BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **35**(4) (2012), 961–968

Composition Operators from Zygmund Spaces to Bloch Spaces in the Unit Ball

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Abstract. Let $H(B)$ denote the space of all holomorphic functions on the unit ball $B \subset \mathbb{C}^n$. Let $\varphi = (\varphi_1, \dots, \varphi_n)$ be a holomorphic self-map of B. The composition operator C_φ on $H(B)$ is defined as follows $(C_\varphi f)(z) = (f \circ \varphi)(z)$. In this paper we investigate the boundedness and compactness of the composition operator C_φ from Zygmund spaces to Bloch spaces in the unit ball.

2010 Mathematics Subject Classification: Primary: 47B33; Secondary: 32A18

Keywords and phrases: Composition operator, Zygmund space, Bloch space, boundedness, compactness.

1. Introduction

Let $H(B)$ denote the space of all holomorphic functions on the unit ball $B \subset \mathbb{C}^n$, $A(B)$ the ball algebra consisting of all functions in $H(B)$ that are continuous up to the boundary of B. Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in $\mathbb{C}^n$, we write

$$\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}, \ |z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

For $f \in H(B)$, let ∇f denote the complex gradient of f, i.e.

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z)\right).$$

The Bloch space $\mathscr{B} = \mathscr{B}(B)$ is the space of all $f \in H(B)$ such that

$$b(f) = \sup_{z\in B}(1-|z|^2)\,|\nabla f(z)| < \infty. \tag{1.1}$$

It is clear that $\mathscr{B}$ is a Banach space under the norm $\|f\|_{\mathscr{B}} = |f(0)| + b(f)$. Let $\mathscr{B}_0$ denote the subspace of $\mathscr{B}$ consisting of those $f \in \mathscr{B}$ for which

$$(1-|z|^2)|\nabla f(z)| \to 0, \tag{1.2}$$

as $|z| \to 1$. This space is called the little Bloch space.

Communicated by V. Ravichandran.
Received: November 15, 2010; *Revised:* February 6, 2011.

For $f \in H(B)$, let $\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z)$ stand for the radial derivative of $f \in H(B)$. From [22], we see that $f \in \mathscr{B}$ if and only if

$$\sup_{z\in B}(1-|z|^2)\,|\Re f(z)| < \infty. \tag{1.3}$$

$f \in \mathscr{B}_0$ if and only if $\lim_{|z|\to 1}(1-|z|^2)|\Re f(z)| = 0$. Moreover, the following asymptotic relation holds (see [3])

$$\|f\|_{\mathscr{B}} \asymp |f(0)| + (1-|z|^2)\,|\Re f(z)|\,.$$

Let $\mathscr{Z} = \mathscr{Z}(B)$ denote the class of all $f \in H(B)$ such that

$$\sup_{z\in B}(1-|z|^2)|\Re^2 f(z)| < \infty. \tag{1.4}$$

It is known that $f \in \mathscr{Z}$ if and only if $f \in A(B)$ and there exists a constant $C > 0$ such that

$$|f(\zeta+h)+f(\zeta-h)-2f(\zeta)| < Ch,$$

for all $\zeta \in \partial B$ and $\zeta \pm h \in \partial B$ (see, for example, [22, p. 261]). For $n = 1$ this result was proved by Zygmund. Hence $\mathscr{Z}$ is called the Zygmund class. Zygmund class with the norm

$$\|f\|_{\mathscr{Z}} = |f(0)| + \sup_{z\in B}(1-|z|^2)|\Re^2 f(z)|$$

will be called the Zygmund space. It is easy to see that $\mathscr{Z}$ is a Banach space with this norm.

Let $\varphi = (\varphi_1,\ldots,\varphi_n)$ be a holomorphic self-map of B. Define a linear operator C_φ on $H(B)$, called the composition operator, by

$$(C_\varphi f)(z) = (f\circ\varphi)(z),$$

where $f \in H(B)$. It is interesting to provide a function theoretic characterization when φ induce a bounded or compact weighted composition operator on various spaces. The book [4] contains much information on this topic.

In the setting of the unit disk, composition operators and Volterra operators on Zygmund spaces were studied respectively in [2,5], composition operators on Bloch spaces was studied in [13,14], weighted composition operators from Zygmund spaces to Bloch spaces was studied in [6]. In the setting of the unit ball, composition operator on Bloch spaces was studied in [3,15] and an integral type operator on Bloch-type spaces was studied in [21]. See also [7–12,16–20] for various operators on Zygmund spaces in the unit disk and the unit ball.

