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On Intuitionistic Fuzzy Prime Bi-Ideals of Semigroups

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Abstract. In this paper, we introduce the notions of intuitionistic fuzzy prime (resp. strongly prime and semiprime) bi-ideals of a semigroup. By using these ideas we characterize those semigroups for which each intuitionistic fuzzy bi-ideal is semiprime and strongly prime.

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1. Introduction

The theory of fuzzy sets proposed by Zadeh [30] in his classic paper of 1965, deal with the applications of fuzzy technology in information processing. The information processing is already important and it will certainly increase in importance in the future. Mordeson, Malik and Kuroki gave a systematic exposition of fuzzy semigroups in [23], where one can find theoretical results on fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines and fuzzy languages. The monograph by Mordeson and Malik [24] deals with the applications of fuzzy approach to the concepts of automata and formal languages. After the introduction of fuzzy sets by Zadeh, there have been a number of generalizations of this fundamental concept. Atanassov [5] introduced the notion of intuitionistic fuzzy sets which is a generalization of fuzzy sets. The fuzzy sets give the degree of membership of an element in a given set while the intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership. As for fuzzy sets, the degree of membership is a real number between 0 and 1, this is also the case for the degree of non-membership, and the sum of these two degrees is not greater than 1. For more details on intuitionistic fuzzy sets, we refer the reader to [5,6,7]. Fuzzy sets are intuitionistic fuzzy sets but the converse is not necessarily true [5]. Infect, there are situations where intuitionistic fuzzy set theory is more

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appropriate to deal with [9]. Intuitionistic fuzzy set theory has been applied in different fields, that is, logic programming, decision making problems, etc. De *et al.* [12] studied the Sanchez's approach for medical diagnosis and extended this concept with the notion of intuitionistic fuzzy set theory. Some authors applied this concept to generalize some notions of algebra, for example Davvaz *et al.* [11], applied this concept in H_v-modules. They introduced the notion of an intuitionistic fuzzy H_v-submodule of an H_v -module, and studied related properties. Kim *et al.* [20], considered the intuitionistic fuzzification of the concept of sub-hyperquasigroups in a hyperquasigroup. In [21, 22], Kim and Jun introduced the concept of intuitionistic fuzzy (interior) ideals of semigroups. Shabir *et al.* in [26], introduced the concept of prime bi-ideals, strongly prime bi-ideal and semiprime bi-ideals of a semigroup and studied those semigroups for which each bi-ideal is semiprime and strongly irreducible.

In this paper we introduce the concept of intuitionistic fuzzy prime, strongly prime and semiprime bi-ideals of semigroups and give characterizations of semigroups in terms of these notions. We characterize those semigroups for which each intuitionistic fuzzy bi-ideal is semiprime and strongly irreducible.

2. Basic concepts of semigroups

A semigroup is a non-empty set S together with an associative binary operation "$\cdot$". An element 0 of a semigroup S with at least two elements is called a zero element of S if $x0 = 0x = 0$ for all x in S. A semigroup which contains a zero element is called a semigroup with zero. If a semigroup S has no zero element then it is easy to adjoin a zero element 0 to the set by defining $0x = x0 = 0 = 00$ for all x in S. We shall use the notation S^0 with the following meanings:

$$S^0 = \begin{cases} S, & \text{if } S \text{ has a zero element} \\ S \cup \{0\}, & \text{otherwise.} \end{cases}$$

For $A, B \subseteq S$, we denote $AB := \{ab \mid a \in A, b \in B\}$. A non-empty subset A of a semigroup S is called a subsemigroup of S if $ab \in A$ for all $a, b \in A$. A subsemigroup B of a semigroup S is called a bi-ideal of S if $BSB \subseteq B$. A non-empty subset A of a semigroup S is called left (right) ideal of S if $SA \subseteq A$ ($AS \subseteq A$). A is called two sided ideal of S if it is both a left and a right ideal of S. Every left (right) ideal of a semigroup is a bi-ideal but the converse is not true. It is well known that the intersection of any number of bi-ideals of a semigroup S is either empty or a bi-ideal of S. Also the product of two bi-ideals of a semigroup S is a bi-ideal of S. Let $a \in S$. Then intersection of all bi-ideals of S which contain a is a bi-ideal of S containing a. Of course this is the smallest bi-ideal of S containing a and is called the bi-ideal generated by a. We shall denote this bi-ideal by $B(a)$. Clearly $B(a) = a \cup a^2 \cup aSa$.

A bi-ideal B of a semigroup S is called prime (strongly prime) if $B_1B_2 \subseteq B$ ($B_1B_2 \cap B_2B_1 \subseteq B$) implies $B_1 \subseteq B$ or $B_2 \subseteq B$ for any bi-ideals B_1 and B_2 of S (cf.[26]). A bi-ideal B of a semigroup S is called semiprime if $B_1^2 \subseteq B$ implies $B_1 \subseteq B$ for any bi-ideal B_1 of S (cf.[26]). An element a of a semigroup S is called a regular element if there exists an element x in S such that $axa = a$. A semigroup S is called regular if every element of S is regular. An element a of a semigroup S is called intra-regular if there exist elements x and y in S such that $xa^2y = a$. A semigroup S is called intra-regular if every element of S is intra regular.

A function f from a non-empty set A to the unit interval $[0,1]$ of real numbers is called a fuzzy subset of A. A fuzzy subset f of a semigroup S is called a fuzzy subsemigroup of S if $f(xy) \geq \min\{f(x), f(y)\}$ for all $x,y \in S$. A fuzzy subsemigroup of S is called a fuzzy bi-ideal of S if $f(xyz) \geq \min\{f(x), f(z)\}$ for all $x,y,z \in S$ (cf. [23]). A fuzzy subset f of a semigroup S is called a fuzzy left (right) ideal of S if $f(xy) \geq f(y)$ $(f(xy) \geq f(x))$ for all $x,y \in S$. Every fuzzy left (right) ideal is a fuzzy bi-ideal but the converse is not true. For any two fuzzy subsets f,g of a non-empty set A we define the fuzzy subsets $(f \wedge g)(a) = f(a) \wedge g(a)$, $(f \vee g)(a) = f(a) \vee g(a)$ for all $a \in A$. Also $f \leq g$ means that $f(a) \leq g(a)$ for all $a \in A$.

3. Intuitionistic fuzzy bi-ideals

As an important generalization of the notion of fuzzy set in S, Atanassov (cf. [5]) introduced the concept of an intuitionistic fuzzy set (IFS for short) defined on a non-empty set S as objects having the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle | x \in S\}$,where the functions $\mu_A : S \longrightarrow [0,1]$ and $\gamma_A : S \longrightarrow [0,1]$ denote the *degree of membership* (namely $\mu_A(x)$) and the *degree of non-membership* (namely $\gamma_A(x)$) of each element $x \in S$ to the set A respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in S$ (cf.[5]). In particular, $0_\sim$ and $1_\sim$ denote the intuitionistic fuzzy empty set and the intuitionistic fuzzy whole set in a set X defined by $0_\sim(x) = (0,1)$ and $1_\sim(x) = (1,0)$ for each $x \in X$, respectively. We shall write IFS for intuitionistic fuzzy sets.

