E(H,S)$ consisting of those ordered pairs $(x,y)$ such that $sx = y$ for some $s \in S$ (cf [7]). By the ordered pair $(x,y)$, we mean that $x \longrightarrow y$. So $x \longrightarrow y$ if $sx = y$, for some $s \in S$. Moreover, if we assume that $x$ is adjacent to $y$ in $\mathrm{Cay}(H,S)$ if and only if $(x,y)$ or $(y,x)$ is an element of the edge-set $E(H,S)$, then we have the undirected Cayley graph $\mathrm{Cay}(H,S)$. Therefore, in an undirected Cayley graph $\mathrm{Cay}(H,S)$, $x$ is adjacent to $y$ if and only if $x \longrightarrow y$ or $y \longrightarrow x$. Now, consider the complement of the cozero-divisor graph $\Gamma'(R)$, denoted by $\overline{\Gamma'(R)}$. For any two distinct vertices $a$ and $b$ in $W^*(R)$, $a$ is adjacent to $b$ if and only if $a \in bR$ or $b \in aR$. Thus the graph $\overline{\Gamma'(R)}$ and the undirected graph Cayley graph $\mathrm{Cay}(W^*(R), R\setminus\{1\})$ coincide. In section three, we study the graph $\overline{\Gamma'(R)}$.

Throughout the paper, $R$ is a commutative ring with non-zero identity. We denote the set of maximal ideals and the Jacobson radical of $R$ by $\max(R)$ and $J(R)$, respectively. In a graph $G$, the distance between two distinct vertices $a$ and $b$, denoted by $\mathrm{d}_G(a,b)$, is the length of a shortest path connecting $a$ and $b$, if such a path exists; otherwise, we set $\mathrm{d}_G(a,b) := \infty$. The diameter of a graph $G$ is $\mathrm{diam}(G) = \sup\{\mathrm{d}_G(a,b) : a \text{ and } b \text{ are distinct vertices of } G\}$. The girth of $G$, denoted by $\mathrm{g}(G)$, is the length of a shortest cycle in $G$, if $G$ contains a cycle; otherwise, $\mathrm{g}(G) := \infty$. Also, $V(G)$ and $E(G)$ are the sets of vertices and edges of $G$, respectively and for two distinct vertices $a$ and $b$ in $V(G)$, the notation $a - b$ means that $a$ and $b$ are adjacent. A graph $G$ is said to be connected if there exists a path between any two distinct vertices, and it is complete if each pair of distinct vertices is joined by an edge. For a positive integer $n$, we use $K_n$ to denote the complete graph with $n$ vertices. Also, we say that $G$ is totally disconnected if no two vertices of $G$ are adjacent. For a positive integer $r$, an $r$-partite graph is one whose vertex set can be partitioned into $r$ subsets so that no edge has both ends in any one subset. A complete $r$-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes $m$ and $n$ is denoted by $K_{m,n}$. Also, the valency of a vertex $a$ is the number of edges of the graph $G$ incident with $a$. The complement $\overline{G}$ of $G$ is the graph with the same vertex-set as $G$, where two distinct vertices are adjacent whenever they are non-adjacent in $G$.

## 2. On the cozero-divisor graphs

Recall that if $R$ is finite, then each element of $R$ is either a unit or a zero-divisor and so $W(R) = Z(R)$. Also, by [6, Theorem 1], $|R| \leqslant |Z(R)|^2$ when $|Z(R)| \geqslant 2$. Moreover, we recall that the union of graphs $G_1$ and $G_2$, which is denoted by $G_1 \cup G_2$, where $G_1$ and $G_2$ are two vertex-disjoint graphs, is a graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Also a graph on $n$ vertices such that $n-1$ of the vertices have valency one, all of which are adjacent only to the remaining vertex $a$, is called a star graph with center $a$. In fact, every star graph with $n$ vertices is isomorphic to $K_{1,n-1}$, the complete bipartite graph with part sizes 1 and $n-1$. We consider the empty graph as a star graph. Also, a double-star graph is a union of two star graphs with centers $a_1$ and $a_2$ such that $a_1$ is adjacent to $a_2$. A unicyclic graph is a connected graph with a unique cycle, or we can regard a unicycle graph as a cycle attached with each vertex a (rooted) tree.

In the following theorem we study the case that $\Gamma'(R)$ is a forest.