Motivated by [2,5–12,16–20], in this paper we study the boundedness and compactness of composition operators C_φ from Zygmund spaces to Bloch spaces and little Bloch spaces.

Throughout the paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \preceq b$ means that there is a positive constant C such that $a \le Cb$. If both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2. Main results and proofs

In this section we state and prove our main results. First we need several auxiliary results which we will use in the proofs of main results.

Lemma 2.1. [9,17] *Suppose that $f \in \mathscr{Z}$. Then the following statements are true.*

(a) *There is a positive constant C independent of f such that*

$$|\Re f(z)| \le C\|f\|_{\mathscr{Z}} \ln \frac{e}{1-|z|} \tag{2.1}$$

and

$$|\nabla f(z)| \le C\|f\|_{\mathscr{Z}} \ln \frac{e}{1-|z|}. \tag{2.2}$$

(b) *There is a positive constant C independent of f such that*

$$\|f\|_\infty = \sup_{|z|<1} |f(z)| \le C\|f\|_{\mathscr{Z}}. \tag{2.3}$$

Lemma 2.2. *Let $\varphi = (\varphi_1,\dots,\varphi_n)$ be a holomorphic self-map of B. Then $C_\varphi : \mathscr{Z} \to \mathscr{B}$ is compact if and only if $C_\varphi : \mathscr{Z} \to \mathscr{B}$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathscr{Z}$ which converges to zero uniformly on compact subset of B as $k \to \infty$, we have $\|C_\varphi f_k\|_{\mathscr{B}} \to 0$ as $k \to \infty$.*

Lemma 2.3. [15] *A closed set K in $\mathscr{B}_0$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} (1-|z|^2)|\nabla f(z)| = 0.$$

To formulate our main result, let us introduce some notations. Let $\varphi = (\varphi_1,\dots,\varphi_n)$ be a holomorphic self-map of B, denote

$$D\varphi(z) = \begin{pmatrix} \frac{\partial\varphi_1(z)}{\partial z_1} & \cdots & \frac{\partial\varphi_1(z)}{\partial z_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial\varphi_n(z)}{\partial z_1} & \cdots & \frac{\partial\varphi_n(z)}{\partial z_n} \end{pmatrix}$$

and $D\varphi(z)^T$, the transpose of the matrix $D\varphi(z)$. Set

$$|D\varphi(z)| = \left(\sum_{k,l=1}^n \left| \frac{\partial\varphi_l}{\partial z_k}(z) \right|^2 \right)^{1/2}. \tag{2.4}$$

Now, we formulate and prove the main results of this section.

Theorem 2.1. *Let $\varphi = (\varphi_1,\dots,\varphi_n)$ be a holomorphic self-map of B.*

(i) *If*

$$K_1 := \sup_{z\in B} (1-|z|^2)|D\varphi(z)| \ln \frac{e}{1-|\varphi(z)|^2} < \infty, \tag{2.5}$$

then $C_\varphi : \mathscr{Z} \to \mathscr{B}$ is bounded.

(ii) *If $C_\varphi : \mathscr{Z} \to \mathscr{B}$ is bounded, then*

$$K_2 := \sup_{z\in B} (1-|z|^2)|D\varphi(z)^T \overline{\varphi(z)^T}| \ln \frac{1}{1-|\varphi(z)|^2} < \infty. \tag{2.6}$$

Proof. (i). Suppose that (2.5) holds. Then for arbitrary $z \in B$ and $f \in \mathscr{Z}$, by Lemma 2.1 we have

$$\begin{aligned}(1-|z|^2)|\nabla(C_\varphi f)(z)| &= (1-|z|^2)|\nabla(f\circ\varphi)(z)| \\ &= (1-|z|^2)\Big(\sum_{k=1}^n \Big| \sum_{l=1}^n \frac{\partial f}{\partial \zeta_l}(\varphi(z)) \frac{\partial \varphi_l}{\partial z_k}(z) \Big|^2 \Big)^{1/2}\end{aligned}$$

$$
\begin{aligned}
&\le (1-|z|^2)\Big(\sum_{k=1}^{n}\sum_{l=1}^{n}\Big|\frac{\partial \varphi_l}{\partial z_k}(z)\Big|^2\Big)^{1/2}\Big(\sum_{l=1}^{n}\Big|\frac{\partial f}{\partial \zeta_l}(\varphi(z))\Big|^2\Big)^{1/2}\\
&\le (1-|z|^2)|D\varphi(z)|\,|(\nabla f)(\varphi(z))|\\
&\le C\|f\|_{\mathscr{Z}}(1-|z|^2)|D\varphi(z)|\ln\frac{e}{1-|\varphi(z)|^2}. \qquad (2.7)
\end{aligned}
$$

