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Characterization of Ordered Semigroups in Terms of Fuzzy Soft Ideals

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Abstract. In this paper, we apply the concept of fuzzy soft sets to ordered semigroup theory. The concepts of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideals are introduced and some related properties are obtained. Three kinds of lattice structures of the set of all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left (right) ideals of an ordered semigroup are derived. The characterization of left quasi-regular ordered semigroups and ordered semigroups that are left quasi-regular and intra-regular in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideals is discussed.

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1. Introduction

Dealing with uncertainties is a major problem in many areas such as economics, engineering, environmental science, medical science and social sciences. These kinds of problems cannot be dealt with by classical methods, because classical methods have inherent difficulties. To overcome these difficulties, Molodtsov [13] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Up to the present, research on soft sets has been very active and many important results have been achieved in the theoretical aspect. Maji *et al.* [12] introduced several algebraic operations in soft set theory and published a detail theoretical study on soft sets. Ali *et al.* [2] further presented and investigated some new algebraic operations for soft sets. Aygünoğlu and Aygün [3] discussed the applications of fuzzy soft sets to group theory and investigated (normal) fuzzy soft groups. Feng *et al.* [4] investigated soft semirings by using the soft set theory. Jun [5] introduced and

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investigated the notion of soft BCK/BCI-algebras. Jun and Park [7] and Jun *et al.* [6] discussed the applications of soft sets in ideal theory of BCK/BCI-algebras and in d-algebras, respectively. Koyuncu and Tanay [1] introduced and studied soft rings. Zhan and Jun [15] characterized the (implicative, positive implicative and fantastic) filteristic soft BL-algebras based on $\in$-soft sets and q-soft sets.

The purpose of this paper is to deal with the algebraic structure of ordered semigroups by applying fuzzy soft set theory. The rest of this paper is organized as follows. In Section 2, we summarize some basic concepts which will be used throughout the paper and study a new ordering relation on the set of all fuzzy soft sets over a universe set. In Section 3, we define and investigate $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideals. The characterization of left quasi-regular ordered semigroups and ordered semigroups that are left quasi-regular and intra-regular in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideals is discussed in Section 4. Some conclusions are given in the last Section.

2. Preliminaries

2.1. Ordered semigroups

In this section, we recall some basic notions and results on ordered semigroups (see [8, 9]). An *ordered semigroup* is an algebraic system $(S, \cdot, \leq)$ consisting of a non-empty set S together with a binary operation "$+$" and a compatible ordering "$\leq$" on S such that $(S, \cdot)$ is a semigroup and $x \leq y$ implies $ax \leq ay$ and $xa \leq ya$ for all $x, y, a \in S$. An *identity* of an ordered semigroup $(S, \cdot, \leq)$ is an element e of S such that $ea = ae = a$ for all $a \in S$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. A subset I of S is called a *left* (resp., *right*) *ideal* of S if it satisfies the following conditions: (1) $SI \subseteq I$ (resp., $IS \subseteq I$); (2) if $x \in I$ and $S \ni y \leq x$, then $y \in I$. If I is both a left and a right ideal of S, then I is called an *ideal* of S. A subset P of S is called a *bi-ideal* if it satisfies the following conditions: (1) $PP \subseteq P$; (2) $PSP \subseteq P$; (3) if $x \in P$ and $S \ni y \leq x$, then $y \in P$.

For $X, Y \subseteq S$, denote $(X] := \{x \in S | x \leq y \text{ for some } y \in X\}$ and $XY := \{xy \in S | x \in X, y \in Y\}$. For $x \in S$, define $A_x = \{(y,z) \in S \times S | x \leq yz\}$. For $X, Y \subseteq S$, we have $X \subseteq (X]$, $(X](Y] \subseteq (XY]$, $((X]] = (X]$ and $(X] \subseteq (Y]$ if $X \subseteq Y$. X is said to be *idempotent* if $(X] = (X^2]$. A subset Q of S is called a *quasi-ideal* if it satisfies the following conditions: (1) $(QS] \cap (SQ] \subseteq Q$; (2) if $x \in Q$ and $S \ni y \leq x$, then $y \in Q$.

2.2. Fuzzy sets

Let X be a non-empty set. A fuzzy subset μ of X is defined as a mapping from X into $[0,1]$, where $[0,1]$ is the usual interval of real numbers. We denote by $\mathscr{F}(X)$ the set of all fuzzy subsets of X. A fuzzy subset μ of X of the form

$$\mu(y) = \begin{cases} r(\neq 0) & \text{if } y = x, \\ 0 & \text{otherwise} \end{cases}$$

is said to be a *fuzzy point with support x and value r* and is denoted by x_r, where $r \in (0,1]$.

In what follows let $\gamma, \delta \in [0,1]$ be such that $\gamma < \delta$. For any $Y \subseteq X$, we define $\chi_{\gamma Y}^{\delta}$ be the fuzzy subset of X by $\chi_{\gamma Y}^{\delta}(x) \geq \delta$ for all $x \in Y$ and $\chi_{\gamma Y}^{\delta}(x) \leq \gamma$ otherwise. Clearly, $\chi_{\gamma Y}^{\delta}$ is the characteristic function of Y if $\gamma = 0$ and $\delta = 1$.

For a fuzzy point x_r and a fuzzy subset μ of X, we say that

(1) $x_r \in_\gamma \mu$ if $\mu(x) \geq r > \gamma$.
(2) $x_r q_\delta \mu$ if $\mu(x) + r > 2\delta$.
(3) $x_r \in_\gamma \vee q_\delta \mu$ if $x_r \in_\gamma \mu$ or $x_r q_\delta \mu$.

Let us now introduce a new ordering relation on $\mathscr{F}(X)$, denoted as " $\subseteq \vee q_{(\gamma,\delta)}$", as follows. For any $\mu, \nu \in \mathscr{F}(X)$, by $\mu \subseteq \vee q_{(\gamma,\delta)} \nu$ we mean that $x_r \in_\gamma \mu$ implies $x_r \in_\gamma \vee q_\delta \nu$ for all $x \in X$ and $r \in (\gamma, 1]$. Moreover, μ and ν are said to be (γ,δ)-*equal*, denoted by $\mu =_{(\gamma,\delta)} \nu$, if $\mu \subseteq \vee q_{(\gamma,\delta)} \nu$ and $\nu \subseteq \vee q_{(\gamma,\delta)} \mu$.

In the sequel, unless otherwise stated, $\overline{\alpha}$ means α does not hold, where $\alpha \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \subseteq \vee q_{(\gamma,\delta)}\}$.

Lemma 2.1. *Let $\mu, \nu \in \mathscr{F}(X)$. Then $\mu \subseteq \vee q_{(\gamma,\delta)} \nu$ if and only if* $\max\{\nu(x), \gamma\} \geq \min\{\mu(x), \delta\}$ *for all $x \in X$.*

Proof. It is straightforward. ∎

Lemma 2.2. *Let $\mu, \nu, \omega \in \mathscr{F}(X)$. If $\mu \subseteq \vee q_{(\gamma,\delta)} \nu$ and $\nu \subseteq \vee q_{(\gamma,\delta)} \omega$, then $\mu \subseteq \vee q_{(\gamma,\delta)} \omega$.*

Proof. It is straightforward by Lemma 2.1. ∎

Lemmas 2.1 and 2.2 give that " $=_{(\gamma,\delta)}$ " is an equivalence relation on $\mathscr{F}(X)$. It is also worth noticing that $\mu =_{(\gamma,\delta)} \nu$ if and only if $\max\{\min\{\mu(x), \delta\}, \gamma\} = \max\{\min\{\nu(x), \delta\}, \gamma\}$ for all $x \in X$ by Lemma 2.1.

Definition 2.1. [10] Let $(S, \cdot, \leq)$ be an ordered semigroup and $\mu, \nu \in \mathscr{F}(S)$. Define the *product* of μ and ν, denoted by $\mu \circ \nu$, by

$$(\mu \circ \nu)(x) = \begin{cases} \sup\limits_{(y,z) \in A_x} \min\{\mu(y), \nu(z)\} & \text{if there exist } y, z \in S \text{ such that } (y,z) \in A_x, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in S$.

Lemma 2.3. *Let $(S, \cdot, \leq)$ be an ordered semigroup and $X, Y \subseteq S$. Then we have*
(1) $X \subseteq Y$ if and only if $\chi^\delta_{\gamma X} \subseteq \vee q_{(\gamma,\delta)} \chi^\delta_{\gamma Y}$.
(2) $\chi^\delta_{\gamma X} \cap \chi^\delta_{\gamma Y} =_{(\gamma,\delta)} \chi^\delta_{\gamma(X \cap Y)}$.
(3) $\chi^\delta_{\gamma X} \circ \chi^\delta_{\gamma Y} =_{(\gamma,\delta)} \chi^\delta_{\gamma(XY]}$.

Proof. It is straightforward. ∎

2.3. Fuzzy soft sets

Let U be an initial universe set and E the set of all possible parameters under consideration with respect to U. As a generalization of soft set introduced in Molodtsov [13], Maji *et al.* [11] defined fuzzy soft set in the following way.

Definition 2.2. A pair $\langle F, A \rangle$ is called a *fuzzy soft set* over U, where $A \subseteq E$ and F is a mapping given by $F : A \to \mathscr{F}(U)$.

In general, for every $\varepsilon \in A$, $F(\varepsilon)$ is a fuzzy set of U and it is called *fuzzy value set* of parameter ε. The set of all fuzzy soft sets over U with parameters from E is called a *fuzzy soft class*, and it is denote by $\mathscr{F}\mathscr{S}(U, E)$.

Definition 2.3. [14] Let $\langle F, A \rangle$ and $\langle G, B \rangle$ be two fuzzy soft sets over U. We say that $\langle F, A \rangle$ is a *fuzzy soft subset* of $\langle G, B \rangle$ and write $\langle F, A \rangle \subset \langle G, B \rangle$ if

(i) *$A \subseteq B$;*
(ii) *For any $\varepsilon \in A$, $F(\varepsilon) \subseteq G(\varepsilon)$.*

$\langle F,A\rangle$ and $\langle G,B\rangle$ are said to be *fuzzy soft equal* and write $\langle F,A\rangle = \langle G,B\rangle$ if $\langle F,A\rangle \subset \langle G,B\rangle$ and $\langle G,B\rangle \subset \langle F,A\rangle$.

Let us now introduce some new concepts on fuzzy soft sets analogous to the concepts introduced in Ali *et al.* [2].

Definition 2.4. The *extended intersection* of two fuzzy soft sets $\langle F,A\rangle$ and $\langle G,B\rangle$ over U is a fuzzy soft set denoted by $\langle H,C\rangle$, where $C = A\cup B$ and

$$H(\varepsilon) = \begin{cases} F(\varepsilon) & \text{if } \varepsilon \in A-B, \\ G(\varepsilon) & \text{if } \varepsilon \in B-A, \\ F(\varepsilon)\cap G(\varepsilon) & \text{if } \varepsilon \in A\cap B, \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $\langle H,C\rangle = \langle F,A\rangle \tilde{\cap} \langle G,B\rangle$.

Definition 2.5. The *extended union* of two fuzzy soft sets $\langle F,A\rangle$ and $\langle G,B\rangle$ over U is a fuzzy soft set denoted by $\langle H,C\rangle$, where $C = A\cup B$ and

$$H(\varepsilon) = \begin{cases} F(\varepsilon) & \text{if } \varepsilon \in A-B, \\ G(\varepsilon) & \text{if } \varepsilon \in B-A, \\ F(\varepsilon)\cup G(\varepsilon) & \text{if } \varepsilon \in A\cap B, \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $\langle H,C\rangle = \langle F,A\rangle \tilde{\cup} \langle G,B\rangle$.