For simplicity we use $A = (\mu_A, \gamma_A)$ for IFS instead of $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle | x \in S\}$. For any two IFSs $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ of a semigroup S we define:

(1) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in S$,
(2) $A^c = \{\langle x, \gamma_A(x), \mu_A(x)\rangle | x \in S\} = (\gamma_A, \mu_A)$,
(3) $A \cap B = \{\langle x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\gamma_A(x), \gamma_B(x)\}\rangle | x \in S\} = (\mu_A \wedge \mu_B, \gamma_A \vee \gamma_B)$,
(4) $A \cup B = \{\langle x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\gamma_A(x), \gamma_B(x)\}\rangle | x \in S\} = (\mu_A \vee \mu_B, \gamma_A \wedge \gamma_B)$,
(5) $1_\sim = (1,0)$ and $0_\sim = (0,1)$.

For any two intuitionistic fuzzy sets $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ of a semigroup S, define $A \circ B = (\mu_{A\circ B}, \gamma_{A\circ B})$ (cf. [19]) where:

$$\mu_{A\circ B}(a) := \begin{cases} \bigvee\limits_{a=yz} \min\{\mu_A(y), \mu_B(z)\} & \text{if } a = yz, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\gamma_{A\circ B}(a) := \begin{cases} \bigwedge\limits_{a=yz} \max\{\gamma_A(y), \gamma_B(z) & \text{if } a = yz, \\ 0 & \text{otherwise.} \end{cases}$$

If $IFS(S)$ denotes the set of all intuitionistic fuzzy sets of a semigroup S. One can easily see that the multiplication "$\circ$ " on $IFS(S)$ is associative. It is clear that for any IFS, A, B, C of a semigroup S if $B \subseteq C$, then $A \circ B \subseteq A \circ C$ and $B \circ A \subseteq C \circ A$. (cf. [1])

Lemma 3.1. *Let S be a semigroup and A, B, C be any intuitionistic fuzzy subsets of S then,*

(1) $A \circ (B \cup C) = (A \circ B) \cup (A \circ C)$, $(B \cup C) \circ A = (B \circ A) \cup (C \circ A)$.
(2) $A \circ (B \cap C) \subseteq (A \circ B) \cap (A \circ C)$, $(B \cap C) \circ A \subseteq (B \circ A) \cap (C \circ A)$.

Let $\{A_i\}_{i\in I}$ be a family of intuitionistic fuzzy subsets of S then $\bigcap\limits_{i\in I} A_i$ is an intuitionistic fuzzy subset of S where $\bigcap\limits_{i\in I} A_i = (\bigwedge\limits_{i\in I} \mu_{A_i}, \bigvee\limits_{i\in I} \gamma_{A_i})$ and

$$\bigwedge_{i\in I} \mu_{A_i} : S \longrightarrow [0,1] | x \longrightarrow \bigwedge_{i\in I} \mu_{A_i}(x) := \inf_{i\in I}\{\mu_{A_i}(x) | x \in S\},$$
$$\bigvee_{i\in I} \gamma_{A_i} : S \longrightarrow [0,1] | x \longrightarrow \bigvee_{i\in I} \gamma_{A_i}(x) := \sup_{i\in I}\{\gamma_{A_i}(x) | x \in S\}.$$

Let A be a non-empty subset of a semigroup S, the intuitionistic characteristic function $\chi_A = (\mu_{\chi_A}, \gamma_{\chi_A})$ is defined as [19]:

$$\mu_{\chi_A}(x) := \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ if } x \notin A, \end{cases} \quad \text{and} \quad \gamma_{\chi_A}(x) := \begin{cases} 0 \text{ if } x \in A, \\ 1 \text{ if } x \notin A. \end{cases}$$

Definition 3.1. *Let S be a semigroup and let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy subsets of S then A is called an:*

(1) *Intuitionistic fuzzy subsemigroup of S, if $\mu_A(xy) \geq min\{\mu_A(x), \mu_A(y)\}$ and $\gamma_A(xy) \leq max\{\gamma_A(x), \gamma_A(y)\}$ for all $x, y \in S$.*
(2) *Intuitionistic fuzzy bi-ideal of S, if A is an intuitionistic fuzzy subsemigroup of S and $\mu_A(xyz) \geq min\{\mu_A(x), \mu_A(z)\}$ and $\gamma_A(xyz) \leq max\{\gamma_A(x), \gamma_A(z)\}$ for all $x, y, z \in S$.*
(3) *Intuitionistic fuzzy left (right) ideal of S, if $\mu_A(xy) \geq \mu_A(y) \quad (\mu_A(xy) \geq \mu_A(x))$ and $\gamma_A(xy) \leq \gamma_A(y) (\gamma_A(xy) \leq \gamma_A(x))$ for all $x, y, z \in S$.*

Every intuitionistic fuzzy left (right) ideal is intuitionistic fuzzy bi-ideal but the converse is not true.

Lemma 3.2. [20] *Let A be a non-empty subset of a semigroup S. Then*

(1) *A is a subsemigroup of S if and only if $\chi_A = (\mu_{\chi_A}, \gamma_{\chi_A})$ is an intuitionistic fuzzy subsemigroup of S.*
(2) *A is a bi-ideal of S if and only if $\chi_A = (\mu_{\chi_A}, \gamma_{\chi_A})$ is an intuitionistic fuzzy bi-ideal of S.*

Lemma 3.3. [1] *Let S be a semigroup and $A, B \subseteq S$ then*

(1) *$A \subseteq B$ if and only if $\chi_A \subseteq \chi_B$.*
(2) *$\chi_A \circ \chi_B = \chi_{A\circ B}$.*

Lemma 3.4. [1] *Let S be a semigroup and $0_\sim \neq A$ be an intuitionistic fuzzy subset of S. Then*

(1) *A is an intuitionistic fuzzy subsemigroup of S if and only if $A \circ A \subseteq A$.*
(2) *A is an intuitionistic fuzzy bi-ideal of S if and only if $A \circ A \subseteq A$ and $A \circ 1_\sim \circ A \subseteq A$.*

Lemma 3.5. [1] *Every intuitionistic fuzzy left (right) ideal of a semigroup is intuitionistic fuzzy bi-ideal.*

Lemma 3.6. [1] *A semigroup S is regular if and only if $A \circ B = A \cap B$ for each intuitionistic fuzzy right ideal A and each intuitionistic fuzzy left ideal B of S.*

Lemma 3.7. [1] *A semigroup S is intra regular if and only if $A \cap B \subseteq B \circ A$ for each intuitionistic fuzzy right ideal A and each intuitionistic fuzzy left ideal B of S.*

4. Intuitionistic fuzzy Prime (strongly prime and semiprime) bi-ideals

In this section we define intuitionistic fuzzy prime (resp. strongly prime and semiprime) bi-ideals of a semigroup S.

Definition 4.1. *Let S be a semigroup and $A=(\mu_A,\gamma_A)$ an intuitionistic fuzzy bi-ideal of S. Then A is called an intuitionistic fuzzy prime (strongly prime) bi-ideal of S if, for any intuitionistic fuzzy bi-ideals $B=(\mu_B,\gamma_B)$ and $C=(\mu_c,\gamma_c)$ of S, $B\circ C\subseteq A$ (resp. $B\circ C\cap C\circ B\subseteq A$) implies $B\subseteq A$ or $C\subseteq A$. An intuitionistic fuzzy bi-ideal $A=(\mu_A,\gamma_A)$ of S is called an intuitionistic fuzzy semiprime bi-ideal of S if, for any intuitionistic fuzzy bi-ideal $B=(\mu_B,\gamma_B)$ of S, $B\circ B\subseteq A$ implies $B\subseteq A$.*

Obviously every intuitionistic fuzzy strongly prime bi-ideal is an intuitionistic fuzzy prime bi-ideal and every intuitionistic fuzzy prime bi-ideal is semiprime but the converse is not true is general.