From this, and since $|C_\varphi f(0)|\le \|f\|_\infty \le \|f\|_{\mathscr{Z}}$, it follows that $C_\varphi:\mathscr{Z}\to\mathscr{B}$ is bounded.

(ii). Assume that $C_\varphi:\mathscr{Z}\to\mathscr{B}$ is bounded. It is easy to see that for every $m=1,\cdots,n$, the functions $f_m(z)=z_m$ belong to $\mathscr{Z}$. Therefore $\varphi_m=f_m\circ\varphi\in\mathscr{B}$, i.e.

$$
\sup_{z\in B}(1-|z|^2)\sqrt{\sum_{i=1}^{n}\Big|\frac{\partial \varphi_m}{\partial z_i}(z)\Big|^2}=\sup_{z\in B}(1-|z|^2)|\nabla\varphi_m(z)|<\infty,\ m=1,\cdots,n. \qquad (2.8)
$$

By (2.8) and the following elementary inequality

$$
\Big(\sum_{i=1}^{n}a_i\Big)^p\le\begin{cases}\sum_{i=1}^{n}a_i^p & ,\quad p\in(0,1]\\ n^{p-1}\sum_{i=1}^{n}a_i^p & ,\quad p\ge 1\end{cases},\ a_i\ge 0, i=1,\cdots,n,
$$

we obtain that

$$
\begin{aligned}
\sup_{z\in B}(1-|z|^2)|D\varphi(z)|&=\sup_{z\in B}(1-|z|^2)\Bigg(\sum_{m=1}^{n}\sum_{k=1}^{n}\Big|\frac{\partial \varphi_m}{\partial z_k}(z)\Big|^2\Bigg)^{1/2}\\
&\le\sup_{z\in B}(1-|z|^2)\sum_{m=1}^{n}\Bigg(\sum_{k=1}^{n}\Big|\frac{\partial \varphi_m}{\partial z_k}(z)\Big|^2\Bigg)^{1/2}<\infty. \qquad (2.9)
\end{aligned}
$$

Set (see e.g. [9])

$$
h(z)=(z-1)\left[\left(1+\ln\frac{1}{1-z}\right)^2+1\right]
$$

and

$$
h_a(z)=\frac{h(\langle z,a\rangle)}{|a|^2}\left(\ln\frac{1}{1-|a|^2}\right)^{-1}
$$

for $a\in B$ such that $|a|>\sqrt{1-1/e}$. Then,

$$
\Re h_a(z)=\frac{\langle z,a\rangle}{|a|^2}\left(\ln\frac{1}{1-\langle z,a\rangle}\right)^2\left(\ln\frac{1}{1-|a|^2}\right)^{-1}
$$

and

$$
\Re^2 h_a(z)=\Re h_a(z)+\frac{2\langle z,a\rangle^2}{|a|^2(1-\langle z,a\rangle)}\left(\ln\frac{1}{1-\langle z,a\rangle}\right)\left(\ln\frac{1}{1-|a|^2}\right)^{-1}.
$$

Thus for $\sqrt{1-1/e}<|a|<1$, we obtain that

$$
M_1=\sup_{\sqrt{1-1/e}<|a|<1}\|h_a\|_{\mathscr{Z}}<\infty.
$$

Therefore we have

$$
\begin{aligned}
M_1\|C_\varphi\|_{\mathscr{Z}\to\mathscr{B}}\ge\|C_\varphi h_{\varphi(\lambda)}\|_{\mathscr{B}}&\ge\sup_{z\in B}(1-|z|^2)|\nabla(h_{\varphi(\lambda)}\circ\varphi)(z)|\\
&\ge(1-|\lambda|^2)\Big|\nabla(h_{\varphi(\lambda)}\circ\varphi)(\lambda)\Big|
\end{aligned}
$$