**Theorem 2.1.** *Let R be a non-local finite ring.*

(i) *If $\Gamma'(R)$ is a forest (contains no cycles), then $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field.*
(ii) *If $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field, then $\Gamma'(R)$ is a star graph.*

*Proof.* (i) Suppose that $\Gamma'(R)$ is a forest. Since $R$ is finite, there exists a positive integer $n$ such that $R \cong R_1 \times \cdots \times R_n$, where $R_i$ is a local ring with maximal ideal $\mathfrak{m}_i$, for $i = 1, \ldots, n$. Whenever $n \geqslant 3$, since $|\max(R)| \geqslant 3$, it is easy to see that $\Gamma'(R)$ contains a cycle and so it is not a forest. Moreover, since $R$ is non-local, $n \geqslant 2$. Hence we have that $n = 2$ and so $R \cong R_1 \times R_2$. Now, suppose that $R_2$ is not a field. Then we have the cycle $(0,1)-(1,0)-(0,1+r)-(1,r)-(0,1)$, where $r \in W^*(R_2)$ and so $\Gamma'(R)$ is not a forest which is impossible. Hence $R_2$ is a field. Similarly, $R_1$ is a field. If neither $R_1$ nor $R_2$ is $\mathbb{Z}_2$, then for any arbitrary elements $r \in R_1 \setminus \{0,1\}$ and $s \in R_2 \setminus \{0,1\}$, we have the cycle $(0,1)-(1,0)-(0,s)-(r,0)-(0,1)$, which is again impossible. This implies that $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field.

(ii) If $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field, then one can easily see that $\Gamma'(R)$ is a star graph with center $(1,0)$. ∎

**Theorem 2.2.** *Let R be a finite ring.*

(i) *If R is non-local, then $\Gamma'(R)$ is a double-star graph if and only if $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field.*
(ii) *If R is local with principal maximal ideal $\mathfrak{m}$, then $\Gamma'(R)$ is a double-star graph if and only if R is either $\mathbb{Z}_4, \mathbb{Z}_2[X]/(x^2\mathbb{Z}_2[X])$ or $\mathbb{F}$, where $\mathbb{F}$ is a field.*
(iii) *If R is local with non-principal maximal ideal $\mathfrak{m}$ and $\Gamma'(R)$ is a double-star graph, then the minimal generating set of $\mathfrak{m}$ has two elements.*

*Proof.* (i) Since every double-star graph is a forest and also every star graph is double-star, the result immediately follows from Theorem 2.1.

(ii) By [1, Theorem 2.7], the graph $\Gamma'(R)$ is totally disconnected and so, in this situation, $\Gamma'(R)$ is a double-star graph if and only if $|\mathfrak{m}| \leqslant 2$. If $|\mathfrak{m}| = 1$, then $R$ is a field. Otherwise, $|\mathfrak{m}| = 2$. Now, since $|R| \leqslant |Z(R)|^2$, one can conclude that $R$ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(x^2\mathbb{Z}_2[X])$.

(iii) Suppose that $\mathfrak{m}$ is not principal and that the graph $\Gamma'(R)$ is a double-star graph. Also, assume to the contrary that $\{r_1, r_2, r_3\}$ is a subset of a minimal generating set of $\mathfrak{m}$. Then we have the triangle $r_1 - r_2 - r_3 - r_1$, which is the required contradiction. ∎

The following corollary is an immediate consequence of Theorems 2.1 and 2.2.

**Corollary 2.1.** *Let R be a finite ring. If the graph $\Gamma'(R)$ is a double-star graph, then either R is local or $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field.*

Recall that a connected forest is called a tree. By slight modifications in the proofs of Theorems 2.1 and 2.2 we have the following consequences.

**Consequences 2.1.** Let $R$ be a finite ring.

(a) If $R$ is non-local, then the following conditions are equivalent.
  (i) $\Gamma'(R)$ is a forest.
  (ii) $\Gamma'(R)$ is a star graph.
  (iii) $\Gamma'(R)$ is a double-star graph.
  (iv) $\Gamma'(R)$ is a tree.

(v) $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field.

(b) If $R$ is local with principal maximal ideal, then $\Gamma'(R)$ is a forest and the following conditions are equivalent.

(i) $\Gamma'(R)$ is a star graph.

(ii) $\Gamma'(R)$ is a double-star graph.

(iii) $R$ is a field or $R$ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(x^2\mathbb{Z}_2[X])$.

(iv) $\Gamma'(R)$ is a tree.

The following Lemma is needed in the sequel.

**Lemma 2.1.** *Suppose that $\Gamma'(R)$ is a unicyclic graph. Then $|\max(R)| \leqslant 3$. In particular, if $\max(R) = \{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3\}$, then $|\mathfrak{m}_i \setminus \bigcup_{i \neq j} \mathfrak{m}_j| = 1$, for all $i = 1,2,3$.*

*Proof.* Assume to the contrary that, for $i = 1,\dots,4$, $\mathfrak{m}_i$ is a maximal ideal of $R$. Let $a_i \in \mathfrak{m}_i \setminus \bigcup_{i \neq j} \mathfrak{m}_j$, where $1 \leqslant i \leqslant 4$. Then the vertices $a_1, a_2, a_3, a_4$ form a complete subgraph of $\Gamma'(R)$. So $\Gamma'(R)$ is not a unicyclic graph, which is a contradiction.