Definition 2.6. Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two fuzzy soft sets over U such that $A\cap B \neq \emptyset$. The *restricted intersection* of $\langle F,A\rangle$ and $\langle G,B\rangle$ is defined to be the fuzzy soft set $\langle H,C\rangle$, where $C = A\cap B$ and $H(\varepsilon) = F(\varepsilon)\cap G(\varepsilon)$ for all $\varepsilon \in C$. This is denoted by $\langle H,C\rangle = \langle F,A\rangle \cap \langle G,B\rangle$.

Definition 2.7. Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two fuzzy soft sets over U such that $A\cap B \neq \emptyset$. The *restricted union* of $\langle F,A\rangle$ and $\langle G,B\rangle$ is defined to be the fuzzy soft set $\langle H,C\rangle$, where $C = A\cap B$ and $H(\varepsilon) = F(\varepsilon)\cup G(\varepsilon)$ for all $\varepsilon \in C$. This is denoted by $\langle H,C\rangle = \langle F,A\rangle \cup \langle G,B\rangle$.

Definition 2.8. Let $V \subseteq U$. A fuzzy soft set $\langle F,A\rangle$ over V is said to be *a relative whole (γ,δ)-fuzzy soft set* (with respect to universe set V and parameter set A), denoted by $\Sigma(V,A)$, if $F(\varepsilon) = \chi^{\delta}_{\gamma V}$ for all $\varepsilon \in A$.

Definition 2.9. Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two fuzzy soft sets over U. We say that $\langle F,A\rangle$ is an *(γ,δ)-fuzzy soft subset* of $\langle G,B\rangle$ and write $\langle F,A\rangle \subset_{(\gamma,\delta)} \langle G,B\rangle$ if

(i) *$A \subseteq B$;*
(ii) *For any $\varepsilon \in A$, $F(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} G(\varepsilon)$.*

$\langle F,A\rangle$ and $\langle G,B\rangle$ are said to be *(γ,δ)-fuzzy soft equal* and write $\langle F,A\rangle \asymp_{(\gamma,\delta)} \langle G,B\rangle$ if $\langle F,A\rangle \subset_{(\gamma,\delta)} \langle G,B\rangle$ and $\langle G,B\rangle \subset_{(\gamma,\delta)} \langle F,A\rangle$.

Clearly, $\langle F,A\rangle \subset \langle G,B\rangle$ implies $\langle F,A\rangle \subseteq \vee q_{(\gamma,\delta)} \langle G,B\rangle$ by Lemma 2.1 and Definition 2.12.

Lemma 2.4. *Let $\langle F,A\rangle$, $\langle G,B\rangle$ and $\langle H,C\rangle$ be fuzzy soft sets over U. If $\langle F,A\rangle \subset_{(\gamma,\delta)} \langle G,B\rangle$ and $\langle G,B\rangle \subset_{(\gamma,\delta)} \langle H,C\rangle$. Then $\langle F,A\rangle \subset_{(\gamma,\delta)} \langle F,C\rangle$.*

Proof. It is straightforward by Lemma 2.2 and Definition 2.9. ∎

Lemmas 2.1, 2.4 and Definition 2.9 give that " $\asymp_{(\gamma,\delta)}$ " is an equivalence relation on $\mathscr{F}\mathscr{S}(U,E)$. Now we introduce the concept of the product of two fuzzy soft sets over an ordered semigroup $(S,\cdot,\leq)$ as follows.

Definition 2.10. The *product* of two fuzzy soft sets $\langle F,A\rangle$ and $\langle G,B\rangle$ over an ordered semigroup $(S,\cdot,\leq)$ is a fuzzy soft set over S, denoted by $\langle F\circ G,C\rangle$, where $C=A\cup B$ and

$$(F\circ G)(\varepsilon)=\begin{cases} F(\varepsilon) & \text{if } \varepsilon\in A-B,\\ G(\varepsilon) & \text{if } \varepsilon\in B-A,\\ F(\varepsilon)\circ G(\varepsilon) & \text{if } \varepsilon\in A\cap B,\end{cases}$$

for all $\varepsilon\in C$. This is denoted by $\langle F\circ G,C\rangle=\langle F,A\rangle\odot\langle G,B\rangle$.

The following results can be easily deduced.

Lemma 2.5. *Let $\langle F_1,A\rangle$, $\langle F_2,A\rangle$, $\langle G_1,B\rangle$ and $\langle G_2,B\rangle$ be fuzzy soft sets over an ordered semigroup $(S,\cdot,\leq)$ such that $\langle F_1,A\rangle\subset_{(\gamma,\delta)}\langle F_2,A\rangle$ and $\langle G_1,B\rangle\subset_{(\gamma,\delta)}\langle G_2,B\rangle$. Then*
(1) $\langle F_1,A\rangle\odot\langle G_1,B\rangle\subset_{(\gamma,\delta)}\langle F_2,A\rangle\odot\langle G_2,B\rangle$.
(2) $\langle F_1,A\rangle\cap\langle G_1,B\rangle\subset_{(\gamma,\delta)}\langle F_2,A\rangle\cap\langle G_2,B\rangle$ and $\langle F_1,A\rangle\widetilde{\cap}\langle G_1,B\rangle\subset_{(\gamma,\delta)}\langle F_2,A\rangle\widetilde{\cap}\langle G_2,B\rangle$.
(3) $\langle F_1,A\rangle\cup\langle G_1,B\rangle\subset_{(\gamma,\delta)}\langle F_2,A\rangle\cup\langle G_2,B\rangle$ and $\langle F_1,A\rangle\widetilde{\cup}\langle G_1,B\rangle\subset_{(\gamma,\delta)}\langle F_2,A\rangle\widetilde{\cup}\langle G_2,B\rangle$.

Lemma 2.6. *Let $\langle F,A\rangle$, $\langle G,B\rangle$ and $\langle H,C\rangle$ be fuzzy soft sets over an ordered semigroup $(S,\cdot,\leq)$. Then $\langle F,A\rangle\odot(\langle G,B\rangle\odot\langle H,C\rangle)=(\langle F,A\rangle\odot\langle G,B\rangle)\odot\langle H,C\rangle$.*

3. $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft ideals over an ordered semigroup

In this section, we will introduce the concepts of $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left (right) ideals, $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft bi-ideals and $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft quasi-ideals over an ordered semigroup and investigate their fundamental properties and mutual relationships.

Definition 3.1. A fuzzy soft set $\langle F,A\rangle$ over an ordered semigroup $(S,\cdot,\leq)$ is called an $(\in_\gamma,\in_\gamma\vee q_\delta)$-*fuzzy soft left* (resp., *right*) *ideal* over S if it satisfies:

(F1a) $\Sigma(S,A)\odot\langle F,A\rangle\subset_{(\gamma,\delta)}\langle F,A\rangle$ (resp., $\langle F,A\rangle\odot\Sigma(S,A)\subset_{(\gamma,\delta)}\langle F,A\rangle$),
(F2a) If $y\leq x$, then $x_r\in_\gamma F(\varepsilon)\Rightarrow y_r\in_\gamma\vee q_\delta\, F(\varepsilon)$ for all $x,y\in S$, $\varepsilon\in A$ and $r\in(\gamma,1]$.

A fuzzy soft set over S is called an $(\in_\gamma,\in_\gamma\vee q_\delta)$-*fuzzy soft ideal* over S if it is both an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft right ideal and an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideal over S.

Definition 3.2. A fuzzy soft set $\langle F,A\rangle$ over an ordered semigroup $(S,\cdot,\leq)$ is called an $(\in_\gamma,\in_\gamma\vee q_\delta)$-*fuzzy soft bi-ideal* over S if it satisfies conditions (F2a) and

(F3a) $\langle F,A\rangle\odot\langle F,A\rangle\subset_{(\gamma,\delta)}\langle F,A\rangle$,
(F4a) $\langle F,A\rangle\odot\Sigma(S,A)\odot\langle F,A\rangle\subset_{(\gamma,\delta)}\langle F,A\rangle$.

Definition 3.3. A fuzzy soft set $\langle F,A\rangle$ over an ordered semigroup $(S,\cdot,\leq)$ is called an $(\in_\gamma,\in_\gamma\vee q_\delta)$-*fuzzy soft quasi-ideal* over S if it satisfies conditions (F2a) and

(F5a) $\langle F,A\rangle\odot\Sigma(S,A)\widetilde{\cap}\Sigma(S,A)\odot\langle F,A\rangle\subset_{(\gamma,\delta)}\langle F,A\rangle$.

Lemma 3.1. *Let $\langle F,A\rangle$ be a fuzzy soft set over an ordered semigroup $(S,\cdot,\leq)$. Then* (F2a) *holds if and only if the following condition holds:*
(F2c) $y\leq x\Rightarrow\max\{F(\varepsilon)(y),\gamma\}\geq\min\{F(\varepsilon)(x),\delta\}$ *for all $x,y\in S$ and $\varepsilon\in A$.*

Proof. Assume that condition (F2a) holds. If there exist $x,y\in S$, $\varepsilon\in A$ and $r\in(\gamma,1]$ such that $\max\{F(\varepsilon)(y),\gamma\}<r<\min\{F(\varepsilon)(x),\delta\}$, then $F(\varepsilon)(x)>r$ and $F(\varepsilon)(y)<r<\delta$, this implies $x_r\in_\gamma F(\varepsilon)$ but $y_r\overline{\in_\gamma\vee q_\delta}F(\varepsilon)$, which contradicts condition (F2a). Hence condition (F2c) is valid.

Conversely, assume that condition (F2c) holds. If there exist $x,y \in S$, $\varepsilon \in A$ and $r \in (\gamma,1]$ with $y \leq x$ and $x_r \in F(\varepsilon)$ such that $y_r \overline{\in_\gamma \vee q_\delta} F(\varepsilon)$, then $F(\varepsilon)(x) \geq r > \gamma$ but $F(\varepsilon)(y) < r$ and $F(\varepsilon)(y) + r < 2\delta$, it follows that $F(\varepsilon)(y) < \delta$. Hence $\max\{F(\varepsilon)(y),\gamma\} < \min\{F(\varepsilon)(x),\delta\}$, a contradiction. Thus condition (F2a) is valid. ∎

Lemma 3.2. *Let $\langle F,A\rangle$ be a fuzzy soft set over an ordered semigroup $(S,\cdot,\leq)$ which satisfies condition* (F2a)*. Then* (F1a) *holds if and only if one of the following conditions holds:*

(F1b) *$x_r \in_\gamma F(\varepsilon)$ implies $(yx)_r \in_\gamma \vee q_\delta F(\varepsilon)$ (resp., $(xy)_r \in_\gamma \vee q_\delta F(\varepsilon)$) for all $x,y \in S, \varepsilon \in A$ and $r \in (\gamma,1]$;*

(F1c) $\max\{F(\varepsilon)(xy),\gamma\} \geq \min\{F(\varepsilon)(y),\delta\}$ *(resp.,* $\max\{F(\varepsilon)(xy),\gamma\} \geq \min\{F(\varepsilon)(x),\delta\}$*) for all $x,y \in S$ and $\varepsilon \in A$.*

Proof. (F1a)⇒(F1b) For any $x,y \in S, \varepsilon \in A$ and $r \in (\gamma,1]$, if $x_r \in_\gamma F(\varepsilon)$, then $F(\varepsilon)(x) > r > \gamma$. By (F1a) and Lemma 2.1, we have

$$\begin{aligned}
\max\{F(\varepsilon)(xy),\gamma\} &\geq \min\{(\chi^\delta_{\gamma S} \circ F(\varepsilon))(xy),\delta\} \\
&= \min\left\{ \sup_{(a,b)\in A_{xy}} \min\{\chi^\delta_{\gamma S}(a), F(\varepsilon)(b)\}, \delta \right\} \\
&= \sup_{(a,b)\in A_{xy}} \min\left\{\chi^\delta_{\gamma S}(a), F(\varepsilon)(b), \delta\right\} \\
&\geq \min\{F(\varepsilon)(y),\delta\} \geq \min\{r,\delta\} > \gamma.
\end{aligned}$$

It follows that $F(\varepsilon)(xy) \geq \min\{r,\delta\}$. We consider the following cases.