$$
\begin{aligned}
&= (1-|\lambda|^2)\Big(\sum_{k=1}^{n}\Big|\sum_{l=1}^{n}\frac{\partial h_{\varphi(\lambda)}}{\partial \zeta_l}(\varphi(\lambda))\frac{\partial \varphi_l}{\partial z_k}(\lambda)\Big|^2\Big)^{1/2}\\
&= (1-|\lambda|^2)\Big(\sum_{k=1}^{n}\Big|\sum_{l=1}^{n}\frac{\overline{\varphi_l(\lambda)}}{|\varphi(\lambda)|^2}\ln\frac{1}{1-|\varphi(\lambda)|^2}\frac{\partial \varphi_l}{\partial z_k}(\lambda)\Big|^2\Big)^{1/2}\\
&= (1-|\lambda|^2)\frac{1}{|\varphi(\lambda)|^2}\ln\frac{1}{1-|\varphi(\lambda)|^2}|D\varphi(\lambda)^T\overline{\varphi(\lambda)^T}|
\end{aligned}
$$

$$
\geq (1-|\lambda|^2)\ln\frac{1}{1-|\varphi(\lambda)|^2}|D\varphi(\lambda)^T\overline{\varphi(\lambda)^T}| \tag{2.10}
$$

for $\lambda \in B$ such that $\sqrt{1-1/e} < |\varphi(\lambda)| < 1$. By (2.9) we obtain that

$$
\begin{aligned}
&(1-|\lambda|^2)\ln\frac{1}{1-|\varphi(\lambda)|^2}|D\varphi(\lambda)^T\overline{\varphi(\lambda)^T}|\\
&\leq (1-|\lambda|^2)\sqrt{\sum_{k=1}^{n}\Big|\sum_{l=1}^{n}\frac{\partial \varphi_l}{\partial z_k}(\lambda)\overline{\varphi_l(\lambda)}\Big|^2}
\end{aligned}
$$

$$
\leq C(1-|\lambda|^2)\sqrt{\sum_{k=1}^{n}\sum_{l=1}^{n}\Big|\frac{\partial \varphi_l}{\partial z_k}(\lambda)\Big|^2} = C(1-|\lambda|^2)|D\varphi(\lambda)| < \infty \tag{2.11}
$$

for $\lambda \in B$ such that $|\varphi(\lambda)| \leq \sqrt{1-1/e}$. (2.10) together with (2.11) implies (2.6). This completes the proof of Theorem 2.1. ∎

Theorem 2.2. *Let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be a holomorphic self-map of B.*

(i) *Suppose that $C_\varphi : \mathscr{Z} \to \mathscr{B}$ is bounded. If*

$$
\lim_{|\varphi(z)|\to 1}(1-|z|^2)|D\varphi(z)|\ln\frac{e}{1-|\varphi(z)|^2} = 0, \tag{2.12}
$$

then $C_\varphi : \mathscr{Z} \to \mathscr{B}$ is compact.

(ii) *If $C_\varphi : \mathscr{Z} \to \mathscr{B}$ is compact, then*

$$
\lim_{|\varphi(z)|\to 1}(1-|z|^2)|D\varphi(z)^T\overline{\varphi(z)^T}|\ln\frac{1}{1-|\varphi(z)|^2} = 0. \tag{2.13}
$$

Proof. (i). We assume that $C_\varphi : \mathscr{Z} \to \mathscr{B}$ is bounded and (2.12) holds. From the proof of Theorem 2.1, we see that

$$
L := \sup_{z\in\mathbb{D}}|D\varphi(z)|(1-|z|^2) < \infty.
$$

Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $\mathscr{Z}$ with $\sup_{k\in\mathbb{N}}\|f_k\|_{\mathscr{Z}} \leq M$ and $f_k \to 0$ uniformly on compact subsets of B as $k \to \infty$. By (2.12) we have that for any $\varepsilon > 0$, there is a constant $\delta \in (0,1)$, such that $\delta < |\varphi(z)| < 1$ implies

$$
(1-|z|^2)|D\varphi(z)|\ln\frac{e}{1-|\varphi(z)|^2} < \varepsilon/M. \tag{2.14}
$$