Now, suppose that $\max(R) = \{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3\}$. Let $a_i \in \mathfrak{m}_i \setminus \bigcup_{i \neq j} \mathfrak{m}_j$, for $i = 1,2,3$. Assume to the contrary that for some $1 \leqslant i \leqslant 3$, there is an element $b_i \in \mathfrak{m}_i \setminus \bigcup_{i \neq j} \mathfrak{m}_j$ with $a_i \neq b_i$. Without loss of generality, we may assume that $i = 1$. Now, we have the cycles

$$a_1 - a_2 - a_3 - a_1 \text{ and } b_1 - a_2 - a_3 - b_1.$$

This means that $\Gamma'(R)$ is not a unicyclic graph which is the required contradiction. ∎

In the next theorem, we characterize the rings whose cozero-divisor graphs are unicyclic.

**Theorem 2.3.** *Let $R$ be a non-local finite ring. Then $\Gamma'(R)$ is a unicyclic graph if and only if $R$ is one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(x^2\mathbb{Z}_2[X])$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$.*

*Proof.* Clearly, if $R$ is one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$, then the cozero-divisor graph $\Gamma'(R)$ is a unicyclic graph. Conversely, suppose that $\Gamma'(R)$ is a unicyclic graph. Since $R$ is non-local and finite, there exists a positive integer $n \geqslant 2$ such that $R \cong R_1 \times \cdots \times R_n$, where $R_i$ is a local ring with maximal ideal $\mathfrak{m}_i$, for $i = 1,\dots,n$. In view of Lemma 2.1, we may assume that $n \leqslant 3$. Now, suppose that $n = 3$. We show that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. To this end, assume to the contrary that there exists $1 \leqslant i \leqslant 3$ such that $R_i \ncong \mathbb{Z}_2$. Without loss of generality, we may assume that $R_2 \ncong \mathbb{Z}_2$. Put $M_1 := \mathfrak{m}_1 \times R_2 \times R_3$, $M_2 := R_1 \times \mathfrak{m}_2 \times R_3$ and $M_3 := R_1 \times R_2 \times \mathfrak{m}_3$. Now, since $R_2 \ncong \mathbb{Z}_2$, there exists an element $a$ in $R_2$ such that $a$ or $1 + a$ is a unit in $R_2$. So one can assume that $a$ is a unit. Then $(0,1,1), (0,a,1) \in M_1 \setminus (M_2 \cup M_3)$, which is impossible by Lemma 2.1. But we have the cycles $(0,1,0) - (0,0,1) - (1,0,0) - (0,1,0)$ and $(1,0,1) - (0,1,1) - (1,1,0) - (1,0,1)$ in $\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$, and so the cozero-divisor graph of the ring $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is not unicyclic. Since $R$ is non-local, we may assume that $n = 2$ and $R \cong R_1 \times R_2$. Now, suppose that one of the rings $R_1$ or $R_2$ has at least four elements. So without loss of generality we may assume that $|R_1| \geqslant 4$. If $R_2 \ncong \mathbb{Z}_2$, there exists a cycle $(0,1) - (1,0) - (0,b) - (a,0) - (0,1)$, where $a \in R_1 \setminus \{0,1\}$ and $b \in R_2 \setminus \{0,1\}$. Also, there exists $c$ in $R_1 \setminus \{0,1,a\}$ such that the vertex $(c,0)$ is adjacent to both vertices $(0,1)$ and $(0,b)$. This means that $\Gamma'(R)$ is not a unicyclic graph. Hence, in this situation, we may assume that $R_2 \cong \mathbb{Z}_2$. Now, if $R_1$ is a field, then $\Gamma'(R)$ is a star graph and so it is not unicyclic. If $R$ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(x^2\mathbb{Z}_2[X])$, then we are done. Otherwise, $|R_1| > 4$ and $R_1$ is not a field. Thus we have the cycle $(0,1) - (1,0) - (a,1) - (b,0) - (0,1)$, where $a \in W^*(R_1)$ and $b = 1 + a \in U(R_1)$. Also, suppose that $c \in R_1 \setminus \{0,1,a,b\}$. Then $c$ or $1 + c$ is a unit in $R_1$. Note that $1 + c \in R_1 \setminus \{0,1,a,b\}$. So, we may assume that $c$ is a unit. Moreover, the vertex $(c,0)$ is adjacent to the vertices $(0,1)$ and $(a,1)$ in $\Gamma'(R)$ which is again impossible. Now, the only remaining case is that the rings $R_1$ and $R_2$ have less than four elements and so $R$ is isomorphic to one of the rings $\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_3$ or $\mathbb{Z}_3\times\mathbb{Z}_3$. But, $\Gamma'(\mathbb{Z}_2\times\mathbb{Z}_2)$ and $\Gamma'(\mathbb{Z}_2\times\mathbb{Z}_3)$ are star graphs. Therefore, we have that $R\cong\mathbb{Z}_3\times\mathbb{Z}_3$. ∎

A refinement of a graph $H$ is a graph $G$ such that the vertex sets of $G$ and $H$ are the same and every edge in $H$ is an edge in $G$.