Case 1: $r \leq \delta$. Then $F(\varepsilon)(xy) \geq r$, that is, $(xy)_r \in_\gamma F(\varepsilon)$.

Case 1: $r > \delta$. Then $F(\varepsilon)(xy) + r > 2\delta$, that is, $(xy)_r q_\delta F(\varepsilon)$.

Thus, in any case, $(xy)_r \in_\gamma \vee q_\delta F(\varepsilon)$. Therefore, condition (F1b) is valid.

(F1b)⇒(F1c) Let $x,y \in S$ and $\varepsilon \in A$. If $\max\{F(\varepsilon)(xy),\gamma\} < r = \min\{F(\varepsilon)(y),\delta\}$, then $y_r \in_\gamma F(\varepsilon)$ but $(xy)_r \overline{\in_\gamma \vee q_\delta} F(\varepsilon)$, which contradicts condition (F1b) and so condition (F1c) holds.

(F1c)⇒(F1a) If $\chi^\delta_{\gamma S} \odot \langle F,A\rangle \overline{\subset_{(\gamma,\delta)}} \langle F,A\rangle$, then there exist $\varepsilon \in A$ and $x_r \in_\gamma \chi^\delta_{\gamma S} \circ F(\varepsilon)$ such that $x_r \overline{\in_\gamma \vee q_\delta} F(\varepsilon)$. Hence $F(\varepsilon)(x) < r$ and $F(\varepsilon)(x) + r \leq 2\delta$, which gives $F(\varepsilon)(x) < \delta$. If there exist $a,b \in S$ with $x \leq ab$, then by conditions (F1c) and (F2c), we have

$$\begin{aligned}
\delta > \max\{F(\varepsilon)(x),\gamma\} &\geq \max\{\min\{F(\varepsilon)(ab),\delta\},\gamma\} = \min\{\max\{F(\varepsilon)(ab),\gamma\},\delta\} \\
&\geq \min\{\min\{F(\varepsilon)(b),\delta\},\delta\} = \min\{F(\varepsilon)(b),\delta\}.
\end{aligned}$$

It follows that $\max\{F(\varepsilon)(x),\gamma\} \geq F(\varepsilon)(b)$. Hence we have

$$\begin{aligned}
r \leq (\chi^\delta_{\gamma S} \circ F(\varepsilon))(x) &= \sup_{(a,b)\in A_x} \min\{\chi^\delta_{\gamma S}(a), F(\varepsilon)(b)\} \leq \sup_{(a,b)\in A_x} F(\varepsilon)(b) \\
&\leq \sup_{(a,b)\in A_x} \max\{F(\varepsilon)(x),\gamma\} = \max\{F(\varepsilon)(x),\gamma\},
\end{aligned}$$

a contradiction. Therefore, condition (F1a) is satisfied. ∎

As a directly consequence of Lemmas 3.1 and 3.2, we have the following results.

Theorem 3.1. *A fuzzy soft set $\langle F,A\rangle$ over an ordered semigroup $(S,\cdot,\leq)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideal over S if and only if it satisfies conditions* (F2a) *and*

(F6b) $y_r \in_\gamma F(\varepsilon)$ *implies* $(xy)_r \in_\gamma \vee q_\delta F(\varepsilon)$ *and* $(yx)_r \in_\gamma \vee q_\delta F(\varepsilon)$ *for all* $x,y \in S$, $\varepsilon \in A$ *and* $r \in (\gamma, 1]$.

Theorem 3.2. *A fuzzy soft set* $\langle F,A\rangle$ *over an ordered semigroup* $(S,\cdot,\leq)$ *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft ideal over S if and only if it satisfies conditions* (F2c) *and*

(F6c) $\max\{F(\varepsilon)(xy),\gamma\} \geq \min\{\max\{F(\varepsilon)(x),F(\varepsilon)(y)\},\delta\}$ *for all* $x,y \in S$ *and* $\varepsilon \in A$.

Lemma 3.3. *Let* $\langle F,A\rangle$ *be a fuzzy soft set over an ordered semigroup* $(S,\cdot,\leq)$ *which satisfies condition* (F1a). *Then* (F3a) *holds if and only if one of the following conditions holds:*

(F3b) $x_r \in_\gamma F(\varepsilon)$ *and* $y_s \in_\gamma F(\varepsilon)$ *imply* $(xy)_{\min\{r,s\}} \in_\gamma \vee q_\delta F(\varepsilon)$ *for all* $x,y \in S, \varepsilon \in A$ *and* $r,s \in (\gamma,1]$;

(F3c) $\max\{F(\varepsilon)(xy),\gamma\} \geq \min\{F(\varepsilon)(x),F(\varepsilon)(y),\delta\}$ *for all* $x,y \in S$ *and* $\varepsilon \in A$.

Proof. The proof is similar to that of Lemma 3.2. ∎

Lemma 3.4. *Let* $\langle F,A\rangle$ *be a fuzzy soft set over an ordered semigroup* $(S,\cdot,\leq)$ *which satisfies condition* (F1a). *Then* (F4a) *holds if and only if one of the following conditions holds:*

(F4b) $x_r \in_\gamma F(\varepsilon)$ *and* $z_s \in_\gamma F(\varepsilon)$ *imply* $(xyz)_{\min\{r,s\}} \in_\gamma \vee q_\delta F(\varepsilon)$ *for all* $x,y,z \in S, \varepsilon \in A$ *and* $r,s \in (\gamma,1]$;

(F4c) $\max\{F(\varepsilon)(xyz),\gamma\} \geq \min\{F(\varepsilon)(x),F(\varepsilon)(z),\delta\}$ *for all* $x,y \in S$ *and* $\varepsilon \in A$.

Proof. (F4a)⇒(F4b) For any $x,y,z \in S, \varepsilon \in A$ and $r,s \in (\gamma,1]$, if $x_r \in_\gamma F(\varepsilon)$ and $z_s \in_\gamma F(\varepsilon)$, then $F(\varepsilon)(x) > r > \gamma$ and $F(\varepsilon)(z) > s > \gamma$. By (F4a) and Lemma 2.1, we have

$$\begin{aligned}
&\max\{F(\varepsilon)(xyz),\gamma\} \geq \min\{(F(\varepsilon)\circ\chi^\delta_{\gamma S}\circ F(\varepsilon))(xyz),\delta\}\\
&= \min\left\{\sup_{(a,b)\in A_{xyz}} \min\{(F(\varepsilon)\circ\chi^\delta_{\gamma S})(a),F(\varepsilon)(b)\},\delta\right\}\\
&\geq \min\{(F(\varepsilon)\circ\chi^\delta_{\gamma S})(xy),F(\varepsilon)(z),\delta\} \quad (\text{since } xyz \leq (xy)z)\\
&= \min\left\{\sup_{(a,b)\in A_{xy}} \min\{F(\varepsilon)(a),\chi^\delta_{\gamma S}(b)\},F(\varepsilon)(z),\delta\right\}\\
&\geq \min\{F(\varepsilon)(x),\chi^\delta_{\gamma S}(y),F(\varepsilon)(z),\delta\}\\
&= \min\{F(\varepsilon)(x),F(\varepsilon)(z),\delta\}\\
&\geq \min\{r,s,\delta\} > \gamma.
\end{aligned}$$

It follows that $F(\varepsilon)(xyz) \geq \min\{r,s,\delta\}$. We consider the following cases.

Case 1: $\min\{r,s\} \leq \delta$. Then $F(\varepsilon)(xyz) \geq \min\{r,s\}$, that is, $(xyz)_{\min\{r,s\}} \in_\gamma F(\varepsilon)$.

Case 2: $\min\{r,s\} > \delta$. Then $F(\varepsilon)(xyz) + \min\{r,s\} > 2\delta$, that is, $(xyz)_{\min\{r,s\}} q_\delta F(\varepsilon)$.

Thus, in any case, $(xyz)_{\min\{r,s\}} \in_\gamma \vee q_\delta F(\varepsilon)$. Therefore, condition (F4b) is valid.

(F4b)⇒(F4c) Let $x,y,z \in S$ and $\varepsilon \in A$. If $\max\{F(\varepsilon)(xyz),\gamma\} < r = \min\{F(\varepsilon)(x),F(\varepsilon)(z),\delta\}$, then $x_r, z_r \in_\gamma F(\varepsilon)$ but $(xyz)_r \overline{\in_\gamma \vee q_\delta} F(\varepsilon)$, which contradicts condition (F4b) and so condition (F4c) holds.

(F4c)⇒(F4a) If $\langle F,A\rangle \odot \Sigma(S,A) \odot \langle F,A\rangle \overline{\subset_{(\gamma,\delta)}} \langle F,A\rangle$, then there exist $\varepsilon \in A$ and $x_r \in_\gamma F(\varepsilon)\circ\chi^\delta_{\gamma S}\circ F(\varepsilon)$ such that $x_r \overline{\in_\gamma \vee q_\delta} F(\varepsilon)$. Hence $F(\varepsilon)(x) < r$ and $F(\varepsilon)(x) + r \leq 2\delta$, which gives $F(\varepsilon)(x) < \delta$. For $a,b,c \in S$ such that $(a,b) \in A_x, (c,d) \in A_a$ $x \leq cdb$, by conditions (F2c) and (F4c), we have

$$\delta > \max\{F(\varepsilon)(x),\gamma\} \geq \max\{\min\{F(\varepsilon)(cdb),\delta\},\gamma\} = \min\{\max\{F(\varepsilon)(cdb),\gamma\},\delta\}$$

$$\geq \min\{\min\{F(\varepsilon)(c), F(\varepsilon)(b), \delta\}, \delta\}$$
$$= \min\{F(\varepsilon)(c), F(\varepsilon)(b), \delta\}.$$

It follows that $\max\{F(\varepsilon)(x),\gamma\} \geq \min\{F(\varepsilon)(c), F(\varepsilon)(b)\}$. Thus we have

$$\begin{aligned} r \leq (F(\varepsilon)\circ\chi^{\delta}_{\gamma S}\circ F(\varepsilon))(x) &= \sup_{(a,b)\in A_x} \min\{(F(\varepsilon)\circ\chi^{\delta}_{\gamma S})(a), F(\varepsilon)(b)\} \\ &= \sup_{(a,b)\in A_x} \min\left\{ \sup_{(c,d)\in A_a} \min\{F(\varepsilon)(c), \chi^{\delta}_{\gamma S}(d)\}, F(\varepsilon)(b)\right\} \\ &\leq \sup_{(a,b)\in A_x} \min\left\{ \sup_{(c,d)\in A_a} F(\varepsilon)(c), F(\varepsilon)(b)\right\} \\ &= \sup_{(a,b)\in A_x,(c,d)\in A_a} \min\{F(\varepsilon)(c), F(\varepsilon)(b)\} \\ &\leq \sup_{(a,b)\in A_x,(c,d)\in A_a} \max\{F(\varepsilon)(x),\gamma\} = \max\{F(\varepsilon)(x),\gamma\}, \end{aligned}$$

a contradiction. Therefore condition (F4a) is satisfied. ∎

For any fuzzy soft set $\langle F,A\rangle$ over an ordered semigroup $(S,\cdot,\leq)$, $\varepsilon\in A$ and $r\in(\gamma,1]$, denote $F(\varepsilon)_r = \{x\in S | x_r \in_\gamma F(\varepsilon)\}$, $\langle F(\varepsilon)\rangle_r = \{x\in S| x_r q_\delta F(\varepsilon)\}$ and $[F(\varepsilon)]_r = \{x\in S | x_r \in_\gamma \vee q_\delta F(\varepsilon)\}$. The next theorem presents the relationships between $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy soft left ideals (resp., right ideals, bi-ideals, quasi-ideals) and crisp left ideals (resp., right ideals, bi-ideals, quasi-ideals) of an ordered semigroup.