Let $K = \{w \in B : |w| \leq \delta\}$. By (2.14), we have

$$
\begin{aligned}
\|C_\varphi f_k\|_{\mathscr{B}} &\leq \sup_{z\in B}(1-|z|^2)|\nabla f_k(\varphi(z))D\varphi(z)| + |f_k(\varphi(0))|\\
&= \sup_{\{z\in B:\,|\varphi(z)|\leq\delta\}}(1-|z|^2)|D\varphi(z)||\nabla f_k(\varphi(z))|
\end{aligned}
$$

$$+ \sup_{\{z\in B:\,\delta<|\varphi(z)|<1\}} (1-|z|^2)|D\varphi(z)||\nabla f_k(\varphi(z))| + |f_k(\varphi(0))|$$

$$= L\sup_{w\in K}|\nabla f_k(w)| + C\|f_k\|_{\mathscr{Z}} \sup_{\{z\in B:\,\delta<|\varphi(z)|<1\}} (1-|z|^2)|D\varphi(z)|\ln\frac{e}{1-|\varphi(z)|^2}$$

$$+ |f_k(\varphi(0))|$$

$$\le L\sup_{w\in K}|\nabla f_k(w)| + C\varepsilon + |f_k(\varphi(0))|. \tag{2.15}$$

By Cauchy's estimate, if f_k is a sequence which converges on compact subsets of B to zero, then the sequence ∇f_k also converges to zero on compact subsets of B as $k\to\infty$. In particular, since K and $\{\varphi(0)\}$ are compact it follows that $\lim_{k\to\infty}\sup_{w\in K}|\nabla f_k(w)| = 0$ and $\lim_{k\to\infty}|f_k(\varphi(0))| = 0$. Using these facts and letting $k\to\infty$ in (2.15), we obtain that $\limsup_{k\to\infty}\|C_\varphi f_k\|_{\mathscr{B}} \le C\varepsilon$. Since ε is an arbitrary positive number it follows that the last limit is equal to zero. Employing Lemma 2.2, the implication follows.

(ii). Suppose that $C_\varphi : \mathscr{Z}\to\mathscr{B}$ is compact. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B such that $|\varphi(z_k)|\to 1$ as $k\to\infty$. We choose test functions $(f_k)_{k\in\mathbb{N}}$ defined by

$$f_k(z) = \frac{\langle z,\varphi(z_k)\rangle - 1}{|\varphi(z_k)|^2}\left[\left(1+\ln\frac{1}{1-\langle z,\varphi(z_k)\rangle}\right)^2+1\right]\left(\ln\frac{1}{1-|\varphi(z_k)|^2}\right)^{-1}. \tag{2.16}$$

From the proof of Theorem 2.1 we see that $\sup_{k\in\mathbb{N}}\|f_k\|_{\mathscr{Z}}\le C$. Moreover, f_k converges to zero uniformly on compact subsets of B. Hence, in view of Lemma 2.2 it follows that $\|C_\varphi f_k\|_{\mathscr{B}}\to 0$, as $k\to\infty$. Similarly to the proof of Theorem 2.1 we have

$$\begin{aligned}\|C_\varphi f_k\|_{\mathscr{B}} &\ge \sup_{z\in B}(1-|z|^2)|\nabla(C_\varphi f_k)(z)|\\ &\ge (1-|z_k|^2)|D\varphi(z_k)^T\overline{\varphi(z_k)^T}|\ln\frac{1}{1-|\varphi(z_k)|^2},\end{aligned}$$

i.e.

$$\lim_{k\to\infty}(1-|z_k|^2)|D\varphi(z_k)^T\overline{\varphi(z_k)^T}|\ln\frac{1}{1-|\varphi(z_k)|^2} = 0.$$

The result follows. ∎

Theorem 2.3. *Let $\varphi=(\varphi_1,\dots,\varphi_n)$ be a holomorphic self-map of B. If*

$$\lim_{|z|\to1}(1-|z|^2)|D\varphi(z)|\ln\frac{e}{1-|\varphi(z)|^2} = 0, \tag{2.17}$$

then $C_\varphi : \mathscr{Z}\to\mathscr{B}_0$ is compact.