**Proposition 2.1.** *The graph $\Gamma'(R)$ is the refinement of a star graph if and only if there exists an element $a$ in $W^*(R)$ such that $|aR|=2$ and, for all $b\in W^*(R)$ with $a\neq b$, $a\notin bR$. In particular, if there exists a maximal ideal $\mathfrak{m}$ of $R$ such that $|\mathfrak{m}|=2$, then $\Gamma'(R)$ is the refinement of a star graph.*

*Proof.* First suppose that $\Gamma'(R)$ is the refinement of a star graph. So there is a vertex $a$ which is adjacent to all the other vertices. This means that $|aR|=2$ and $a\notin bR$, for all $b\in W^*(R)\setminus\{a\}$. Conversely, if there exists an element $a$ in $W^*(R)$ such that $|aR|=2$ and for all $b\in W^*(R)$ with $a\neq b$, $a\notin bR$, then clearly the vertex $a$ is adjacent to all vertices in $W^*(R)\setminus\{a\}$. This implies that $\Gamma'(R)$ is the refinement of a star graph. ∎

We recall that a cycle graph is a graph which consists of a single cycle, and the number of edges in a cycle is called its length.

**Lemma 2.2.** *If $\Gamma'(R)$ is a union of cycle graphs, then $|\max(R)|\leqslant 3$. In particular, if $\max(R)=\{\mathfrak{m}_1,\mathfrak{m}_2,\mathfrak{m}_3\}$, then $|\mathfrak{m}_i\setminus\bigcup_{j\neq i}\mathfrak{m}_j|=1$, for all $i=1,2,3$.*

*Proof.* If $|\max(R)|\geqslant 4$, then $\Gamma'(R)$ contains a subgraph isomorphic to $K_4$ and so it can't be a union of cycle graphs. Hence we have that $|\max(R)|<4$. Now, suppose that $\max(R)=\{\mathfrak{m}_1,\mathfrak{m}_2,\mathfrak{m}_3\}$ and $m_i$ is an arbitrary element in $\mathfrak{m}_i\setminus\bigcup_{j\neq i}\mathfrak{m}_j$, where $1\leqslant i\leqslant 3$. Also, assume to the contrary that there exists $m_i'\in\mathfrak{m}_i\setminus\bigcup_{j\neq i}\mathfrak{m}_j$ with $m_i\neq m_i'$, for some integer $i$ with $1\leqslant i\leqslant 3$. Without loss of generality, we may assume that $i=1$. Thus, we have the cycles $m_1-m_2-m_3-m_1$ and $m_1'-m_2-m_3-m_1'$ which is impossible. ∎

**Theorem 2.4.** *Let $R$ be a non-local finite ring. Then $\Gamma'(R)$ is a union of cycle graphs if and only if $R\cong\mathbb{Z}_3\times\mathbb{Z}_3$.*

*Proof.* If $R\cong\mathbb{Z}_3\times\mathbb{Z}_3$, then $\Gamma'(R)$ is isomorphic to $C_4$, a cycle graph of length four. Conversely, assume that $\Gamma'(R)$ is the union of cycle graphs. Since $R$ is finite, there exists a positive integer $n$ such that $R\cong R_1\times\cdots\times R_n$, where $R_i$ is a local ring with maximal ideal $\mathfrak{m}_i$, for $i=1,\ldots,n$. Since $R$ is non-local, in light of Lemma 2.2, $n=2$ and so $R\cong R_1\times R_2$. Now, suppose that one of the rings $R_1$ or $R_2$ has more than three elements, say $R_1$. Then, for $r,s\in R_1\setminus\{0,1\}$, the vertex $(0,1)$ is adjacent to the vertices $(1,0)$, $(r,0)$ and $(s,0)$. This implies that $\Gamma'(R)$ is not a union of cycle graphs. Therefore, $|R_1|$, $|R_2|\leqslant 3$. This means that $R$ is isomorphic to one of the following rings:

$$\mathbb{Z}_2\times\mathbb{Z}_2,\ \mathbb{Z}_2\times\mathbb{Z}_3 \text{ or } \mathbb{Z}_3\times\mathbb{Z}_3.$$