Theorem 3.3. *Let $(S,\cdot,\leq)$ be an ordered semigroup and $\langle F,A\rangle$ a fuzzy soft set over S. Then:*

(1) *$\langle F,A\rangle$ is an $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy soft left ideal (resp., right ideal, bi-ideal, quasi-ideal) over S if and only if non-empty subset $F(\varepsilon)_r$ is a left ideal (resp., right ideal, bi-ideal, quasi-ideal) of S for all $\varepsilon\in A$ and $r\in(\gamma,\delta]$.*

(2) *If $2\delta = 1+\gamma$, then $\langle F,A\rangle$ is an $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy soft left ideal (resp., right ideal, bi-ideal, quasi-ideal) over S if and only if non-empty subset $\langle F(\varepsilon)\rangle_r$ is a left ideal (resp., right ideal, bi-ideal, quasi-ideal) of S for all $\varepsilon\in A$ and $r\in(\delta,1]$.*

(3) *$\langle F,A\rangle$ is an $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy soft left ideal (resp., right ideal, bi-ideal, quasi-ideal) over S if and only if non-empty subset $[F(\varepsilon)]_r$ is a left ideal (resp., right ideal, bi-ideal, quasi-ideal) of S for all $\varepsilon\in A$ and $r\in(\gamma,\min\{2\delta-\gamma,1\}]$.*

Proof. We only prove (2) and (3). (1) can be easily proved.

(2) Assume that $2\delta = 1+\gamma$. Let $\langle F,A\rangle$ be an $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy soft left ideal over S and assume that $\langle F(\varepsilon)\rangle_r \neq \emptyset$ for some $\varepsilon\in A$ and $r\in(\delta,1]$. Let $x\in S$ and $y\in\langle F(\varepsilon)\rangle_r$. Then $y_r q_\delta F(\varepsilon)$, that is, $F(\varepsilon)(y)+r > 2\delta$. Since $\langle F,A\rangle$ is an (γ,δ)-fuzzy soft left ideal over S, we have $\max\{F(\varepsilon)(xy),\gamma\} \geq \min\{F(\varepsilon)(y),\delta\}$. Hence, by $r>\delta$,

$$\begin{aligned} \max\{F(\varepsilon)(xy)+r, \gamma+r\} &= \max\{F(\varepsilon)(xy),\gamma\}+r \geq \min\{F(\varepsilon)(y),\delta\}+r \\ &= \min\{F(\varepsilon)(y)+r, \delta+r\} > 2\delta. \end{aligned}$$

From $r\leq 1 = 2\delta-\gamma$, that is, $r+\gamma\leq 2\delta$, we have $F(\varepsilon)(xy)+r > 2\delta$ and so $xy\in\langle F(\varepsilon)\rangle_r$. Similarly, we can show that $y\leq x$ for $y\in S$ and $x\in\langle F(\varepsilon)\rangle_r$ implies $y\in\langle F(\varepsilon)\rangle_r$. Therefore, $\langle F(\varepsilon)\rangle_r$ is a left ideal of S.

Conversely, assume that the given conditions hold. If there exist $\varepsilon\in A$ and $x,y\in S$ such that $\max\{F(\varepsilon)(xy),\gamma\} < \min\{F(\varepsilon)(y),\delta\}$. Take $r = 2\delta - \max\{F(\varepsilon)(xy),\gamma\}$. Then

$r \in (\delta, 1]$, $F(\varepsilon)(xy) \leq 2\delta - r$, $F(\varepsilon)(y) > \max\{G(\varepsilon)(xy), \gamma\} = 2\delta - r$, that is, $y \in \langle F(\varepsilon)\rangle_r$ but $xy \notin \langle F(\varepsilon)\rangle_r$, a contradiction. Hence $\langle F,A\rangle$ satisfies condition (F2c). Similarly we may show that $\langle F,A\rangle$ satisfies condition (F1c). Therefore, $\langle F,A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal over S.

(3) Let $\langle F,A\rangle$ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal over S and assume that $[F(\varepsilon)]_r \neq \emptyset$ for some $\varepsilon \in A$ and $r \in (\gamma, \min\{2\delta - \gamma, 1\}]$. Let $x \in S$ and $y \in [F(\varepsilon)]_r$. Then $y_r \in_\gamma \vee q_\delta F(\varepsilon)$, that is, $F(\varepsilon)(y) \geq r > \gamma$ or $F(\varepsilon)(y) > 2\delta - r \geq 2\delta - (2\delta - \gamma) = \gamma$. Since $\langle F,A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal over S, we have $\max\{F(\varepsilon)(xy), \gamma\} \geq \min\{F(\varepsilon)(y), \delta\}$ and so $F(\varepsilon)(xy) \geq \min\{F(\varepsilon)(y), \delta\}$ since $\gamma < \min\{F(\varepsilon)(y), \delta\}$ in any case. Now we consider the following cases.

Case 1: $r \in (\gamma, \delta]$. Then $2\delta - r \geq \delta \geq r$. It follows from $F(\varepsilon)(y) \geq r$ or $F(\varepsilon)(y) > 2\delta - r$ that $F(\varepsilon)(xy) \geq \min\{F(\varepsilon)(y), \delta\} \geq r$. Hence $(x+y)_r \in_\gamma F(\varepsilon)$.

Case 2: $r \in (\delta, \min\{2\delta - \gamma, 1\}]$. Then $r > \delta > 2\delta - r$. It follows from $F(\varepsilon)(y) \geq r$ or $F(\varepsilon)(y) > 2\delta - r$ that $F(\varepsilon)(xy) \geq \min\{F(\varepsilon)(y), \delta\} > 2\delta - r$. Hence $(xy)_r q_\delta F(\varepsilon)$.

Thus, in any case, $(xy)_r \in_\gamma \vee q_\delta F(\varepsilon)$, that is, $xy \in [F(\varepsilon)]_r$. Similarly, we can show that $y \leq x$ for $y \in S$ and $x \in [F(\varepsilon)]_r$ implies $y \in [F(\varepsilon)]_r$. Therefore, $[F(\varepsilon)]_r$ is a left ideal of S.

Conversely, assume that the given conditions hold. If there exist $\varepsilon \in A$ and $x, y \in S$ such that $\max\{F(\varepsilon)(xy), \gamma\} < r = \min\{F(\varepsilon)(y), \delta\}$. Then $y_r \in_\gamma F(\varepsilon)$ but $(xy)_r \overline{\in_\gamma \vee q_\delta} F(\varepsilon)$, that is, $y \in [F(\varepsilon)]_r$ but $xy \notin [F(\varepsilon)]_r$, a contradiction. Hence $\langle F,A\rangle$ satisfies condition (F1c). Similarly we may show that $\langle F,A\rangle$ satisfies condition (F2c). Therefore, $\langle F,A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal over S.

The case for $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft right ideals (bi-ideals, quasi-ideals) over S can be similarly proved. ∎

As a direct consequence of Theorem 3.3, we have the following results.

Corollary 3.1. *Let $(S, \cdot, \leq)$ be an ordered semigroup and $\gamma, \gamma', \delta, \delta' \in [0,1]$ such that $\gamma < \delta$, $\gamma' < \delta'$, $\gamma < \gamma'$ and $\delta' < \delta$. Then any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal (resp., right ideal, bi-ideal, quasi-ideal) of S is an $(\in_{\gamma'}, \in_{\gamma'} \vee q_{\delta'})$-fuzzy soft left ideal (resp., right ideal, bi-ideal, quasi-ideal) over S.*

Corollary 3.2. *Let $(S, \cdot, \leq)$ be an ordered semigroup and $P \subseteq S$. Then P is a left ideal (resp., right ideal, bi-ideal, quasi-ideal) of S if and only if $\Sigma(P,A)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal (resp., right ideal, bi-ideal, quasi-ideal) over S for any $A \subseteq E$.*

Theorem 3.4. *Let $(S, \cdot, \leq)$ be an ordered semigroup, $\langle F,A\rangle$ and $\langle G,B\rangle$ fuzzy soft sets over S. If $\langle F,A\rangle$ and $\langle G,B\rangle$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideals (right ideals, bi-ideals, quasi-ideals) over S, then so are $\langle F,A\rangle \cap \langle G,B\rangle$ and $\langle F,A\rangle \widetilde{\cap} \langle G,B\rangle$. Moreover, if $\langle F,A\rangle$ and $\langle G,B\rangle$ are an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft right ideal over S and an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal over S, then $\langle F,A\rangle \odot_h \langle G,B\rangle \subset_{(\gamma,\delta)} \langle F,A\rangle \widetilde{\cap} \langle G,B\rangle$.*

Proof. It is straightforward. ∎

Theorem 3.5. *Let $(S, \cdot, \leq)$ be an ordered semigroup, $\langle F,A\rangle$ and $\langle G,B\rangle$ fuzzy soft sets over S. If $\langle F,A\rangle$ and $\langle G,B\rangle$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left (right) ideals over S, then so are $\langle F,A\rangle \cup \langle G,B\rangle$ and $\langle F,A\rangle \widetilde{\cup} \langle G,B\rangle$.*

Proof. It is straightforward. ∎

Denote by $\mathscr{FSI}(S,E)$ the set of all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left (right) ideals over S. From Theorems 3.4 and 3.5, we have the following results.