Theorem 4.1. *A bi-ideal B of a semigroup S is prime if and only if the intuitionistic characteristic function $\chi_B=(\mu_{\chi_B},\gamma_{\chi_B})$ of B is an intuitionistic fuzzy prime bi-ideal of S.*

Proof. Suppose B is a prime bi-ideal of S, then by Lemma 3.2, χ_B is an intuitionistic fuzzy bi-ideal of S. Let A, C be any intuitionistic fuzzy bi-ideals of S such that $A\circ C\subseteq\chi_B$ but $A\nsubseteq\chi_B$ and $C\nsubseteq\chi_B$, then there exist $x,y\in S$ such that $\mu_A(x)\neq 0,\gamma_A(x)\neq 1$ and $\mu_c(y)\neq 0,\gamma_c(y)\neq 1$ but $\mu_{\chi_B}(x)=0,\gamma_{\chi_B}(x)=1$ and $\mu_{\chi_B}(y)=0,\gamma_{\chi_B}(y)=1$. Hence $x\notin B$ and $y\notin B$. Since B is a prime bi-ideal of S, therefore we have $B(x)B(y)\nsubseteq B$.

Since $\mu_A(x)\neq 0,\gamma_A(x)\neq 1$ and $\mu_C(y)\neq 0,\gamma_C(y)\neq 1$, therefore $\min\{\mu_A(x),\mu_C(y)\}\neq 0$ and $\max\{\gamma_A(x),\gamma_C(y)\}\neq 1$. Since $B(x)B(y)\nsubseteq B$, therefore there exists $a\in S$ such that $a\in B(x)B(y)$ but $a\notin B$. Thus we have $\mu_{\chi_B}(a)=0,\gamma_{\chi_B}(a)=1$ and hence $\mu_{A\circ C}(a)=0$ and $\gamma_{A\circ C}(a)=1$. Since $a\in B(x)B(y)$ we have $a=x_1y_1$ for some $x_1\in B(x)$ and $y_1\in B(y)$. Thus

$$\mu_{A\circ C}(a)=\bigvee_{a=x_1y_1}\min\{\mu_A(x_1),\mu_c(y_1)\}\geq\min\{\mu_A(x_1),\mu_c(y_1)\}$$

and

$$\gamma_{A\circ C}(a)=\bigwedge_{a=x_1y_1}\max\{\gamma_A(x_1),\gamma_c(y_1)\}\leq\max\{\gamma_A(x_1),\gamma_c(y_1)\}.$$

Since, $x_1\in B(x)=\{x\}\cup\{x^2\}\cup xSx$, we have $x_1=x$ or $x_1=x^2$ or $x_1=xyx$ for some $y\in S$. If $x_1=x$, then$\mu_A(x_1)=\mu_A(x)$, and $\gamma_A(x_1)=\gamma_A(x)$ If $x_1=x^2$, then $\mu_A(x_1)=\mu_A(x^2)\geq\min\{\mu_A(x),\mu_A(x)\}=\mu_A(x)$ and $\gamma_A(x_1)=\gamma_A(x^2)\leq\max\{\gamma_A(x),\gamma_A(x)\}=\gamma_A(x)$. If $x_1=xyx$, then $\mu_A(x_1)=\mu_A(xyx)\geq\min\{\mu_A(x),\mu_A(x)\}=\mu_A(x)$ and $\gamma_A(x_1)=\gamma_A(xyx)\leq\max\{\gamma_A(x),\gamma_A(x)\}=\gamma_A(x)$. Also since $y_1\in B(y)=\{y\}\cup\{y^2\}\cup ySy$, we have $y_1=y$ or $y_1=y^2$ or $y_1=yxy$ for some $x\in S$. If $y_1=y$ then $\mu_C(y_1)=\mu_C(y)$, and $\gamma_C(y_1)=\gamma_C(y)$. If $y_1=y^2$, then $\mu_C(y_1)=\mu_C(y^2)\geq\min\{\mu_C(y),\mu_C(y)\}=\mu_C(y)$ and $\gamma_C(y_1)=\gamma_C(y^2)\leq\max\{\gamma_C(y),\gamma_C(y)\}=\gamma_C(y)$. If $y_1=yxy$, then $\mu_C(y_1)=\mu_C(yxy)\geq\min\{\mu_C(y),\mu_C(y)\}=\mu_C(y)$ and $\gamma_C(y_1)=\gamma_C(yxy)\leq\max\{\gamma_C(y),\gamma_C(y)\}=\gamma_C(y)$. Thus $\mu_{A\circ C}(a)\geq\min\{\mu_A(x_1),\mu_c(y_1)\}\geq\min\{\mu_A(x),\mu_c(y)\}\neq 0$ and $\gamma_{A\circ C}(a)\leq\max\{\gamma_A(x_1),\gamma_c(y_1)\}\leq\max\{\gamma_A(x),\gamma_c(y)\}\neq 1$ which is a contradiction to the fact that $\mu_{A\circ C}(a)=0$ and $\gamma_{A\circ C}(a)=1$. Thus for any intuitionistic fuzzy bi-ideals A,C of S, $A\circ C\subseteq\chi_B\Longrightarrow A\subseteq\chi_B$ or $C\subseteq\chi_B$.

Conversely, assume that χ_B is an intuitionistic fuzzy prime bi-ideal of S. Let B_1,B_2 be any bi-ideals of S such that $B_1B_2\subseteq B$, then by Lemma 3.2, $\chi_{B_1}$and χ_{B_2} are intuitionistic fuzzy bi-ideals of S. By hypothesis, $\chi_{B_1\circ B_2}\subseteq\chi_B$. By Lemma 3.3, $\chi_{B_1\circ B_2}=\chi_{B_1}\circ\chi_{B_2}$. Thus

$\chi_{B_1} \circ \chi_{B_2} \subseteq \chi_B$. Since χ_B is prime, we have $\chi_{B_1} \subseteq \chi_B$ or $\chi_{B_2} \subseteq \chi_B$. By Lemma 3.3, we have $B_1 \subseteq B$ or $B_2 \subseteq B$. ∎

Similarly we can prove the following:

Theorem 4.2. *A bi-ideal B of a semigroup S is semiprime if and only if the intuitionistic characteristic function χ_B of B is intuitionistic fuzzy semiprime bi-ideal of S.*

Theorem 4.3. *A bi-ideal B of a semigroup S is strongly prime if and only if the intuitionistic characteristic function $\chi_B = (\mu_{\chi_B}, \gamma_{\chi_B})$ of B is an intuitionistic fuzzy strongly prime bi-ideal of S.*

Proof. Suppose B is a strongly prime bi-ideal of S then by Lemma 3.2, χ_B is an intuitionistic fuzzy bi-ideal of S. Let A, C be any intuitionistic fuzzy bi-ideals of S such that $A \circ C \cap C \circ A \subseteq \chi_B$ but $A \nsubseteq \chi_B$ and $C \nsubseteq \chi_B$, then there exist $x, y \in S$ such that $\mu_A(x) \neq 0, \gamma_A(x) \neq 1$ but $\mu_{\chi_B}(x) = 0, \gamma_{\chi_B}(x) = 1$ and $\mu_{\chi_B}(y) = 0, \gamma_{\chi_B}(y) = 1$. Hence $x \notin B$ and $y \notin B$. Since B is a strongly prime bi-ideal of S, we have $B(x)B(y) \cap B(y)B(x) \nsubseteq B$.