On the other hand, in view of part (a) in Consequences 2.1, the cozero-divisor graph of the rings $\mathbb{Z}_2\times\mathbb{Z}_2$ and $\mathbb{Z}_2\times\mathbb{Z}_3$ are star graphs. Hence $R$ is isomorphic to $\mathbb{Z}_3\times\mathbb{Z}_3$ as required. ∎

**Theorem 2.5.** *Suppose that $R$ is a Noetherian ring. Then $\Gamma'(R)$ is totally disconnected if and only if $R$ is a local ring with principal maximal ideal.*

*Proof.* If $R$ is a local ring with principal maximal ideal, then by [1, Theorem 2.7], $\Gamma'(R)$ is totally disconnected. Conversely, assume that $\Gamma'(R)$ is totally disconnected. It is easy to see that $R$ is local. Let $\mathfrak{m}$ be the maximal ideal of $R$. Assume to the contrary that $\mathfrak{m}$ is not principal and that $aR$ is a maximal principal ideal in $\mathfrak{m}$. Since $\mathfrak{m}$ is not principal, there exists an element $b$ in $\mathfrak{m}$ such that $b \notin aR$. This implies that the vertices $a$ and $b$ are adjacent, which is a contradiction. Therefore $\mathfrak{m}$ is a principal ideal. ∎

In the rest of this section, we study the subgraph $\Gamma'(R) \setminus J(R)$ of the cozero-divisor graph of $\Gamma'(R)$. Recall that an Eulerian graph is a graph which has a path that visits each edge exactly once which starts and ends on the same vertex. By [5, Theorem 4.1], a connected non-empty graph is Eulerian if and only if the valency of each vertex is even.

**Theorem 2.6.** *Suppose that $R$ contains a principal maximal ideal $\mathfrak{m}$ such that $|W(R)\backslash\mathfrak{m}|$ is an odd number. Then $\Gamma'(R) \setminus J(R)$ is not Eulerian.*

*Proof.* Assume that $\mathfrak{m} = aR$ is a principal maximal ideal of $R$. Hence, for all $b \in \mathfrak{m} \setminus \{a\}$, the vertices $a$ and $b$ are not adjacent. Also, for all $c \in W(R) \setminus \mathfrak{m}$, since $a \notin cR$, the vertices $a$ and $c$ are adjacent. This means that the valency of the vertex $a$ is equal to $|W(R)\backslash\mathfrak{m}|$, which is an odd number. Hence, by [5, Theorem 4.1], $\Gamma'(R)\backslash J(R)$ is not an Eulerian graph. ∎

**Example 2.1.** The ring $\mathbb{Z}_{10}$ satisfies the condition of Theorem 2.6 and so the graph $\Gamma'(\mathbb{Z}_{10}) \setminus J(\mathbb{Z}_{10})$ is not an Eulerian graph.

**Theorem 2.7.** *Assume that $R$ is a non-local ring. Then the following conditions are equivalent:*

(i) *$\Gamma'(R) \setminus J(R)$ is complete bipartite.*
(ii) *$\Gamma'(R) \setminus J(R)$ is bipartite.*
(iii) *$\Gamma'(R) \setminus J(R)$ contains no triangles.*

*Proof.* The implications (i)$\Longrightarrow$(ii) and (ii)$\Longrightarrow$(iii) are clear.

(iii)$\Longrightarrow$ (ii) Since $\Gamma'(R) \setminus J(R)$ has no triangles and $R$ is non-local, it has exactly two maximal ideals, say $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Suppose to the contrary that the graph $\Gamma'(R) \setminus J(R)$ is not bipartite. So it contains a cycle of odd length. Therefore, there are vertices $a$ and $b$ in $\mathfrak{m}_1$ (or $\mathfrak{m}_2$) which are adjacent. This implies that every element in $\mathfrak{m}_2 \setminus \mathrm{J}(R)$ (or $\mathfrak{m}_1 \setminus \mathrm{J}(R)$) forms a triangle with vertices $a$ and $b$ which is a contradiction.

(ii)$\Longrightarrow$(i) If $\Gamma'(R) \setminus J(R)$ is bipartite, then in view of [1, Proposition 2.13], we have that $\max(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$. Also, it is easy to see that $V_1 = \mathfrak{m}_1 \setminus \mathfrak{m}_2$ and $V_2 = \mathfrak{m}_2 \setminus \mathfrak{m}_1$ are the parts of the bipartite graph $\Gamma'(R) \setminus J(R)$. Moreover, every vertex in $V_1$ is adjacent to all vertices in $V_2$ and also every vertex in $V_2$ is adjacent to all vertices in $V_1$. Hence $\Gamma'(R) \setminus J(R)$ is a complete bipartite graph. ∎

Recall that a graph is Hamiltonian if it contains a cycle which visits each vertex exactly once and also returns to the starting vertex.