Proof. From Lemma 2.3, we know that $C_\varphi : \mathscr{Z}\to\mathscr{B}_0$ is compact if and only if

$$\lim_{|z|\to1}\sup_{\|f\|_{\mathscr{Z}}\le1}(1-|z|^2)|\nabla(C_\varphi f)(z)| = 0. \tag{2.18}$$

By (2.7) we have

$$(1-|z|^2)|\nabla(C_\varphi f)(z)| \le C\|f\|_{\mathscr{Z}}(1-|z|^2)|D\varphi(z)|\ln\frac{e}{1-|\varphi(z)|^2}. \tag{2.19}$$

Taking the supremum in (2.19) over the unit ball of the space $\mathscr{Z}$, then letting $|z|\to1$, we obtain that (2.18) holds, from which the compactness of $C_\varphi : \mathscr{Z}\to\mathscr{B}_0$ follows. ∎

Theorem 2.4. *Let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be a holomorphic self-map of B. If $C_\varphi : \mathscr{Z} \to \mathscr{B}_0$ is bounded, then*

$$\lim_{|\varphi(z)|\to 1} (1-|z|^2)|D\varphi(z)^T \overline{\varphi(z)^T}| \ln \frac{1}{1-|\varphi(z)|^2} = 0. \tag{2.20}$$

Proof. Suppose that $C_\varphi : \mathscr{Z} \to \mathscr{B}_0$ is bounded. Now assume that the condition (2.20) does not hold. If it were, then it would exist $\varepsilon_0 > 0$ and a sequence $(z^m)_{m\in\mathbb{N}} \in B$, such that $\varphi(z^m) \to \partial B$ and

$$(1-|z^m|^2)|D\varphi(z^m)^T \overline{\varphi(z^m)^T}| \ln \frac{1}{1-|\varphi(z^m)|^2} \geq \varepsilon_0 > 0$$

for sufficiently large m.

We may assume that $\lim_{m\to\infty} z^m = (1, 0, \ldots, 0)$, and also

$$\frac{1-|\varphi(z^{m-1})_1|}{2} > 1-|\varphi(z^m)_1|, \qquad m \in \mathbb{N}.$$

Then, for every non-negative integer s there is at most one $\varphi(z^m)_1$ such that

$$1-\frac{1}{2^s} \leq |\varphi(z^m)_1| < 1-\frac{1}{2^{(s+1)}}.$$

Hence, there is $M \in \mathbb{N}$ such that for any Carleson window

$$Q = \{re^{i\theta} \mid 0 < 1-r < l(Q),\ |\theta-\theta_0| < l(Q)\}$$

and $s \in \mathbb{N}$, there is at most M elements in

$$\{\varphi(z^m)_1 \in Q \mid 2^{-(s+1)} l(Q) < 1-|\varphi(z^m)_1| < 2^{-s} l(Q)\}.$$

Therefore, $(\varphi(z^m)_1)$ is an interpolating sequence for $\mathscr{B}(U)$, the Bloch space of the unit disk, in sense of [1].

By [1] we have some $u(z_1) \in \mathscr{B}(U)$ such that

$$u(\varphi(z^m)_1) = \ln \frac{1}{1-|\varphi(z^m)_1|^2}, \qquad m \in \mathbb{N}.$$

Now define $f(z_1, z_2, \ldots, z_n) = u(z_1) \in H(B)$. It is easy to see that $f \in \mathscr{B}(B)$.

Let F be the antiderivative of f, from the relationship between Bloch and Zygmund functions, we see that $F \in \mathscr{Z}$ such that $\nabla F = f$. We have

$$\begin{aligned}
(1-|z^m|^2)|\nabla(C_\varphi F)(z^m) &\geq (1-|z^m|^2)|D\varphi(z^m)^T \overline{\varphi(z^m)^T}||\nabla F(\varphi(z_n))| \\
&= (1-|z^m|^2)|D\varphi(z^m)^T \overline{\varphi(z^m)^T}||f(\varphi(z^m))| \\
&= (1-|z^m|^2)|D\varphi(z^m)^T \overline{\varphi(z^m)^T}| \ln \frac{1}{1-|\varphi(z^m)_1|^2} \\
&\geq (1-|z^m|^2)|D\varphi(z^m)^T \overline{\varphi(z^m)^T}| \ln \frac{1}{1-|\varphi(z^m)|^2} \\
&\geq \varepsilon_0 > 0
\end{aligned}$$

for sufficiently large m. Since $\varphi(z^m) \to \partial B$ implies that $z^m \to \partial B$, from the above inequality we obtain that $C_\varphi F \notin \mathscr{B}_0$, which is a contradiction. ∎

Remark 2.1. We have not been able to obtain the sufficient and necessary conditions for the boundedness and compactness of composition operators from Zygmund spaces to Bloch spaces in the unit ball. We hope interested readers can continue this project. The interested readers can also study the operator norm and essential norm of composition operators from Zygmund spaces to Bloch spaces and the corresponding problems for weighted composition operators.