Since $\mu_A(x) \neq 0, \gamma_A(x) \neq 1$ and $\mu_C(y) \neq 0, \gamma_C(y) \neq 1$ we have $\min\{\mu_A(x), \mu_C(y)\} \neq 0$ and $max\{\gamma_A(x), \gamma_C(y)\} \neq 1$. Since $B(x)B(y) \cap B(y)B(x) \nsubseteq B$, so there exists $a \in S$ such that $a \in B(x)B(y)$ and $a \in B(y)B(x)$, but $a \notin B$. Thus we have $\mu_{\chi_B}(a) = 0, \gamma_{\chi_B}(a) = 1$ and hence $\mu_{A\circ C}(a) \wedge \mu_{C\circ A}(a) = 0$ and $\gamma_{A\circ C}(a) \vee \gamma_{C\circ A}(a) = 1$. Since $a \in B(x)B(y)$ and $a \in B(y)B(x)$, we have $a = x_1y_1$ and $a = y_2x_2$ for some $x_1, x_2 \in B(x)$ and $y_1, y_2 \in B(y)$. Thus

$$\mu_{A\circ C}(a) = \bigvee_{a=x_1y_1} \min\{\mu_A(x_1), \mu_c(y_1)\} \geq \min\{\mu_A(x_1), \mu_c(y_1)\}$$

and

$$\gamma_{A\circ C}(a) = \bigwedge_{a=x_1y_1} \max\{\gamma_A(x_1), \gamma_c(y_1)\} \leq \max\{\nu_A(x_1), \gamma_c(y_1)\}.$$

Since, $x_1 \in B(x) = \{x\} \cup \{x^2\} \cup xSx$, we have $x_1 = x$ or $x_1 = x^2$ or $x_1 = xyx$ for some $y \in S$.

If $x_1 = x$, then $\mu_A(x_1) = \mu_A(x)$, and $\gamma_A(x_1) = \gamma_A(x)$. If $x_1 = x^2$, then $\mu_A(x_1) = \mu_A(x^2) \geq \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x)$ and $\gamma_A(x_1) = \gamma_A(x^2) \leq \max\{\gamma_A(x), \gamma_A(x)\} = \gamma_A(x)$. If $x_1 = xyx$, then $\mu_A(x_1) = \mu_A(xyx) \geq \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x)$ and $\gamma_A(x_1) = \gamma_A(xyx) \leq \max\{\gamma_A(x), \gamma_A(x)\} = \gamma_A(x)$. Also since $y_1 \in B(y) = \{y\} \cup \{y^2\} \cup ySy$, we have $y_1 = y$ or $y_1 = y^2$ or $y_1 = yxy$ for some $x \in S$. If $y_1 = y$ then $\mu_C(y_1) = \mu_C(y)$, and $\gamma_C(y_1) = \gamma_C(y)$. If $y_1 = y^2$, then $\mu_C(y_1) = \mu_C(y^2) \geq \min\{\mu_C(y), \mu_C(y)\} = \mu_C(y)$ and $\gamma_C(y_1) = \gamma_C(y^2) \leq \max\{\gamma_C(y), \gamma_C(y)\} = \gamma_C(y)$. If $y_1 = yxy$, then $\mu_C(y_1) = \mu_C(yxy) \geq \min\{\mu_C(y), \mu_C(y)\} = \mu_C(y)$ and $\gamma_C(y_1) = \gamma_C(yxy) \leq \max\{\gamma_C(y), \gamma_C(y)\} = \gamma_C(y)$. Thus $\mu_{A\circ C}(a) \geq \min\{\mu_A(x_1), \mu_c(y_1)\} \geq \min\{\mu_A(x), \mu_c(y)\} \neq 0$ and $\gamma_{A\circ C}(a) \leq \max\{\gamma_A(x_1), \gamma_c(y_1)\} \leq \max\{\gamma_A(x), \gamma_c(y)\} \neq 1$.

Similarly we can show that $\mu_{C\circ A}(a) > 0$ and $\gamma_{C\circ A}(a) \neq 1$. Thus, $\mu_{A\circ C}(a) \wedge \mu_{C\circ A}(a) > 0$ and $\gamma_{A\circ C}(a) \vee \gamma_{C\circ A}(a) \neq 1$ which is a contradiction to the fact that $\mu_{A\circ C}(a) \wedge \mu_{C\circ A}(a) = 0$ and $\gamma_{A\circ C}(a) \vee \gamma_{C\circ A}(a) = 1$. Thus, for any intuitionistic fuzzy bi-ideals A, C of S, $A \circ C \cap C \circ A \subseteq \chi_B \Longrightarrow A \subseteq \chi_B$ or $C \subseteq \chi_B$.

Conversely, assume that χ_B is an intuitionistic fuzzy strongly prime bi-ideal of S. Let B_1, B_2 be bi-ideals of S such that $B_1B_2 \cap B_2B_1 \subseteq B$, then by Lemma 3.2, $\chi_{B_1}$and χ_{B_2} are intuitionistic fuzzy bi-ideals of S. By hypothesis, $\chi_{B_1\circ B_2} \cap \chi_{B_2\circ B_1} \subseteq \chi_B$. By Lemma 3.3, we have $\chi_{B_1\circ B_2} = \chi_{B_1} \circ \chi_{B_2}$. Thus, $\chi_{B_1} \circ \chi_{B_2} \cap \chi_{B_2} \circ \chi_{B_1} \subseteq \chi_B$. Since χ_B is strongly prime, we have $\chi_{B_1} \subseteq \chi_B$ or $\chi_{B_2} \subseteq \chi_B$. By Lemma 3.3, we have $B_1 \subseteq B$ or $B_2 \subseteq B$. ∎

Definition 4.2. *An intuitionistic fuzzy bi-ideal B of S is called idempotent if $B = B^2 = B \circ B$, that is, $\mu_{B\circ B} = \mu_B \circ \mu_B = \mu_B$, $\gamma_{B\circ B} = \gamma_B \circ \gamma_B = \gamma_B$.*

Lemma 4.1. [21] *Let* $\{A_i\}_{i\in I}$ *be a family of intuitionistic fuzzy bi-ideals of S, then* $\bigcap_{i\in I} A_i$ *is an intuitionistic fuzzy bi-ideal of S.*

Definition 4.3. *Let S be a semigroup and* $A = (\mu_A, \gamma_A)$ *an intuitionistic fuzzy bi-ideal of S. Then A is called an intuitionistic fuzzy irreducible (resp. strongly irreducible) bi-ideal of S if, for any intuitionistic fuzzy bi-ideals* $B = (\mu_B, \gamma_B)$ *and* $C = (\mu_c, \gamma_c)$ *of S,* $B\cap C = A$ *(resp.* $B\cap C \subseteq A$*) implies* $B = A$ *or* $C = A$ *(resp.* $B \subseteq A$ *or* $C \subseteq A$*).*

The proof of the following lemma is straight forward, so we omit it.

Lemma 4.2. *A bi-ideal B of a semigroup S is an irreducible (resp. strongly irreducible) if and only if the intuitionistic characteristic function* $\chi_B = (\mu_{\chi_B}, \gamma_{\chi_B})$ *of B is an intuitionistic fuzzy irreducible (resp. strongly irreducible) bi-ideal of S.*

5. Semigroup in which each intuitionistic fuzzy bi-ideal is strongly prime

In this section we study those semigroups in which each intuitionistic fuzzy bi-ideal is semiprime and also those semigroups in which each intuitionistic fuzzy bi-ideal is strongly prime.