Theorem 3.6. *$(\mathscr{FSI}(S,E),\widetilde{\cup},\cap)$ is a complete distributive lattice under the ordering relation "$\subset_{(\gamma,\delta)}$".*

Proof. For any $\langle F,A\rangle,\langle G,B\rangle\in\mathscr{FSI}(S,E)$, by Theorems 3.4 and 3.5, $\langle F,A\rangle\widetilde{\cup}\langle G,B\rangle\in\mathscr{FSI}(S,E)$ and $\langle F,A\rangle\cap\langle G,B\rangle\in\mathscr{FSI}(S,E)$. It is obvious that $\langle F,A\rangle\widetilde{\cup}\langle G,B\rangle$ and $\langle F,A\rangle\cap\langle G,B\rangle$ are the least upper bound and the greatest lower bound of $\langle F,A\rangle$ and $\langle G,B\rangle$, respectively. There is no difficulty in replacing $\{\langle F,A\rangle,\langle G,B\rangle\}$ with an arbitrary family of $\mathscr{FSI}(S,E)$ and so $(\mathscr{FSI}(S,E),\widetilde{\cup},\cap)$ is a complete lattice. Now we prove that the following distributive law

$$\langle F,A\rangle\cap(\langle G,B\rangle\widetilde{\cup}\langle H,C\rangle)=(\langle F,A\rangle\cap\langle G,B\rangle)\widetilde{\cup}(\langle F,A\rangle\cap\langle H,C\rangle)$$

holds for all $\langle F,A\rangle,\langle G,B\rangle,\langle H,C\rangle\in\mathscr{FSI}(S,E)$. Suppose that

$\langle F,A\rangle\cap(\langle G,B\rangle\widetilde{\cup}\langle H,C\rangle)=\langle I,A\cap(B\cup C)\rangle$,

$(\langle F,A\rangle\cap\langle G,B\rangle)\widetilde{\cup}(\langle F,A\rangle\cap\langle H,C\rangle)=\langle J,(A\cap B)\cup(A\cap C)\rangle=\langle J,A\cap(B\cup C)\rangle$.

Now for any $\varepsilon\in A\cap(B\cup C)$, it follows that $\varepsilon\in A$ and $\varepsilon\in B\cup C$. We consider the following cases.

Case 1: $\varepsilon\in A,\varepsilon\notin B$ and $\varepsilon\in C$. Then $I(\varepsilon)=F(\varepsilon)\cap H(\varepsilon)=J(\varepsilon)$.

Case 2: $\varepsilon\in A,\varepsilon\in B$ and $\varepsilon\notin C$. Then $I(\varepsilon)=F(\varepsilon)\cap G(\varepsilon)=J(\varepsilon)$.

Case 3: $\varepsilon\in A,\varepsilon\in B$ and $\varepsilon\in C$. Then $I(\varepsilon)=F(\varepsilon)\cap(G(\varepsilon)\cup H(\varepsilon))=(F(\varepsilon)\cap G(\varepsilon))\cup(F(\varepsilon)\cap H(\varepsilon))=J(\varepsilon)$.

Therefore, I and J are the same operators, and so $\langle F,A\rangle\cap(\langle G,B\rangle\widetilde{\cup}\langle H,C\rangle)=(\langle F,A\rangle\cap\langle G,B\rangle)\widetilde{\cup}(\langle F,A\rangle\cap\langle H,C\rangle)$. It follows that $\langle F,A\rangle\cap(\langle G,B\rangle\widetilde{\cup}\langle H,C\rangle)=_{(\gamma,\delta)}(\langle F,A\rangle\cap\langle G,B\rangle)\widetilde{\cup}(\langle F,A\rangle\cap\langle H,C\rangle)$. This completes the proof. ∎

Theorem 3.7. *$(\mathscr{FSI}(S,E),\cup,\widetilde{\cap})$ is a complete distributive lattice under the ordering relation "$\subset'_{(\gamma,\delta)}$", where for any $\langle F,A\rangle,\langle G,B\rangle\in\mathscr{FSI}(S,E)$, $\langle F,A\rangle\subset'_{(\gamma,\delta)}\langle G,B\rangle$ if and only if $B\subseteq A$ and $F(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}G(\varepsilon)$ for any $\varepsilon\in B$.*

Proof. The proof is similar to that of Theorem 3.6. ∎

Now we consider the fuzzy soft sets over a definite parameter set. Let $A\subseteq E$, $(S,\cdot,\leq)$ be an ordered semigroup and

$$\mathscr{FS}_A(S)=\{\langle F,A\rangle\in\mathscr{FSI}(S,E)|F:A\to\mathscr{F}(S)\}$$

the set of fuzzy soft sets over S and the parameter set A. It is trivial to verify that $\langle F,A\rangle\widetilde{\cup}\langle G,A\rangle$, $\langle F,A\rangle\widetilde{\cap}\langle G,A\rangle,\langle F,A\rangle\cup\langle G,A\rangle,\langle F,A\rangle\cap\langle G,A\rangle\in\mathscr{FS}_A(S)$ for all $\langle F,A\rangle,\langle G,A\rangle\in\mathscr{FS}_A(S)$.

Corollary 3.3. *$(\mathscr{FS}_A(S),\widetilde{\cup},\cap)$ and $(\mathscr{FS}_A(S),\cup,\widetilde{\cap})$ are sublattices of $(\mathscr{FSI}(S,E),\widetilde{\cup},\cap)$ and $(\mathscr{FSI}(S,E),\cup,\widetilde{\cap})$, respectively.*

Theorem 3.8. *Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left (right) ideals over an ordered semigroup $(S,\cdot,\leq)$. Then so is $\langle F,A\rangle\odot\langle G,B\rangle$.*

Proof. Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideals over S. Then for any $\varepsilon\in A\cup B$, we consider the following cases.

Case 1: $\varepsilon\in A-B$. Then $(F\circ G)(\varepsilon)=F(\varepsilon)$. It follows that $(F\circ G)(\varepsilon)$ satisfies conditions (F1c) and (F2c) since $\langle F,A\rangle$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideal over S.

Case 2: $\varepsilon\in B-A$. Then $(F\circ G)(\varepsilon)=G(\varepsilon)$. It follows that $(F\circ G)(\varepsilon)$ satisfies conditions (F1c) and (F2c) since $\langle G,B\rangle$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideal over S.

Case 3: $\varepsilon\in A\cap B$. Then $(F\circ G)(\varepsilon)=F(\varepsilon)\circ G(\varepsilon)$. Now we show that $F(\varepsilon)\circ G(\varepsilon)$ satisfies conditions (F1c) and (F2c).

(1) For any $x,y \in S$, we have

$$\begin{aligned}
\min\{(F(\varepsilon)\circ G(\varepsilon))(y),\delta\} &= \min\left\{\sup_{(a,b)\in A_y}\min\{F(\varepsilon)(a),G(\varepsilon)(b)\},\delta\right\}\\
&= \sup_{(a,b)\in A_y}\min\{\min\{F(\varepsilon)(a),\delta\},\min\{G(\varepsilon)(b),\delta\}\}\\
&\leq \sup_{(xa,b)\in A_{xy}}\min\{\max\{F(\varepsilon)(xa),\gamma\},\max\{G(\varepsilon)(b),\gamma\}\}\\
&= \sup_{(xa,b)\in A_{xy}}\max\{\min\{F(\varepsilon)(xa),G(\varepsilon)(b)\},\gamma\}\\
&\leq \sup_{(c,b)\in A_{xy}}\max\{\min\{F(\varepsilon)(c),G(\varepsilon)(b)\},\gamma\}\\
&= \max\{(F(\varepsilon)\circ G(\varepsilon))(xy),\gamma\}.
\end{aligned}$$

(2) Let $x,y \in S$ be such that $y \leq x$. Then it is easy to see that $(F(\varepsilon)\circ G(\varepsilon))(y) \geq (F(\varepsilon)\circ G(\varepsilon))(x)$. Hence $\max\{(F(\varepsilon)\circ G(\varepsilon))(y),\gamma\} \geq \min\{(F(\varepsilon)\circ G(\varepsilon))(x),\delta\}$.

Summing up the above statements, $F(\varepsilon)\circ G(\varepsilon)$ satisfies conditions (F1c) and (F2c).

Therefore, $\langle F,A\rangle \odot \langle G,B\rangle$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideal over S.

The case for $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft right ideals over S can be similarly proved. ∎

Let $(S,\cdot,\leq)$ be an ordered semigroup with an identity e. Denote by $\mathscr{F}\mathscr{S}\mathscr{I}(S,E)$ the set of all $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left (right) ideals over S such that $F(\varepsilon)(e) \geq \delta$ for all $\langle F,A\rangle \in \mathscr{F}\mathscr{S}\mathscr{I}(S,E)$. Then we have the following result.

Theorem 3.9. *Let $(S,\cdot,\leq)$ be an ordered semigroup with an identity e. Then $(\mathscr{F}\mathscr{S}\mathscr{I}(S,E), \odot,\cap)$ is a complete lattice under the relation "$\subset_{(\gamma,\delta)}$".*

Proof. Let $\langle F,A\rangle,\langle G,B\rangle \in \mathscr{F}\mathscr{S}\mathscr{I}(S,E)$. It follows from Theorems 3.4 and 3.8 that $\langle F,A\rangle\cap\langle G,B\rangle \in \mathscr{F}\mathscr{S}\mathscr{I}(S,E)$ and $\langle F,A\rangle\odot\langle G,B\rangle \in \mathscr{F}\mathscr{S}\mathscr{I}(S,E)$. It is clear that $\langle F,A\rangle\cap\langle G,B\rangle$ is the greatest lower bound of $\langle F,A\rangle$ and $\langle G,B\rangle$. We now show that $\langle F,A\rangle\odot\langle G,B\rangle$ is the least upper bound of $\langle F,A\rangle$ and $\langle G,B\rangle$. For any $\varepsilon \in A$ and $x \in S$, we consider the following cases.

Case 1: $\varepsilon \in A-B$. Then $(F\circ G)(\varepsilon) = F(\varepsilon)$.

Case 2: $\varepsilon \in A\cap B$. Then $(F\circ G)(\varepsilon) = F(\varepsilon)\circ G(\varepsilon)$. Since $G(\varepsilon)(e) \geq \delta$, we have

$$\begin{aligned}
&\max\{(F(\varepsilon)\circ G(\varepsilon))(x),\gamma\} = \max\left\{\sup_{(a,b)\in A_x}\min\{F(\varepsilon)(a),G(\varepsilon)(b)\},\gamma\right\}\\
&\geq \max\{\min\{F(\varepsilon)(x),G(\varepsilon)(e)\},\gamma\} = \min\{\max\{F(\varepsilon)(x),\gamma\},\max\{G(\varepsilon)(e),\gamma\}\}\\
&\geq \min\{F(\varepsilon)(x),\delta\},
\end{aligned}$$

hence $F(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} F(\varepsilon)\circ G(\varepsilon)$. Therefore, $\langle F,A\rangle \subset_{(\gamma,\delta)} \langle F,A\rangle\odot\langle G,B\rangle$. Similarly, we have $\langle G,B\rangle \subset_{(\gamma,\delta)} \langle F,A\rangle\odot\langle G,B\rangle$. Now, let $\langle H,C\rangle \in \mathscr{F}\mathscr{S}\mathscr{I}(S,E)$ be such that $\langle F,A\rangle \subset_{(\gamma,\delta)} \langle H,C\rangle$ and $\langle G,B\rangle \subset_{(\gamma,\delta)} \langle H,C\rangle$. Then, we have $\langle F,A\rangle\odot\langle G,B\rangle \subset_{(\gamma,\delta)} \langle H,C\rangle\odot\langle H,C\rangle \subset_{(\gamma,\delta)} \langle H,C\rangle$. Hence $\langle F,A\rangle\vee\langle G,B\rangle = \langle F,A\rangle\odot\langle G,B\rangle$. There is no difficulty in replacing $\{\langle F,A\rangle, \langle G,B\rangle\}$ with an arbitrary family of $\mathscr{F}\mathscr{S}\mathscr{I}(S,E)$ and so $(\mathscr{F}\mathscr{S}\mathscr{I}(S,E),\odot,\cap)$ is a complete lattice under the relation "$\subset_{(\gamma,\delta)}$". ∎

The following theorems present the relationships among $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left (right) ideals, $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft bi-ideals and $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft quasi-ideals.