**Theorem 2.8.** *Let $R$ be a finite ring with two maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$ such that $|\mathfrak{m}_1| = |\mathfrak{m}_2|$. Then $\Gamma'(R) \setminus J(R)$ is Hamiltonian.*

*Proof.* For $i = 1,2$, put $\mathfrak{m}_i \setminus \mathrm{J}(R) := \{a_{i1}, \ldots, a_{it}\}$, where $t := |\mathfrak{m}_1 \setminus J(R)|$. Then it is easy to see that $a_{11} - a_{21} - \cdots - a_{1t} - a_{2t} - a_{11}$ is a Hamiltonian cycle in $\Gamma'(R) \setminus J(R)$. ∎

We close this section with the following observation that compares the chromatic and clique numbers of the graph $\Gamma'(R) \setminus J(R)$. To this end, we recall some basic definitions. The chromatic number of a graph $G$, denoted by $\chi(G)$, is the minimal number of colors which can be assigned to the vertices of $G$ in such a way that every two adjacent vertices have different colors. Also, a clique of a graph is a complete subgraph and the number of vertices in a largest clique of $G$, denoted by $\omega(G)$, is called the clique number of $G$. Obviously $\chi(G) \geqslant \omega(G)$.

**Theorem 2.9.** *Assume that R is non-local. Then* $\chi(\Gamma'(R)\setminus J(R)) = 2$ *if and only if* $\omega(\Gamma'(R)\setminus J(R)) = 2$.

*Proof.* Clearly, $\omega(\Gamma'(R)\setminus J(R)) \leqslant \chi(\Gamma'(R)\setminus J(R))$. Now, if $\chi(\Gamma'(R)\setminus J(R)) = 2$, then, since $R$ is non-local, we have that $\omega(\Gamma'(R)\setminus J(R)) = 2$. Conversely, assume that $\omega(\Gamma'(R)\setminus J(R)) = 2$. Thus $|\max(R)| = 2$. If $\chi(\Gamma'(R)\setminus J(R)) > 2$, then $\Gamma'(R)\setminus J(R)$ is not bipartite and so by Theorem 2.7, it contains some triangles. This means that $\omega(\Gamma'(R)\setminus J(R)) \geqslant 3$ which is impossible. Thus, $\chi(\Gamma'(R)\setminus J(R)) = 2$. ∎

## 3. Complement of the cozero-divisor graph

As we mentioned in the introduction, the complement of the cozero-divisor graph $\Gamma'(R)$, is the Cayley graph $\mathrm{Cay}(W^*(R), R\setminus\{1\})$. In our first result we provide a connection between two graphs $\Gamma(R)$ and $\overline{\Gamma'(R)}$.

**Proposition 3.1.** *Let R be a finite ring such that* $\Gamma(R)$ *is not a refinement of a complete r-partite graph, where r is a positive integer. Then* $\overline{\Gamma'(R)}$ *is connected.*

*Proof.* Assume to the contrary that $\overline{\Gamma'(R)}$ is not connected and let $C_1,\ldots,C_r$ be its connected components. Hence, for $1 \leqslant i,j \leqslant r$ with $i \neq j$ and for every two vertices $a \in C_i$ and $b \in C_j$, we have that $ab = 0$. This means that $\Gamma(R)$ has a complete $r$-partite graph as a subgraph. In other words, $\Gamma(R)$ is a refinement of a complete $r$-partite graph, which is the required contradiction. ∎

The following corollary is an immediate consequence of Proposition 3.1.

**Corollary 3.1.** *If* $\overline{\Gamma'(R)}$ *is disconnected, then* $\Gamma(R)$ *is a refinement of a complete r-partite graph, where r is the number of connected components of* $\overline{\Gamma'(R)}$.

**Proposition 3.2.** $\overline{\Gamma'(R)}$ *is complete if and only if the set of all principal ideals of R is totally ordered by inclusion.*

*Proof.* The graph $\overline{\Gamma'(R)}$ is complete if and only if for every distinct vertices $a$ and $b$, $a$ is adjacent to $b$. This means that $aR \subseteq bR$ or $bR \subseteq aR$. So it is equivalent to the set of all principal ideals of $R$ is totally ordered by inclusion. ∎

The following corollary is an immediate consequence of Proposition 3.2 in conjunction with [1, Theorem 2.7].