Lemma 5.1. *Let S be a semigroup. Then the intersection of any family of intuitionistic fuzzy prime bi-ideals of S is an intuitionistic fuzzy semiprime bi-ideal of S.*

Proof. Straight forward. ∎

Proposition 5.1. *An intuitionistic fuzzy strongly irreducible, semiprime bi-ideal of a semigroup S is an intuitionistic fuzzy strongly prime bi-ideal of S.*

Proof. Let A be an intuitionistic fuzzy strongly irreducible, semiprime bi-ideal of S. Let B and C be intuitionistic fuzzy bi-ideals of S such that $B\circ C\cap C\circ B \subseteq A$. Since $(B\cap C)^2 \subseteq B\circ C$ and also $(B\cap C)^2 \subseteq C\circ B$, so $(B\cap C)^2 \subseteq B\circ C\cap C\circ B \subseteq A$. Since A is an intuitionistic fuzzy semiprime bi-ideal, so $(B\cap C) \subseteq A$. As A is irreducible so $B \subseteq A$ or $C \subseteq A$, that is A is an intuitionistic fuzzy strongly prime bi-ideal. ∎

Proposition 5.2. *Let* $A = (\mu_A, \gamma_A)$ *be an intuitionistic fuzzy bi-ideal of a semigroup S with* $\mu_A(a) = t$ *and* $\gamma_A(a) = 1-t$ *where* $a \in S$ *and* $t \in (0,1]$*, then there exist an intuitionistic fuzzy irreducible bi-ideal* $B = (\mu_B, \gamma_B)$ *of S such that* $A \subseteq B$ *and* $\mu_B(a) = t$*,* $\gamma_B(a) = 1-t$*.*

Proof. Let $X = \{C|C$ is an intuitionistic fuzzy bi-ideal of S, $\mu_C(a) = t$, $\gamma_C(a) = 1-t$ and $A \subseteq C\}$, then $X \neq \phi$ because $A \in X$. Then the collection X is a partially ordered set under inclusion. If Y is any totally order subset of X, say $Y = \{C_i|i \in I\}$, then $\bigcup_{i\in I} C_i$ is an intuitionistic fuzzy bi-ideal of S. Indeed: Let $x, y \in S$, then

$$\begin{aligned}(\bigvee_{i\in I}\mu_{C_i})(xy) &= \bigvee_{i\in I}(\mu_{C_i}(xy)) \geq \bigvee_{i\in I}(\mu_{C_i}(x)\wedge\mu_{C_i}(y))\\ &= \bigvee_{i\in I}(\mu_{C_i}(x))\wedge\bigvee_{i\in I}(\mu_{C_i}(y)) = (\bigvee_{i\in I}\mu_{C_i})(x)\wedge(\bigvee_{i\in I}\mu_{C_i})(y)\end{aligned}$$

and

$$(\bigwedge_{i\in I}\gamma_{C_i})(xy) = \bigwedge_{i\in I}(\gamma_{C_i}(xy)) \leq \bigwedge_{i\in I}(\gamma_{C_i}(x)\vee\gamma_{C_i}(y))$$

$$= \bigwedge_{i\in I}(\gamma_{C_i}(x)) \vee \bigwedge_{i\in I}(\gamma_{C_i}(y)) = (\bigwedge_{i\in I}\gamma_{C_i})(x) \vee (\bigwedge_{i\in I}\gamma_{C_i})(y)$$

Hence $\bigcup_{i\in I} C_i$ is an intuitionistic fuzzy subsemigroup of S. Let $x, y, z \in S$, then

$$(\bigvee_{i\in I}\mu_{C_i})(xyz) = \bigvee_{i\in I}(\mu_{C_i}(xyz)) \geq \bigvee_{i\in I}(\mu_{C_i}(x) \wedge \mu_{C_i}(z))$$
$$= \bigvee_{i\in I}(\mu_{C_i}(x)) \wedge \bigvee_{i\in I}(\mu_{C_i}(z)) = (\bigvee_{i\in I}\mu_{C_i})(x) \wedge (\bigvee_{i\in I}\mu_{C_i})(z)$$

and

$$(\bigwedge_{i\in I}\gamma_{C_i})(xyz) = \bigwedge_{i\in I}(\gamma_{C_i}(xyz)) \leq \bigwedge_{i\in I}(\gamma_{C_i}(x) \vee \gamma_{C_i}(z))$$
$$= \bigwedge_{i\in I}(\gamma_{C_i}(x)) \vee \bigwedge_{i\in I}(\gamma_{C_i}(z)) = (\bigwedge_{i\in I}\gamma_{C_i})(x) \vee (\bigwedge_{i\in I}\gamma_{C_i})(z).$$

Thus, $\bigcup_{i\in I} C_i$ is an intuitionistic fuzzy bi-ideal of S. As $A \subseteq C_i$ for each $i \in I$, so $A \subseteq \bigcup_{i\in I} C_i$. Also, $(\bigvee_{i\in I}\mu_{C_i})(a) = \bigvee_{i\in I}\mu_{C_i}(a) = t$ and $(\bigwedge_{i\in I}\gamma_{C_i})(a) = \bigwedge_{i\in I}\gamma_{C_i}(a) = 1-t$. Thus, $\bigcup_{i\in I} C_i$ is the upper bound of Y. Thus, by Zorn's lemma, their exists an intuitionistic fuzzy bi-ideal $B = (\mu_B, \gamma_B)$ of S which is maximal with respect to the property that $A \subseteq B$ and $\mu_B(a) = t, \gamma_B(a) = 1-t$. We now show that B is an intuitionistic fuzzy irreducible bi-ideal of S. Suppose $B = B_1 \cap B_2$ where B_1 and B_2 are intuitionistic fuzzy bi-ideal of S. Then $B \subseteq B_1$ and $B \subseteq B_2$. We claim that $B = B_1$ or $B = B_2$. Suppose on the contrary $B \neq B_1$ and $B \neq B_2$. Since B is a maximal bi-ideal with respect to the property that $\mu_B(a) = t$ and $\gamma_B(a) = 1-t$ and $B \neq B_1$ and $B \neq B_2$, it follows that $\mu_{B_1}(a) \neq t$ or $\gamma_{B_1}(a) \neq 1-t$ and $\mu_{B_2}(a) \neq t$ or $\gamma_{B_2}(a) \neq 1-t$. Hence $t = \mu_B(a) = \mu_{B_1\wedge B_2}(a) = \mu_{B_1}(a) \wedge \mu_{B_2}(a) \neq t\, textor 1-t = \gamma_B(a) = \gamma_{B_1\vee B_2}(a) = \gamma_{B_1}(a) \vee \gamma_{B_2}(a) \neq 1-t$, which is a contradiction. Hence either $B = B_1$ or $B = B_2$. Thus, B is an irreducible intuitionistic fuzzy bi-ideal of S. ∎

Theorem 5.1. [1] *A semigroup S is regular if and only if for every intuitionistic fuzzy bi-ideal A of S we have,$A = A \circ 1_\sim \circ A$.*

Lemma 5.2. *Let S be a semigroup and A, B be intuitionistic fuzzy bi-ideals of S. Then $A \circ B$ is an intuitionistic fuzzy bi-ideal of S.*

Proof. Let A and B be intuitionistic fuzzy bi-ideals of S. Then

$$(A \circ B) \circ (A \circ B) = (A \circ B \circ A) \circ B \leq (A \circ 1_\sim \circ A) \circ B \leq A \circ B.$$

Also,

$$(A \circ B) \circ 1_\sim \circ (A \circ B) \leq (A \circ 1_\sim) \circ 1_\sim \circ (A \circ B) = A \circ (1_\sim \circ 1_\sim) \circ (A \circ B)$$
$$\leq A \circ 1_\sim \circ (A \circ B) = (A \circ 1_\sim \circ A) \circ B \leq A \circ B.$$

Thus $A \circ B$ is an intuitionistic fuzzy bi-ideal of S. ∎

Next theorem characterizes those semigroups in which each intuitionistic fuzzy bi-ideal is semiprime.