Acknowledgement. The author thanks the referees for their useful comments.

References

[1] B. Bøe and A. Nicolau, Interpolation by functions in the Bloch space, *J. Anal. Math.* **94** (2004), 171–194.

[2] B. R. Choe, H. Koo and W. Smith, Composition operators on small spaces, *Integral Equations Operator Theory* **56** (2006), no. 3, 357–380.

[3] D. D. Clahane and S. Stević, Norm equivalence and composition operators between Bloch/Lipschitz spaces of the ball, *J. Inequal. Appl.* **2006**, Art. ID 61018, 11 pp.

[4] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, CRC, Boca Raton, FL, 1995.

[5] S. Li and S. Stević, Volterra-type operators on Zygmund spaces, *J. Inequal. Appl.* **2007**, Art. ID 32124, 10 pp.

[6] S. Li and S. Stević, Weighted composition operators from Zygmund spaces into Bloch spaces, *Appl. Math. Comput.* **206** (2008), no. 2, 825–831.

[7] S. Li and S. Stević, Generalized composition operators on Zygmund spaces and Bloch type spaces, *J. Math. Anal. Appl.* **338** (2008), no. 2, 1282–1295.

[8] S. Li and S. Stević, Products of Volterra type operator and composition operator from H^∞ and Bloch spaces to Zygmund spaces, *J. Math. Anal. Appl.* **345** (2008), no. 1, 40–52.

[9] S. Li and S. Stević, Cesàro-type operators on some spaces of analytic functions on the unit ball, *Appl. Math. Comput.* **208** (2009), no. 2, 378–388.

[10] S. Li and S. Stević, Integral-type operators from Bloch-type spaces to Zygmund-type spaces, *Appl. Math. Comput.* **215** (2009), no. 2, 464–473.

[11] S. Li and S. Stević, On an integral-type operator from ω-Bloch spaces to μ-Zygmund spaces, *Appl. Math. Comput.* **215** (2010), no. 12, 4385–4391.

[12] S. Li and S. Stević, Products of composition and differentiation operators from Zygmund spaces to Bloch spaces and Bers spaces, *Appl. Math. Comput.* **217** (2010), no. 7, 3144–3154.

[13] Z. Lou, Composition operators on Bloch type spaces, *Analysis (Munich)* **23** (2003), no. 1, 81–95.

[14] K. Madigan and A. Matheson, Compact composition operators on the Bloch space, *Trans. Amer. Math. Soc.* **347** (1995), no. 7, 2679–2687.

[15] J. Shi and L. Luo, Composition operators on the Bloch space of several complex variables, *Acta Math. Sin. (Engl. Ser.)* **16** (2000), no. 1, 85–98.

[16] S. Stević, Composition operators from the Hardy space to the Zygmund-type space on the upper half-plane, *Abstr. Appl. Anal.* **2009**, Art. ID 161528, 8 pp.

[17] S. Stević, On an integral operator from the Zygmund space to the Bloch-type space on the unit ball, *Glasg. Math. J.* **51** (2009), no. 2, 275–287.

[18] S. Stević, Composition operators from the weighted Bergman space to the nth weighted-type space on the upper half-plane, *Appl. Math. Comput.* **217** (2010), no. 7, 3379–3384.

[19] S. Stević, On an integral-type operator from Zygmund-type spaces to mixed-norm spaces on the unit ball, *Abstr. Appl. Anal.* **2010**, Art. ID 198608, 7 pp.

[20] S. Stević, Weighted differentiation composition operators from the mixed-norm space to the nth weighted-type space on the unit disk, *Abstr. Appl. Anal.* **2010**, Art. ID 246287, 15 pp.

[21] S. Stević, On an integral operator between Bloch-type spaces on the unit ball, *Bull. Sci. Math.* **134** (2010), no. 4, 329–339.

[22] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Graduate Texts in Mathematics, 226, Springer, New York, 2005.