**Corollary 3.2.** *Let R be a Noetherian local ring such that its maximal ideal is principal. Then* $\overline{\Gamma'(R)}$ *is complete.*

**Proposition 3.3.** *Let R be a Noetherian ring. If* $\overline{\Gamma'(R)}$ *has an infinite clique, then R has a principal ideal with infinite order which contains all vertices of the clique.*

*Proof.* Let $K$ be an infinite clique in $\overline{\Gamma'(R)}$ and $a_1$ be a vertex of $K$. Assume to the contrary that there is no principal ideal in $R$ that contains all vertices of $K$. Since the principal ideal

$a_1R$ doesn't contain all vertices of $K$, there exists a vertex $a_2$ in $K$ such that $a_2 \notin a_1R$. As $a_1$ and $a_2$ are adjacent and $a_2 \notin a_1R$, we have $a_1 \in a_2R$. Therefore, $a_1R \subsetneq a_2R$. Again since the principal ideal $a_2R$ doesn't contain all vertices of $K$, there exists a vertex $a_3$ in $K$ such that $a_3 \notin a_2R$. Also, $a_2$ and $a_3$ are adjacent. This implies that $a_2 \in a_3R$ and so $a_2R \subsetneq a_3R$. By continuing this method, we find an increasing sequence of principal ideals of $R$ which doesn't stop and this is a contradiction. ∎

Assume that $R_1$ and $R_2$ are two commutative rings with non-zero identities. Note that $\overline{\Gamma'(R_1 \times R_2)}$ is not connected, in general. For example, $\overline{\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_2)}$ is disconnected. In the following theorem we study the girth of $\overline{\Gamma'(R_1 \times R_2)}$.

**Theorem 3.1.** $g(\overline{\Gamma'(R_1 \times R_2)}) = 3, 6$ *or* $\infty$.

*Proof.* Set $R := R_1 \times R_2$. If $|U(R_1)| \geqslant 3$, then $(1,0) - (u,0) - (v,0) - (1,0)$ is a cycle in $\overline{\Gamma'(R)}$, where $u$ and $v$ are non-identity distinct elements in $U(R_1)$. So, $g(\overline{\Gamma'(R)}) = 3$. Similarly if $|U(R_2)| \geqslant 3$, then $g(\overline{\Gamma'(R)}) = 3$. Hence, $|U(R_1)|, |U(R_2)| \leqslant 2$. Now assume that $|R_1| \geqslant 4$ or $|R_2| \geqslant 4$. Without loss of generality, suppose that $|R_1| \geqslant 4$. If $|U(R_1)| = 2$, then $(1,0) - (u,0) - (z,0) - (1,0)$ is a cycle in $\overline{\Gamma'(R)}$, where $u$ is a non-identity element in $U(R_1)$ and $z \in W^*(R_1)$. So $g(\overline{\Gamma'(R)}) = 3$. If $|U(R_1)| = 1$ and there is some adjacency in $\overline{\Gamma'(R_1)}$, then one can consider the cycle $(1,0) - (a,0) - (b,0) - (1,0)$, where $a$ and $b$ are adjacent in $\overline{\Gamma'(R_1)}$ and so $g(\overline{\Gamma'(R)}) = 3$. Otherwise, there is no adjacency in $\overline{\Gamma'(R_1)}$. Now, if $R_2 \ncong \mathbb{Z}_2$, then $(a,0) - (a,1) - (a,b) - (a,0)$, where $a \in W^*(R_1)$ and $b \in R_2 \setminus \{0,1\}$, is a cycle in $\overline{\Gamma'(R)}$ and so $g(\overline{\Gamma'(R)}) = 3$. If $R_2 \cong \mathbb{Z}_2$, then $(0,1) - (a,1) - (a,0) - (1,0) - (b,0) - (b,1) - (0,1)$ is a cycle of length six, where $a, b \in W^*(R_1)$ and in this case, one can easily check that all cycles have length six. Therefore in this situation, we have $g(\overline{\Gamma'(R)}) = 6$. Now, it is enough to consider the case $|R_1|, |R_2| \leqslant 3$. Then, in this situation, $\overline{\Gamma'(R_1 \times R_2)}$ has no cycles and hence $g(\overline{\Gamma'(R)}) = \infty$. ∎

If $R$ is a commutative ring with a non-trivial idempotent, then $R = R_1 \times R_2$, for some commutative rings $R_1$ and $R_2$. Now, the following consequences follow from the proof of Theorem 3.1.

**Consequences 3.1.**

(i) Let $R \cong R_1 \times R_2$, where neither $R_1$ nor $R_2$ is $\mathbb{Z}_2$. Then

$$g(\overline{\Gamma'(R_1 \times R_2)}) = 3 \text{ or } \infty.$$

(ii) Let $R \cong R_1 \times R_2$. Then $g(\overline{\Gamma'(R_1 \times R_2)}) = \infty$ if and only if $R$ is isomorphic to one of the following rings:

$$\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3 \text{ or } \mathbb{Z}_3 \times \mathbb{Z}_3.$$

(iii) Assume that $R \cong R_1 \times R_2$. If $|U(R_1)| > 1$ and $R_1 \ncong \mathbb{Z}_3$, then $g(\overline{\Gamma'(R_1 \times R_2)}) = 3$. Similarly, if $|U(R_2)| > 1$ and $R_2 \ncong \mathbb{Z}_3$, then $g(\overline{\Gamma'(R_1 \times R_2)}) = 3$.