Theorem 5.2. *Let S be a semigroup. Then the following are equivalent:*

(1) *S is both regular and intra-regular.*
(2) *$A \circ A = A$ for every intuitionistic fuzzy bi-ideal A of S.*

(3) $A \cap B = (A \circ B) \cap (B \circ A)$ *for every intuitionistic fuzzy bi-ideals A and B of S.*
(4) *Each intuitionistic fuzzy bi-ideal of S is intuitionistic fuzzy semiprime.*
(5) *Each proper intuitionistic fuzzy bi-ideal of S is the intersection of irreducible intuitionistic fuzzy semiprime bi-ideals of S which contain it.*

Proof. (1) $\Longrightarrow$ (2) Let S be both regular and intra-regular semigroup and $A = (\mu_A, \gamma_B)$ be an intuitionistic fuzzy bi-ideal of S. Let $a \in S$. Since S is regular and intra-regular, so there exist $x, y, z \in S$ such that $a = axa$ and $a = ya^2z$. Thus we have $a = axa = axaxa = ax(ya^2z)xa = (axya)(azxa)$. Hence,

$$\mu_{A\circ A}(a) = \bigvee_{a=yz} \min\{\mu_A(y), \mu_A(z)\} \geq \min\{\mu_A(axya), \mu_A(azxa)\}$$
$$\geq \min\{(\mu_A(a) \wedge \mu_A(a)), \mu_A(a) \wedge \mu_A(a)\} = \mu_A(a).$$

and

$$\gamma_{A\circ A}(a) = \bigwedge_{a=yz} \max\{\gamma_A(y), \gamma_A(z)\} \leq \max\{\gamma_A(axya), \gamma_A(azxa)\}$$
$$\leq \max\{(\gamma_A(a) \vee \gamma_A(a)), (\gamma_A(a) \vee \gamma_A(a))\} = \gamma_A(a).$$

Hence $\mu_{A\circ A}(a) \geq \mu_A(a)$ and $\gamma_{A\circ A}(a) \leq \gamma_A(a)$. Thus $A \subseteq A \circ A$. For the reverse inclusion, since A is an intuitionistic fuzzy subsemigroup of S, we have $\mu_{A\circ A} \leq \mu_A$ and $\gamma_{A\circ A} \geq \gamma_A$, that is $A \circ A \subseteq A$. Therefore, $A \circ A = A$.

(2) $\Longrightarrow$ (3). Let A, B be intuitionistic fuzzy bi-ideals of S , then $A \cap B$ is an intuitionistic fuzzy bi-ideal of S. By (2), we have $A \cap B = (A \cap B) \circ (A \cap B) \subseteq A \circ B$. Similarly, $A \cap B \subseteq B \circ A$. Thus $A \cap B \subseteq A \circ B \cap B \circ A$.

For the reverse inclusion, since $A \circ B$ and $B \circ A$ are intuitionistic fuzzy bi-ideals of S, so $A \circ B \cap B \circ A$ is an intuitionistic fuzzy bi-ideal S. Hence by hypothesis, we have, by Lemma 3.4 and $B \circ B = B$,

$$A \circ B \cap B \circ A = (A \circ B \cap B \circ A) \circ (A \circ B \cap B \circ A) \subseteq A \circ B \circ B \circ A$$
$$= A \circ (B \circ B) \circ A = A \circ B \circ A \subseteq A \circ 1_\sim \circ A = A.$$

Hence $A \circ B \cap B \circ A \subseteq A$. Similarly, we can prove that $A \circ B \cap B \circ A \subseteq B$. Hence $A \circ B \cap B \circ A \subseteq A \cap B$. Therefore, $A \circ B \cap B \circ A = A \cap B$.

(3) $\Longrightarrow$ (4). Let A and B be intuitionistic fuzzy bi-ideals of S such that $A \circ A \subseteq B$. Then by hypothesis, $A = A \cap A = A \circ A \cap A \circ A = A \circ A$. Thus $A \subseteq B$. Hence each intuitionistic fuzzy bi-ideal of S is semiprime.

(4) $\Longrightarrow$ (5). Let A be a proper intuitionistic fuzzy bi-ideal of S and $\{A_i : i \in I\}$ be the collection of all irreducible intuitionistic fuzzy bi-ideals of S which contain A. Proposition 5.2 guarantees the existence of such irreducible intuitionistic fuzzy bi-ideals. Hence $A \subseteq \bigcap_{i\in I} A_i$. Let $a \in S$, then by Proposition 5.2, their exist an irreducible intuitionistic fuzzy bi-ideal A_α of S such that $A \subseteq A_\alpha$ and $\mu_A(a) = \mu_{A_\alpha}(a)$, $\gamma_A(a) \leq \gamma_{A_\alpha}(a)$. Thus $A_\alpha \in \{A_i : i \in I\}$. Hence $\bigcap_{i\in I} A_i \subseteq A_\alpha$, that is $\bigwedge_{i\in I} \mu_{A_i}(a) \leq \mu_{A_\alpha^{(a)}} = \mu_{A^{(a)}}$ and $\bigvee_{i\in I} \gamma_{A_i}(a) \geq \gamma_{A_\alpha^{(a)}} \geq \gamma_{A^{(a)}}$. Hence $\bigcap_{i\in I} A_i \subseteq A$. Consequently, $\bigcap_{i\in I} A_i = A$. By hypothesis, each intuitionistic fuzzy bi-ideal of S is semiprime, so each intuitionistic fuzzy bi-ideal of S is the intersection of irreducible semiprime intuitionistic fuzzy bi-ideals of S which contain it.

(5) $\Longrightarrow$ (2). Let A be an intuitionistic fuzzy bi-ideal of S . Then $A \circ A$ is also an intuitionistic fuzzy bi-ideal of S by Lemma 5.5. Then $A \circ A \subseteq A$, because A is an intuitionistic fuzzy

subsemigroup of S. Also $A \circ A \subseteq A \circ A$, by hypothesis $A \circ A$ is semiprime so $A \subseteq A \circ A$. Thus $A \circ A = A$.

$(2) \Rightarrow (1)$. Let A be an intuitionistic fuzzy right ideal and B an intuitionistic fuzzy left ideal of S, $A \circ B \subseteq A \cap B$. On the other hand, $A \cap B$ is an intuitionistic fuzzy bi-ideal of S, so by hypothesis $A \cap B = (A \cap B) \circ (A \cap B) \subseteq A \circ B$. Thus $A \cap B = A \circ B$, which implies that S is regular. Also $A \cap B = (A \cap B) \circ (A \cap B) \subseteq B \circ A$ which implies that S is intra regular. ∎

Proposition 5.3. *Let S be a regular and intra regular semigroup. Then the following assertions for an intuitionistic fuzzy bi-ideal A of S are equivalent.*

(1) *A is strongly irreducible.*
(2) *A is strongly prime.*

Proof. $(1) \Longrightarrow (2)$. Let S be a regular and intra regular semigroup and A be a strongly irreducible intuitionistic fuzzy bi-ideal of S. Suppose B, C are intuitionistic fuzzy bi-ideal of S such that $(B \circ C) \cap (C \circ B) \subseteq A$. Since S is both regular and intra regular so by Theorem 5.2, $(B \circ C) \cap (C \circ B) = B \cap C$. Hence $B \cap C \subseteq A$. Since A is strongly irreducible, we have $B \subseteq A$ or $C \subseteq A$.