Theorem 3.10. *Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft right ideal and an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideal over an ordered semigroup $(S,\cdot,\leq)$, respectively. Then both $\langle F,A\rangle\cap\langle G,B\rangle$ and $\langle F,A\rangle\tilde{\cap}\langle G,B\rangle$ are $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft quasi-ideals over S.*

Proof. It is straightforward. ▮

Theorem 3.11. *Let $(S,\cdot,\leq)$ be an ordered semigroup. Then:*

(1) *Every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left (right) ideal over S is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft quasi-ideal over S.*
(2) *Every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft quasi-ideal over S is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft bi-ideal over S.*

Proof. The proof of (1) is straightforward. We show (2). Let $\langle F,A\rangle$ be any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft quasi-ideal over S. To show that $\langle F,A\rangle$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft bi-ideal over S, it is sufficient to show $\langle F,A\rangle\odot\langle F,A\rangle\subset_{(\gamma,\delta)}\langle F,A\rangle$ and $\langle F,A\rangle\odot\Sigma(S,A)\odot\langle F,A\rangle\subset_{(\gamma,\delta)}\langle F,A\rangle$. In fact, since $\langle F,A\rangle$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft quasi-ideal over S, by Lemma 2.5, we have

$\langle F,A\rangle\odot\langle F,A\rangle\subset_{(\gamma,\delta)}\Sigma(S,A)\odot\langle F,A\rangle$, $\langle F,A\rangle\odot\langle F,A\rangle\subset_{(\gamma,\delta)}\langle F,A\rangle\odot\Sigma(S,A)$,
$\langle F,A\rangle\odot\Sigma(S,A)\odot\langle F,A\rangle\subset_{(\gamma,\delta)}\Sigma(S,A)\odot\langle F,A\rangle$ and $\langle F,A\rangle\odot\Sigma(S,A)\odot\langle F,A\rangle\subset_{(\gamma,\delta)}\langle F,A\rangle\odot\Sigma(S,A)$.

Hence $\langle F,A\rangle\odot\langle F,A\rangle\subset_{(\gamma,\delta)}\Sigma(S,A)\odot\langle F,A\rangle\tilde{\cap}\langle F,A\rangle\odot\Sigma(S,A)\subset_{(\gamma,\delta)}\langle F,A\rangle$
and
$\langle F,A\rangle\odot\Sigma(S,A)\odot\langle F,A\rangle\subset_{(\gamma,\delta)}\Sigma(S,A)\odot\langle F,A\rangle\tilde{\cap}\langle F,A\rangle\odot\Sigma(S,A)\subset_{(\gamma,\delta)}\langle F,A\rangle$.
This completes the proof. ▮

Note that the converse of Theorem 3.21(1)–(2) does not hold in general as shown in the following examples.

Example 3.1. *Define on the set $S=\{a,b,c,d\}$ an ordering relation by $a\leq b, a\leq c, a\leq d$ and a multiplication operation "$\cdot$" by the table:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	c	a
c	a	a	b	a
d	a	d	b	b

Then $(S,\cdot,\leq)$ is an ordering semigroup. Let $E=\{2,3,4\}$. Define a fuzzy soft set $\langle F,A\rangle$ over S as follows.

$$F(\varepsilon)(x)=\begin{cases}\frac{1}{\varepsilon} & \text{if } x\in\{a,b\},\\ \frac{1}{5} & \text{otherwise.}\end{cases}$$

Then $\langle F,A\rangle$ is an $(\in_{0.2},\in_{0.2}\vee q_{0.5})$-fuzzy soft quasi-ideal and it is not an $(\in_{0.2},\in_{0.2}\vee q_{0.5})$-fuzzy soft left (right) ideal over S.

Example 3.2. *Define on the set $S=\{a,b,c,d\}$ an ordered relation by $a\leq c, a\leq b\leq d$ and a multiplication operation "$\cdot$" by the table:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	c

Then $(S,\cdot,\leq)$ is an ordered semigroup. Let $E=(0.2,0.6]$. Define a fuzzy soft set $\langle F,A\rangle$ over S as follows.

$$F(\varepsilon)(x)=\begin{cases}\varepsilon & \text{if } x\in\{a,c\},\\ 0.2 & \text{otherwise}.\end{cases}$$

Then $\langle F,A\rangle$ is an $(\in_{0.2},\in_{0.2}\vee q_{0.6})$-fuzzy soft bi-ideal and it is not an $(\in_{0.2},\in_{0.2}\vee q_{0.6})$-fuzzy soft quasi-ideal over S.

4. Left quasi-regular and intra-regular ordered semigroups

In this section, we will provide the concept of left (right) quasi-regular ordered semigroups and concentrate our study on the characterization of left quasi-regular ordered semigroups and ordered semigroups that are left quasi-regular and intra-regular in terms of $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left (right) ideals, $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy bi-ideals and $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy quasi-ideals.

We start by introducing the following definition.

Definition 4.1. [9] An ordered semigroup $(S,\cdot,\leq)$ is called *intra-regular* if for every $x\in S$ there exists $y,z\in S$ such that $x\leq yx^2z$. Equivalent definitions: (1) $x\in(Sx^2S]$ $\forall x\in S$, (2) $A\subseteq(SA^2S]$ $\forall A\subseteq S$.

Lemma 4.1. [9] *Let $(S,\cdot,\leq)$ be an ordered semigroup. Then S is intra-regular if and only if $L\cap R\subseteq(LR]$ for every left ideal L and every right ideal R of S.*

Theorem 4.1. *Let $(S,\cdot,\leq)$ be an ordered semigroup. Then S is intra-regular if and only if $\langle F,A\rangle\widetilde{\cap}\langle G,B\rangle\subset_{(\gamma,\delta)}\langle F,A\rangle\odot_h\langle G,B\rangle$ for any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideal $\langle F,A\rangle$ and any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft right ideal $\langle G,B\rangle$ over S.*

Proof. Let S be intra-regular, $\langle F,A\rangle$ any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideal and $\langle G,B\rangle$ any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft right ideal over S, respectively. Now let x be any element of S, $\varepsilon\in A\cup B$ and $\langle F,A\rangle\widetilde{\cap}\langle G,B\rangle=\langle H,A\cup B\rangle$. We consider the following cases.

Case 1: $\varepsilon\in A-B$. Then $H(\varepsilon)=F(\varepsilon)=(F\circ G)(\varepsilon)$.

Case 2: $\varepsilon\in B-A$. Then $H(\varepsilon)=G(\varepsilon)=(F\circ G)(\varepsilon)$.

Case 3: $\varepsilon\in A\cap B$. Then $H(\varepsilon)=F(\varepsilon)\cap G(\varepsilon)$ and $(F\circ G)(\varepsilon)=F(\varepsilon)\circ G(\varepsilon)$.

Now we show that $F(\varepsilon)\cap G(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}F(\varepsilon)\circ G(\varepsilon)$. Since S is intra-regular, there exist $y,z\in S$ such that $x\leq yx^2z$. Then we have

$$\begin{aligned}\max\{(F(\varepsilon)\circ G(\varepsilon))(x),\gamma\}&=\max\left\{\sup_{(a,b)\in A_x}\min\{F(\varepsilon)(a),G(\varepsilon)(b)\},\gamma\right\}\\&\geq\max\{\min\{F(\varepsilon)(yx),G(\varepsilon)(xz)\},\gamma\}\\&=\min\{\max\{F(\varepsilon)(yx),\gamma\},\max\{G(\varepsilon)(xz),\gamma\}\}\\&\geq\min\{\min\{F(\varepsilon)(x),\delta\},\min\{G(\varepsilon)(x),\delta\}\}=\min\{(F(\varepsilon)\cap G(\varepsilon)(x),\delta\}.\end{aligned}$$

It follows that $F(\varepsilon)\cap G(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}F(\varepsilon)\circ G(\varepsilon)$, that is, $H(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}(F\circ G)(\varepsilon)$.

Thus, in any case, we have $H(\varepsilon) \subseteq\vee q_{(\gamma,\delta)}(F\circ G)(\varepsilon)$ and so $\langle F,A\rangle\widetilde{\cap}\langle G,B\rangle \subset_{(\gamma,\delta)} \langle F,A\rangle \odot_h \langle G,B\rangle$.

Now, assume that the given condition holds, let L and R be any left ideal and any right ideal of S, respectively. Then by Corollary 3.2, $\Sigma(L,E)$ and $\Sigma(R,E)$ are an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal and an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft right ideal over S, respectively. Now, by the assumption, we have $\Sigma(L,E)\widetilde{\cap}\Sigma(R,E) \subset_{(\gamma,\delta)} \Sigma(L,E)\odot_h \Sigma(R,E)$. Hence by Lemma 2.3, we have

$$\chi^\delta_{\gamma(L\cap R)} =_{(\gamma,\delta)} \chi^\delta_{\gamma L} \cap \chi^\delta_{\gamma R} \subseteq\vee q_{(\gamma,\delta)} \chi^\delta_{\gamma L} \odot_h \chi^\delta_{\gamma R} =_{(\gamma,\delta)} \chi^\delta_{\gamma \overline{LR}}.$$

It follows from Lemma 2.3 that $L\cap R \subseteq \overline{LR}$. Therefore S is intra-regular by Lemma 4.1. ∎

Definition 4.2. An ordered semigroup $(S,\cdot,\leq)$ is called *left (*resp.*, right) quasi-regular* if every left (resp., right) ideal of S is idempotent, and is called *quasi-regular* if every left ideal and every right ideal of S is idempotent.

Lemma 4.2. *An ordered semigroup $(S,\cdot,\leq)$ is left quasi-regular if and only if one of the following conditions holds:*

(1) *There exist $y,z\in S$ such that $x\leq yxzx$ for all $x\in S$.*
(2) *$x\in (SxSx]$ for all $x\in S$.*
(3) *$A\subseteq (SASA]$ for all $A\subseteq S$.*
(4) *$I\cap L = (IL]$ for every ideal I and every left ideal L of S.*

Proof. Assume that S is left quasi-regular. Let x be any element of S. Then $(x\cup Sx]$, is the principal left ideal of S generated by x. By the assumption, we have

$$x \in (x\cup Sx] = (x\cup Sx](x\cup Sx] \subseteq ((x\cup Sx)(x\cup Sx)] = (x^2\cup xSx\cup Sx^2\cup SxSx].$$

We consider the following cases.

Case 1: $x\in (x^2]$. Then $x\leq x^2\leq x^4$.
Case 2: $x\in (xSx]$. Then there exists $y\in S$ such that $x\leq xyx\leq xyxyx$.
Case 3: $x\in (Sx^2]$. Then there exists $y\in S$ such that $x\leq yx^2\leq yxyx^2$.
Case 4: $x\in (SxSx]$. Then there exist $y,z\in S$ such that $x\leq yxzx$.

Thus, in any case, (1) is satisfied. It is clear that (1) ⇔ (2) ⇔ (3). Now assume that (3) holds. Let L be any left ideal of S, by assumption, we have $L\subseteq (SLSL]\subseteq (LL] = (L^2]$. The converse inclusion always holds, and so we have $L = (L^2]$. Therefore S is left quasi-regular.