(iv) Let $R$ be a ring such that it has a non-trivial idempotent element. Then $g(\overline{\Gamma'(R)}) = 3, 6$ or $\infty$.

We need the following lemma in the sequel.

**Lemma 3.1.** *Suppose that $R_1$ and $R_2$ are non-trivial commutative rings with identities. Then $\overline{\Gamma'(R_1 \times R_2)}$ contains a subgraph isomorphic to $K_t$, where $t$ is the number of unit elements* *of $R_i$, for some $i = 1,2$. Moreover, if $|W(R_1)| > 1$ (or $|W(R_2)| > 1$), then $K_{t+1}$ is isomorphic to a subgraph of $\overline{\Gamma'(R_1 \times R_2)}$.*

*Proof.* Suppose that $U(R_1) = \{u_1, \ldots, u_t\}$. Then the vertices $(u_1, 0), \ldots, (u_t, 0)$ form a complete subgraph of $\overline{\Gamma'(R_1 \times R_2)}$ which is isomorphic to $K_t$. Now, if there exists an element $w$ in $W^*(R_1)$, then the vertices $(u_1, 0), \ldots, (u_t, 0), (w, 0)$ form the complete graph $K_{t+1}$. ∎

In the next proposition, which immediately follows from Lemma 3.1, we study the clique number of the graph $\overline{\Gamma'(R_1 \times R_2)}$.

**Proposition 3.4.**

(i) *If $|W(R_1)| = 1 = |W(R_2)|$, then*

$$\omega(\overline{\Gamma'(R_1 \times R_2)}) \geqslant \max\{|U(R_1)|, |U(R_2)|\}.$$

(ii) *If $|W(R_1)| > 1$ and $|W(R_2)| > 1$, then*

$$\omega(\overline{\Gamma'(R_1 \times R_2)}) \geqslant \max\{|U(R_1)|, |U(R_2)|\} + 1.$$

(iii) *If $|W(R_1)| > 1$ and $|W(R_2)| = 1$, then*

$$\omega(\overline{\Gamma'(R_1 \times R_2)}) \geqslant \max\{|U(R_1)| + 1, |U(R_2)|\}.$$

*A similar result holds in the case that $|W(R_1)| = 1$ and $|W(R_2)| > 1$.*

We end this section by investigating the planarity of $\overline{\Gamma'(R_1 \times R_2)}$. Recall that a graph is said to be planar if it can be drawn in the plane, so that its edges intersect only at their ends. Also a subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. A remarkable simple characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of $K_5$ or $K_{3,3}$ (cf. [5, p. 153]).

**Proposition 3.5.** *$\overline{\Gamma'(R_1 \times R_2)}$ is not planar if one of the following conditions holds:*

(i) *$|U(R_1)| \geqslant 5$,*

(ii) *$|U(R_1)| \geqslant 4$ and $|W(R_1)| > 1$.*

*Proof.* Assume that (i) or (ii) holds. Then by Lemma 3.1, $K_5$ is in the structure of $\overline{\Gamma'(R_1 \times R_2)}$ and so by Kuratowski's Theorem, $\overline{\Gamma'(R_1 \times R_2)}$ is not planar. ∎

**Corollary 3.3.** *Assume that $|U(R_1)| = 4$ and $\overline{\Gamma'(R_1 \times R_2)}$ is planar. Then $R_1 \cong \mathbb{Z}_5$.*

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## References

[1] M. Afkhami and K. Khashyarmanesh, The cozero-divisor graph of a commutative ring, *Southeast Asian Bull. Math.* (to appear).

[2] D. F. Anderson, A. Frazier, A. Lauve and P. S. Livingston, The zero-divisor graph of a commutative ring. II, in *Ideal Theoretic Methods in Commutative Algebra (Columbia, MO, 1999)*, 61–72, Lecture Notes in Pure and Appl. Math., 220 Dekker, New York.

[3] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra* **217** (1999), no. 2, 434–447.

[4] I. Beck, Coloring of commutative rings, *J. Algebra* **116** (1988), no. 1, 208–226.

[5] J. A. Bondy and U. S. R. Murty, *Graph theory with applications*, American Elsevier Publishing Co. Inc., New York, 1976.

[6] N. Ganesan, Properties of rings with a finite number of zero divisors, *Math. Ann.* **157** (1964), 215–218.
[7] A. V. Kelarev and C. E. Praeger, On transitive Cayley graphs of groups and semigroups, *European J. Combin.* **24** (2003), no. 1, 59–72.