$(2) \Longrightarrow (1)$. Suppose A is strongly prime intuitionistic fuzzy bi-ideal of S. Let B, C be any intuitionistic fuzzy bi-ideals of S such that $B \cap C \subseteq A$. As $B \circ C \cap C \circ B = B \cap C \subseteq A$ and A is strongly prime, so we have $B \subseteq A$ or $C \subseteq A$. Thus A is strongly irreducible intuitionistic fuzzy bi-ideal of S. ∎

Theorem 5.3. *Each intuitionistic fuzzy bi-ideal of a semigroup S is strongly prime if and only if S is regular and intra regular and the set of intuitionistic fuzzy bi-ideals of S is totally ordered, by inclusion.*

Proof. Suppose that each intuitionistic fuzzy bi-ideal of a semigroup S is strongly prime, then each intuitionistic fuzzy bi-ideal of the semigroup S is semiprime, then by Theorem 5.2, S is both regular and intra regular. We show that the set of intuitionistic fuzzy bi-ideal of S is totally ordered by inclusion. Let B, C be intuitionistic fuzzy bi-ideal of S, then by Theorem 5.2, $(B \circ C) \cap (C \circ B) = B \cap C$. As each intuitionistic fuzzy bi-ideal of S is strongly prime so $B \cap C$ is strongly prime, hence either $B \subseteq B \cap C$ or $C \subseteq B \cap C$, that is either $B \subseteq C$ or $C \subseteq B$.

Conversely, assume that S is regular and intra-regular and the set of intuitionistic fuzzy bi-ideals of S is totally ordered under inclusion, then we show that each intuitionistic fuzzy bi-ideal of the semigroup S is strongly prime. Let A be an intuitionistic fuzzy bi-ideal of S and B, C any intuitionistic fuzzy bi-ideals of S such that $(B \circ C) \cap (C \circ B) \subseteq A$. Since S is both regular and intra-regular, so by Theorem 5.2, $(B \circ C) \cap (C \circ B) = B \cap C$. Hence $B \cap C \subseteq A$. Since the set of intuitionistic fuzzy bi-ideals of S is totally ordered, so either $B \subseteq C$ or $C \subseteq B$, that is either $B \cap C = B$ or $B \cap C = C$. Thus either $B \subseteq A$ or $C \subseteq A$. ∎

Theorem 5.4. *If the set of intuitionistic fuzzy bi-ideals of a semigroup S is totally order by inclusion, then S is both regular and intra regular if and only if each intuitionistic fuzzy bi-ideal of S is intuitionistic fuzzy prime bi-ideal of S.*

Proof. Suppose S is both regular and intra regular and A any intuitionistic fuzzy bi-ideal of S. Let B, C be intuitionistic fuzzy bi-ideals of S such that $B \circ C \subseteq A$. Since the set of intuitionistic fuzzy bi-ideal of S is totally ordered, therefore either $B \subseteq C$ or $C \subseteq B$. Suppose that $B \subseteq C$ then $B \circ B \subseteq B \circ C \subseteq A$. By Theorem 5.2, A is semiprime, so $B \subseteq A$. Hence A is a prime intuitionistic fuzzy bi-ideal of S.

Conversely, assume that every intuitionistic fuzzy bi-ideal of S is prime. Since every prime intuitionistic fuzzy bi-ideal is semiprime, so by Theorem 5.2, S is both regular and intra-regular. ∎

Theorem 5.5. *For a semigroup S,the following assertion are equivalent.*

(1) *Set of intuitionistic fuzzy bi-ideals of S is totally ordered by inclusion.*
(2) *Each intuitionistic fuzzy bi-ideal of S is strongly irreducible.*
(3) *Each intuitionistic fuzzy bi-ideal of S is irreducible.*

Proof. $(1) \Longrightarrow (2)$. Let A be an arbitrary intuitionistic fuzzy bi-ideal of S. Let B,C be intuitionistic fuzzy bi-ideals of S such that $B \cap C \subseteq A$. Since the set of intuitionistic fuzzy bi-ideals of S is totally ordered by inclusion, so either $B \subseteq C$ or $C \subseteq B$. Thus either $B \subseteq A$ or $C \subseteq A$. Hence A is strongly irreducible.

$(2) \Longrightarrow (3)$. Let A be an arbitrary intuitionistic fuzzy bi-ideal of S. Let B,C be intuitionistic fuzzy bi-ideals of S such that $B \cap C = A$. Then $A \subseteq B$ and $A \subseteq C$. By hypothesis either $B \subseteq A$ or $C \subseteq A$. Thus either $B = A$ or $C = A$.

$(3) \Longrightarrow (1)$. Suppose each intuitionistic fuzzy bi-ideal of S is irreducible. Let A,B be intuitionistic fuzzy bi-ideals of S, then $A \cap B$ is an intuitionistic fuzzy bi-ideal of S. Also $A \cap B = A \cap B$. By hypothesis either $A = A \cap B$ or $B = A \cap B$, that is either $A \subseteq B$ or $B \subseteq A$. Hence the set of intuitionistic fuzzy bi-ideals of S is totally ordered by inclusion . ∎

The next example shows that if each bi-ideal of a semigroup is prime, then it is not necessary that its each intuitionistic fuzzy bi-ideal is prime.

Example 5.1. Consider the semigroup $S = \{0,a,b\}$

$\cdot$	0	a	b
0	0	0	0
a	0	a	a
b	0	b	b

It is evident that S is both regular and intra regular. Bi-ideals in S are $\{0\},\{0,a\},\{0,b\}$ and S. All bi-ideals are prime and hence semiprime bi-ideals.

Fact 5.1. Each intuitionistic fuzzy subset of S is an intuitionistic fuzzy subsemigroup of S.

Fact 5.2. Each intuitionistic fuzzy subset $A = (\mu_A, \gamma_A)$ of S is an intuitionistic fuzzy bi-ideal of S if and only if $\mu_A(0) \geq \mu_A(x)$ and $\gamma_A(0) \leq \gamma_A(x)$ for all $x \in S$.

Proof. Suppose $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy bi-ideal of S, then by definition $\mu_A(0) = \mu_A(x0x) \geq \mu_A(x) \wedge \mu_A(x) = \mu_A(x)$ and $\gamma_A(0) = \gamma_A(x0x) \leq \gamma_A(x) \vee \gamma_A(x) = \gamma_A(x)$ for all $x \in S$.

Conversely, assume that $\mu_A(0) \geq \mu_A(x)$ and $\gamma_A(0) \leq \gamma_A(x)$ for all $x \in S$. Then $xyz = x$ if $x,y,z \in \{a,b\}$ and $xyz = 0$ if one of them is zero. Thus $\mu_A(xyz) \geq \mu_A(x) \wedge \mu_A(z)$ and $\gamma_A(xyz) \leq \gamma_A(x) \vee \gamma_A(z)$. ∎

Fact 5.3. Since S is regular and intra regular, so every intuitionistic fuzzy bi-ideal of S is intuitionistic fuzzy semiprime bi-ideal.