Next we show (1)⇔(4). Assume that (1) holds. Let I and L be any ideal and any left ideal of S, respectively, and $x\in I\cap L$. Then there exist $y,z\in S$ such that $x\leq yxzx$. It follows that $yx\in I$ and $zx\in L$, and so $x\in (IL]$. Hence $I\cap L\subseteq (IL]$. On the other hand, it is clear $(IL]\subseteq (I]\cap(L] = I\cap L$. Hence $I\cap L = (IL]$ and so (4) holds. Now assume that (4) holds. Let x be any element of S. Then $(x\cup Sx\cup xS\cup SxS]$ is the principal ideal of S generated by x. By assumption, we have

$$\begin{aligned} x &\in (x\cup Sx\cup xS\cup SxS]\cap(x\cup Sx] = (x\cup Sx\cup xS\cup SxS](x\cup Sx] \\ &\subseteq ((x\cup Sx\cup xS\cup SxS)(x\cup Sx)] = (x^2\cup xSx\cup Sx^2\cup SxSx\cup xSx\cup xSSx\cup SxSx\cup SxSSx]. \end{aligned}$$

Analogous to the above proof, there exist $y,z\in S$ such that $x\leq yxzx$ so (1) holds. This completes the proof. ∎

Definition 4.3. *A fuzzy soft set $\langle F,A\rangle$ over an ordered semigroup $(S,\cdot,\leq)$ is said to be (γ,δ)-fuzzy idempotent* if $\langle F,A\rangle\odot\langle F,A\rangle \asymp_{(\gamma,\delta)} \langle F,A\rangle$.

Theorem 4.2. *An ordered semigroup $(S,\cdot,\leq)$ is left (resp., right) quasi-regular if and only if every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left (resp., right) ideal over S is (γ,δ)-fuzzy idempotent.*

Proof. Let S be a left quasi-regular ordered semigroup and $\langle F,A\rangle$ any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideal over S. Now let x be any element of S and $\varepsilon\in A$. Then, by Lemma 4.2, there exist $y,z\in S$ such that $x\leq yxzx$. Thus we have

$$\begin{aligned}\max\{(F(\varepsilon)\circ F(\varepsilon))(x),\gamma\} &= \max\left\{\sup_{(a,b)\in A_x}\min\{F(\varepsilon)(a),F(\varepsilon)(b)\},\gamma\right\}\\ &\geq \max\{\min\{F(\varepsilon)(yx),F(\varepsilon)(zx)\},\gamma\}\\ &= \min\{\max\{F(\varepsilon)(yx),\gamma\},\max\{F(\varepsilon)(zx),\gamma\}\}\\ &\geq \min\{\min\{F(\varepsilon)(x),\delta\},\min\{F(\varepsilon)(x),\delta\}\}\\ &= \min\{F(\varepsilon)(x),\delta\}.\end{aligned}$$

It follows that $F(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}F(\varepsilon)\circ F(\varepsilon)$. Hence $\langle F,A\rangle\subset_{(\gamma,\delta)}\langle F,A\rangle\odot\langle F,A\rangle$. On the other hand, since $\langle F,A\rangle$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideal over S, we have $\langle F,A\rangle\odot\langle F,A\rangle\subset_{(\gamma,\delta)}\Sigma(S,A)\odot\langle F,A\rangle\subset_{(\gamma,\delta)}\langle F,A\rangle$. Therefore, $\langle F,A\rangle\asymp_{(\gamma,\delta)}\langle F,A\rangle\odot\langle F,A\rangle$.

Conversely, let L be any left ideal of S. Then by Corollary 3.2, $\Sigma(L,E)$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideal over S. Now, by the assumption, we have $\Sigma(L,E)\odot_h\Sigma(L,E)\asymp_{(\gamma,\delta)}\Sigma(L,E)$. Hence by Lemma 2.3, we have

$$\chi^\delta_{\gamma(LL]}=_{(\gamma,\delta)}\chi^\delta_{\gamma L}\circ\chi^\delta_{\gamma L}=_{(\gamma,\delta)}\chi^\delta_{\gamma L}.$$

It follows from Lemma 2.3 that $L=(LL]$. Therefore S is left quasi-regular.

The case for the $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft right ideals over S can be similarly proved. ∎

Theorem 4.3. *Let $(S,\cdot,\leq)$ be an ordered semigroup. Then the following conditions are equivalent.*

(1) *S is left quasi-regular.*

(2) $\langle F,A\rangle\tilde{\cap}\langle G,B\rangle\asymp_{(\gamma,\delta)}\langle F,A\rangle\odot_h\langle G,B\rangle$ *for any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft ideal $\langle F,A\rangle$ and any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideal $\langle G,B\rangle$ over S.*

(3) $\langle F,A\rangle\tilde{\cap}\langle G,B\rangle\subset_{(\gamma,\delta)}\langle F,A\rangle\odot_h\langle G,B\rangle$ *for any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft ideal $\langle F,A\rangle$ and any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft bi-ideal $\langle G,B\rangle$ over S.*

(4) $\langle F,A\rangle\tilde{\cap}\langle G,B\rangle\subset_{(\gamma,\delta)}\langle F,A\rangle\odot_h\langle G,B\rangle$ *for any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft ideal $\langle F,A\rangle$ and any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft quasi-ideal $\langle G,B\rangle$ over S.*

Proof. Assume that (1) holds. Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft ideal and any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft bi-ideal over S, respectively. Now let x be any element of S, $\varepsilon\in A$ and $\langle F,A\rangle\tilde{\cap}\langle G,B\rangle=\langle H,A\cup B\rangle$. We consider the following cases.

Case 1: $\varepsilon\in A-B$. Then $H(\varepsilon)=F(\varepsilon)=(F\circ G)(\varepsilon)$.

Case 2: $\varepsilon\in B-A$. Then $H(\varepsilon)=G(\varepsilon)=(F\circ G)(\varepsilon)$.

Case 3: $\varepsilon\in A\cap B$. Then $H(\varepsilon)=F(\varepsilon)\cap G(\varepsilon)$ and $(F\circ G)(\varepsilon)=F(\varepsilon)\circ G(\varepsilon)$. Now we show that $F(\varepsilon)\cap G(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}F(\varepsilon)\circ G(\varepsilon)$. Since S is left quasi-regular, by Lemma 4.2, there exist $y,z\in S$ such that $x\leq yxzx\leq yxzyxzx$. Then we have

$$\begin{aligned}\max\{(F(\varepsilon)\circ G(\varepsilon))(x),\gamma\} &= \max\left\{\sup_{(a,b)\in A_x}\min\{F(\varepsilon)(a),G(\varepsilon)(b)\},\gamma\right\}\\ &\geq \max\{\min\{F(\varepsilon)(yxzy),G(\varepsilon)(xzx)\},\gamma\}\end{aligned}$$

$$= \min\{\max\{F(\varepsilon)(yxzy),\gamma\},\max\{G(\varepsilon)(xzx),\gamma\}\}$$
$$\geq \min\{\min\{F(\varepsilon)(x),\delta\},\min\{G(\varepsilon)(x),\delta\}\}$$
$$= \min\{(F(\varepsilon)\cap G(\varepsilon)(x),\delta\}.$$

It follows that $F(\varepsilon)\cap G(\varepsilon)\subseteq \vee q_{(\gamma,\delta)}F(\varepsilon)\circ G(\varepsilon)$, that is, $H(\varepsilon)\subseteq \vee q_{(\gamma,\delta)}(F\circ G)(\varepsilon)$. Thus, in any case, we have $H(\varepsilon)\subseteq \vee q_{(\gamma,\delta)}(F\circ G)(\varepsilon)$ and so $\langle F,A\rangle\widetilde{\cap}\langle G,B\rangle \subset_{(\gamma,\delta)} \langle F,A\rangle \odot_h \langle G,B\rangle$.

It is clear that (3) $\Rightarrow$ (4) $\Rightarrow$ (2) by Theorem 3.10. Now assume that (2) holds. Let I and L be any ideal and any left ideal of S, respectively. Then by Corollary 3.2, $\Sigma(I,E)$ and $\Sigma(L,E)$ are an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft ideal and an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideal over S, respectively. Now, by the assumption, we have $\Sigma(I,E)\widetilde{\cap}\Sigma(L,E)\subset_{(\gamma,\delta)}\Sigma(I,E)\odot_h\Sigma(L,E)$. Hence by Lemma 2.3, we have

$$\chi^\delta_{\gamma(I\cap L)} =_{(\gamma,\delta)} \chi^\delta_{\gamma I}\cap\chi^\delta_{\gamma L} =_{(\gamma,\delta)} \chi^\delta_{\gamma I}\odot_h\chi^\delta_{\gamma L} =_{(\gamma,\delta)} \chi^\delta_{\gamma\overline{IL}}.$$

It follows from Lemma 2.3 that $I\cap L=(IL]$. Therefore S is left quasi-regular by Lemma 4.2. ∎

Theorem 4.4. *Let $(S,\cdot,\leq)$ be an ordered semigroup. Then the following conditions are equivalent.*

(1) *S is left quasi-regular.*
(2) *$\langle F,A\rangle\widetilde{\cap}\langle G,A\rangle\widetilde{\cap}\langle H,B\rangle \subset_{(\gamma,\delta)} \langle F,A\rangle\odot\langle G,A\rangle\odot\langle H,B\rangle$ for any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft ideal $\langle F,A\rangle$, any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft right ideal $\langle G,A\rangle$ and any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft bi-ideal $\langle H,B\rangle$ over S.*
(3) *$\langle F,A\rangle\widetilde{\cap}\langle G,A\rangle\widetilde{\cap}\langle H,B\rangle \subset_{(\gamma,\delta)} \langle F,A\rangle\odot\langle G,A\rangle\odot\langle H,B\rangle$ for any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft ideal $\langle F,A\rangle$, any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft right ideal $\langle G,A\rangle$ and any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft quasi-ideal $\langle H,B\rangle$ over S.*

Proof. Assume that (1) holds. Let $\langle F,A\rangle$ be any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft ideal, $\langle G,A\rangle$ any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft right ideal and $\langle H,B\rangle$ any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft bi-ideal over S. Now let x be any element of S, $\langle F,A\rangle\widetilde{\cap}\langle G,A\rangle\widetilde{\cap}\langle H,B\rangle=\langle K_1,A\cup B\rangle$ and $\langle F,A\rangle\odot\langle G,A\rangle\odot\langle H,B\rangle=\langle K_2,A\cup B\rangle$. For any $\varepsilon\in A\cup B$. We consider the following cases.

Case 1: $\varepsilon\in A-B$. Then $K_1(\varepsilon)=F(\varepsilon)\cap G(\varepsilon)$ and $K_2(\varepsilon)=(F\circ G)(\varepsilon)$. Analogous to the proof Theorem 4.3, we have $K_1(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}K_2(\varepsilon)$.

Case 2: $\varepsilon\in B-A$. Then $K_1(\varepsilon)=H(\varepsilon)=K_1(\varepsilon)$.