Consider the intuitionistic fuzzy bi-ideals A,B,C of S,

$\mu_A(0) = 0.7,\ \mu_A(a) = 0.6,\ \mu_A(b) = 0.4\ and \gamma_A(0) = 0.3,\ \gamma_A(a) = 0.4,\ \gamma_A(b) = 0.4$

$\mu_B(0) = 1.0,\ \mu_B(a) = 0.5,\ \mu_B(b) = 0.4\ \text{and} \gamma_B(0) = 0,\ \gamma_B(a) = 0.5,\ \gamma_B(b) = 0.4$

$\mu_C(0) = 0.7,\ \mu_C(a) = 0.65,\ \mu_C(b) = 0.3$ and$\gamma_C(0) = 0.3,\ \gamma_C(a) = 0.35\ \gamma_C(b) = 0.7$

$$\mu_{B\circ C}(0) = \bigvee_{0=xy} \{\mu_B(x) \wedge \mu_C(y)\} = \bigvee_{0=xy} \{1 \wedge 0.7, 1 \wedge 0.65, 1 \wedge 0.3, 0.5 \wedge 0.7, 0.4 \wedge 0.7\} = 0.7$$

$$\mu_{B\circ C}(a) = \bigvee_{a=xy} \{\mu_B(x) \wedge \mu_C(y)\} = \bigvee_{a=xy} \{0.5 \wedge 0.65, 0.5 \wedge 0.3\} = 0.5$$

$$\mu_{B\circ C}(b) = \bigvee_{b=xy} \{\mu_B(x) \wedge \mu_C(y)\} = \bigvee_{b=xy} \{0.4 \wedge 0.65, 0.4 \wedge 0.3\} = 0.4$$

$$\gamma_{B\circ C}(0) = \bigwedge_{0=xy} \{\gamma_B(x) \vee \gamma_C(y)\} = \bigwedge_{0=xy} \{0 \vee 0.3, 0 \vee 0.35, 0 \vee 0.7, 0.5 \vee 0.3, 0.4 \vee 0.3\} = 0.3$$

$$\gamma_{B\circ C}(a) = \bigwedge_{a=xy} \{\gamma_B(x) \vee \gamma_C(y)\} = \bigwedge_{a=xy} \{0.5 \vee 0.35, 0.5 \vee 0.7\} = 0.5$$

$$\gamma_{B\circ C}(b) = \bigwedge_{b=xy} \{\gamma_B(x) \vee \gamma_C(y)\} = \bigwedge_{b=xy} \{0.4 \vee 0.35, 0.4 \vee 0.7\} = 0.4$$

Thus $\mu_{B\circ C} \leq \mu_A$ and $\gamma_{B\circ C} \geq \gamma_A$, that is $B \circ C \subseteq A$ but neither $B \subseteq A$ nor $C \subseteq A$. Thus A is not intuitionistic fuzzy prime bi-ideal of S.

Example 5.2. Consider the semigroup $S = \{0, x, 1\}$

$\cdot$	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

It is evident that S is both regular and intra regular. Bi-ideals in S are $\{0\}, \{0, x\}$ and S. All bi-ideals are strongly prime.

Fact 5.4. Each intuitionistic fuzzy subset of S is an intuitionistic fuzzy subsemigroup of S.

Fact 5.5. Each intuitionistic fuzzy subset $A = (\mu_A, \gamma_A)$ of S is an intuitionistic fuzzy bi-ideal of S if and only if $\mu_A(0) \geq \mu_A(x) \geq \mu_A(1)$ and $\gamma_A(0) \leq \gamma_A(x) \leq \gamma_A(1)$.

Proof. Suppose $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy bi-ideal of S, then by definition $\mu_A(0) = \mu_A(x0x) \geq \mu_A(x) \wedge \mu_A(x) = \mu_A(x)$ and $\gamma_A(0) = \gamma_A(x0x) \leq \gamma_A(x) \vee \gamma_A(x) = \gamma_A(x)$. Also $\mu_A(x) = \mu_A(1x1) \geq \mu_A(1) \wedge \mu_A(1) = \mu_A(1)$ and $\gamma_A(x) = \gamma_A(1x1) \leq \gamma_A(1) \vee \gamma_A(1) = \gamma_A(1)$. Hence $\mu_A(0) \geq \mu_A(x) \geq \mu_A(1)$ and $\gamma_A(0) \leq \gamma_A(x) \leq \gamma_A(1)$.

Conversely, assume that $\mu_A(0) \geq \mu_A(x) \geq \mu_A(1)$ and $\gamma_A(0) \leq \gamma_A(x) \leq \gamma_A(1)$.

Then by Fact 5.4 $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subsemigroup of S. Also $abc = x$ if one of a, b, c is x and non of them is 0, and $abc = 1$ if all of a, b, c are 1 and non of them is 0 and $abc = 0$ if one of them is zero. Thus $\mu_A(abc) \geq \mu_A(a) \wedge \mu_A(c)$ and $\gamma_A(abc) \leq \gamma_A(a) \vee \gamma_A(c)$. Hence $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy bi-ideal of S. ∎

Fact 5.6. Since S is regular and intra regular, so every intuitionistic fuzzy bi-ideal of S is intuitionistic fuzzy semiprime bi-ideal.

Consider the intuitionistic fuzzy bi-ideals A, B, C of S,

$\mu_A(0) = 0.7,\ \mu_A(x) = 0.6,\ \mu_A(0) = 0.5$ and$\gamma_A(0) = 0.3,\ \gamma_A(x) = 0.4,\ \gamma_A(1) = 0.5$

$\mu_B(0) = 1.0,\ \mu_B(x) = 0.5,\ \mu_B(1) = 0.4$ and$\gamma_B(0) = 0,\ \gamma_B(x) = 0.5,\ \gamma_B(1) = 0.47$

$\mu_C(0) = 0.7,\ \mu_C(x) = 0.65,\ \mu_C(1) = 0.3$ and$\gamma_C(0) = 0.3,\ \gamma_C(x) = 0.35,\ \gamma_C(1) = 0.7$

$$\mu_{B\circ C}(0) = \bigvee_{0=ab} \{\mu_B(a) \wedge \mu_C(b)\} = \bigvee_{0=ab} \{1\wedge 0.7, 1\wedge 0.65, 1\wedge 0.3, 0.5\wedge 0.7, 0.4\wedge 0.7\} = 0.7$$

$$\mu_{B\circ C}(x) = \bigvee_{x=ab} \{\mu_B(a) \wedge \mu_C(b)\} = \bigvee_{x=ab} \{0.5\wedge 0.65, 0.5\wedge 0.3, 0.4\wedge 0.65\} = 0.5$$

$$\mu_{B\circ C}(1) = \bigvee_{1=ab} \{\mu_B(a) \wedge \mu_C(b)\} = \bigvee_{1=ab} \{0.4\wedge 0.3\} = 0.4$$

$$\gamma_{B\circ C}(0) = \bigwedge_{0=ab} \{\gamma_B(a) \vee \gamma_C(b)\} = \bigwedge_{0=ab} \{0\vee 0.3, 0\vee 0.35, 0\vee 0.7, 0.5\vee 0.3, 0.4\vee 0.3\} = 0.3$$

$$\gamma_{B\circ C}(x) = \bigwedge_{x=ab} \{\gamma_B(a) \vee \gamma_C(b)\} = \bigwedge_{x=ab} \{0.5\vee 0.35, 0.5\vee 0.7, 0.4\vee 0.65\} = 0.5$$

$$\gamma_{B\circ C}(1) = \bigwedge_{1=ab} \{\gamma_B(a) \vee \gamma_C(b)\} = \bigwedge_{1=ab} \{0.4\vee 0.7\} = 0.7$$

Thus $\mu_{B\circ C} \leq \mu_A$ and $\gamma_{B\circ C} \geq \gamma_A$, that is, $B\circ C \subseteq A$, but neither $B \subseteq A$ nor $C \subseteq A$. Thus A is not intuitionistic fuzzy prime bi-ideal of S.

Example 5.3. Let S be a group. Then by [21] every intuitionistic fuzzy bi-ideal of S is constant. Since every group is regular and intra regular, so S is regular and intra regular. Since the set of all intuitionistic fuzzy bi-ideals of S is totally ordered, so by Theorem 5.3 every intuitionistic fuzzy bi-ideal of S is strongly prime.

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