Case 3: $\varepsilon\in A\cap B$. Then $K_1(\varepsilon)=F(\varepsilon)\cap G(\varepsilon)\cap H(\varepsilon)$ and $K_2(\varepsilon)=F(\varepsilon)\circ G(\varepsilon)\circ H(\varepsilon)$. Now we show that $F(\varepsilon)\cap G(\varepsilon)\cap H(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}F(\varepsilon)\circ G(\varepsilon)\circ H(\varepsilon)$. Since S is left quasi-regular, by Lemma 4.2, there exist $y,z\in S$ such that $x\leq yxzx\leq yxzyxzx$. Then we have

$$\max\{(F(\varepsilon)\circ G(\varepsilon)\circ H(\varepsilon))(x),\gamma\}$$
$$=\max\left\{\sup_{(a,b)\in A_x}\min\{F(\varepsilon)(a),(G(\varepsilon)\circ H(\varepsilon))(b)\},\gamma\right\}$$
$$\geq\max\{\min\{F(\varepsilon)(yxzy),(G(\varepsilon)\circ H(\varepsilon))(xzx)\},\gamma\}$$
$$=\max\left\{\min\left\{F(\varepsilon)(yxzy),\sup_{(a,b)\in A_{xzx}}\min\{G(\varepsilon)(a),H(\varepsilon)(b)\},\gamma\right\}\right.$$
$$\geq\max\{\min\{F(\varepsilon)(yxzy),G(\varepsilon)(xzyxzy),H(\varepsilon)(xzx)\},\gamma\}$$
(since $xzx\leq xzyxzyxzx$)

$$\geq \min\{\max\{F(\varepsilon)(yxzy),\gamma\},\max\{G(\varepsilon)(xzyxzy),\gamma\},\max\{H(\varepsilon)(xzx),\gamma\}\}$$
$$\geq \min\{F(\varepsilon)(x),G(\varepsilon)(x),H(\varepsilon)(x),\delta\}$$
$$= \min\{(F(\varepsilon)\cap G(\varepsilon)\cap H(\varepsilon))(x),\delta\}.$$

This implies $F(\varepsilon)\cap G(\varepsilon)\cap H(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}\, F(\varepsilon)\odot G(\varepsilon)\odot H(\varepsilon)$.

Thus, in any case, $K_1(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} K_2(\varepsilon)$ and so (2) holds.

It is clear that (2) $\Rightarrow$ (3). Now assume that (3) holds. Let $\langle F,A\rangle$ be any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft ideal and $\langle H,B\rangle$ any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft quasi-ideal over S, respectively. Since $\Sigma(S,A)$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft right ideal over S, by the assumption, we have

$$\langle F,A\rangle\widetilde{\cap}\langle H,B\rangle = \langle F,A\rangle\widetilde{\cap}\Sigma(S,A)\widetilde{\cap}\langle H,B\rangle \subset_{(\gamma,\delta)} \langle F,A\rangle\odot\Sigma(S,A)\odot\langle H,B\rangle \subset_{(\gamma,\delta)} \langle F,A\rangle\odot\langle H,B\rangle.$$

It follows from Theorem 4.3 that S is left quasi-regular and so (1) holds. ∎

As for right quasi-regular ordered semigroups, we can obtain similar results as Theorems 4.3 and 4.4. Now we give the characterization of quasi-regular ordered semigroups.

Theorem 4.5. *An ordered semigroup $(S,\cdot,\leq)$ is quasi-regular if and only if*

$$\langle F,A\rangle \asymp_{(\gamma,\delta)} (\Sigma(S,A)\odot\langle F,A\rangle)^2\widetilde{\cap}(\langle F,A\rangle\odot\Sigma(S,A))^2$$

for any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft quasi-ideal $\langle F,A\rangle$ over S.

Proof. Assume that S is a quasi-regular ordered semigroup. Let $\langle F,A\rangle$ be an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft quasi-ideal over S. Analogous to the proof of Theorem 3.8, $\Sigma(S,A)\odot\langle F,A\rangle$ and $\langle F,A\rangle\odot\Sigma(S,A)$ are an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideal and an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft right ideal over S, respectively, and so $\Sigma(S,A)\odot\langle F,A\rangle$ and $\langle F,A\rangle\odot\Sigma(S,A)$ are (γ,δ)-fuzzy idempotent by Theorem 4.2. Hence we have

$$(\Sigma(S,A)\odot\langle F,A\rangle)^2\widetilde{\cap}(\langle F,A\rangle\odot\Sigma(S,A))^2 \asymp_{(\gamma,\delta)} \Sigma(S,A)\odot\langle F,A\rangle\widetilde{\cap}\langle F,A\rangle\odot\Sigma(S,A) \subset_{(\gamma,\delta)} \langle F,A\rangle.$$

Now let x be any element of S and $\varepsilon\in A$. Since S is left quasi-regular, there exist $y,z\in S$ such that $x\leq yxzx$. Thus we have

$$\max\{(\chi^\delta_{\gamma S}\circ F(\varepsilon))^2(x),\gamma\} = \sup_{(a,b)\in A_x}\min\{(\chi^\delta_{\gamma S}\circ F(\varepsilon))(a),(\chi^\delta_{\gamma S}\circ F(\varepsilon))(b)\}$$
$$\geq \min\{(\chi^\delta_{\gamma S}\circ F(\varepsilon))(yx),(\chi^\delta_{\gamma S}\circ F(\varepsilon))(zx)\}$$
$$\geq \min\{F(\varepsilon)(x),\delta\}.$$

This implies $F(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}(\chi^\delta_{\gamma S}\circ F(\varepsilon))^2$. It can be similarly prove that $F(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}(F(\varepsilon)\circ \chi^\delta_{\gamma S})^2$. Thus $F(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}(\chi^\delta_{\gamma S}\circ F(\varepsilon))^2\cap(F(\varepsilon)\circ\chi^\delta_{\gamma S})^2$. Therefore $\langle F,A\rangle\subset_{(\gamma,\delta)}(\Sigma(S,A)\odot \langle F,A\rangle)^2\widetilde{\cap}(\langle F,A\rangle\odot\Sigma(S,A))^2$ and so $\langle F,A\rangle\asymp_{(\gamma,\delta)}(\Sigma(S,A)\odot\langle F,A\rangle)^2\widetilde{\cap}(\langle F,A\rangle\odot\Sigma(S,A))^2$.

Conversely, assume that the given condition holds. Let $\langle F,A\rangle$ be any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideal over S. Then $\langle F,A\rangle$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft quasi-ideal over S by Theorem 3.11. Thus we have

$$\langle F,A\rangle \asymp_{(\gamma,\delta)} (\Sigma(S,A)\odot\langle F,A\rangle)^2\widetilde{\cap}(\langle F,A\rangle\odot\Sigma(S,A))^2$$
$$\subset_{(\gamma,\delta)} (\Sigma(S,A)\odot\langle F,A\rangle)^2 \subset_{(\gamma,\delta)} \langle F,A\rangle\odot\langle F,A\rangle$$
$$\subset_{(\gamma,\delta)} \Sigma(S,A)\odot\langle F,A\rangle \subset_{(\gamma,\delta)} \langle F,A\rangle,$$

and so $\langle F,A\rangle \asymp_{(\gamma,\delta)} \langle F,A\rangle \odot \langle F,A\rangle$. Then it follows from Theorem 4.2 that S is left quasi-regular. Similarly, we may prove that S is right quasi-regular. Therefore S is quasi-regular. ∎

Next, we investigate the characterization of left quasi-regular and intra-regular ordered semigroup. Let us first give an useful lemma as follows.

Lemma 4.3. *An ordered semigroup $(S,\cdot,\leq)$ is left quasi-regular and intra-regular if and only if for any $x \in S$, there exist $y,z \in S$ such that $x \leq yx^2zx$.*

Proof. Assume that S is left quasi-regular and intra-regular. Then by Lemma 4.2 and Definition 4.1, we have $x \in (SxSx]$ and $x \in (Sx^2S]$, thus

$$x \in (SxSx] \subseteq (S(Sx^2S]Sx] \subseteq ((S](Sx^2S](Sx]] \subseteq (SSx^2SSx] \subseteq (Sx^2Sx],$$

this implies that there exist $y,z \in S$ such that $x \leq yx^2zx$.

Conversely, if the given condition holds, it is clear that S is left quasi-regular and intra-regular. ∎

Theorem 4.6. *Let $(S,\cdot,\leq)$ be an ordered semigroup. Then the following conditions are equivalent.*

(1) *S is left quasi-regular and intra-regular.*
(2) *$\langle F,A\rangle \widetilde{\cap} \langle G,B\rangle \subset_{(\gamma,\delta)} \langle F,A\rangle \odot_h \langle G,B\rangle$ for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal $\langle F,A\rangle$ and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft bi-ideal $\langle G,B\rangle$ over S.*
(3) *$\langle F,A\rangle \widetilde{\cap} \langle G,B\rangle \subset_{(\gamma,\delta)} \langle F,A\rangle \odot_h \langle G,B\rangle$ for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal $\langle F,A\rangle$ and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft quasi-ideal $\langle G,B\rangle$ over S.*

Proof. The proof is similar to that of Theorem 4.3. ∎

Theorem 4.7. *Let $(S,\cdot,\leq)$ be an ordered semigroup. Then the following conditions are equivalent.*

(1) *S is left quasi-regular and intra-regular.*
(2) *$\langle F,A\rangle \widetilde{\cap} \langle G,A\rangle \widetilde{\cap} \langle H,B\rangle \subset_{(\gamma,\delta)} \langle F,A\rangle \odot \langle G,A\rangle \odot \langle H,B\rangle$ for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal $\langle F,A\rangle$, any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft right ideal $\langle G,A\rangle$ and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft bi-ideal $\langle H,B\rangle$ over S.*
(3) *$\langle F,A\rangle \widetilde{\cap} \langle G,A\rangle \widetilde{\cap} \langle H,B\rangle \subset_{(\gamma,\delta)} \langle F,A\rangle \odot \langle G,A\rangle \odot \langle H,B\rangle$ for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal $\langle F,A\rangle$, any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft right ideal $\langle G,A\rangle$ and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft quasi-ideal $\langle H,B\rangle$ over S.*
(4) *$\langle F,A\rangle \widetilde{\cap} \langle G,A\rangle \widetilde{\cap} \langle H,B\rangle \subset_{(\gamma,\delta)} \langle F,A\rangle \odot \langle G,A\rangle \odot \langle H,B\rangle$ for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal $\langle F,A\rangle$, any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft bi-ideal $\langle G,A\rangle$ and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft bi-ideal $\langle H,B\rangle$ over S.*
(5) *$\langle F,A\rangle \widetilde{\cap} \langle G,A\rangle \widetilde{\cap} \langle H,B\rangle \subset_{(\gamma,\delta)} \langle F,A\rangle \odot \langle G,A\rangle \odot \langle H,B\rangle$ for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal $\langle F,A\rangle$, any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft bi-ideal $\langle G,A\rangle$ and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft quasi-ideal $\langle H,B\rangle$ over S.*
(6) *$\langle F,A\rangle \widetilde{\cap} \langle G,A\rangle \widetilde{\cap} \langle H,B\rangle \subset_{(\gamma,\delta)} \langle F,A\rangle \odot \langle G,A\rangle \odot \langle H,B\rangle$ for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal $\langle F,A\rangle$, any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft quasi-ideal $\langle G,A\rangle$ and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft bi-ideal $\langle H,B\rangle$ over S.*
(7) *$\langle F,A\rangle \widetilde{\cap} \langle G,A\rangle \widetilde{\cap} \langle H,B\rangle \subset_{(\gamma,\delta)} \langle F,A\rangle \odot \langle G,A\rangle \odot \langle H,B\rangle$ for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal $\langle F,A\rangle$, any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft quasi-ideal $\langle G,A\rangle$ and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft quasi-ideal $\langle H,B\rangle$ over S.*

Proof. The proof is similar to that of Theorem 4.4. ∎

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