

Quasi-Koszulity and Minimal Horseshoe Lemma

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Abstract. In this paper, we mainly concentrate on the criteria for minimal Horseshoe Lemma to be true in the category of quasi- δ -Koszul modules, denoted by $\mathcal{Q}^\delta(R)$. More precisely, for a given short exact sequence $\xi : 0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ in $\mathcal{Q}^\delta(R)$, we show that $JK = K \cap JM$ if and only if minimal Horseshoe Lemma holds with respect to ξ . Moreover, some applications of minimal Horseshoe Lemma are also given.

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1. Introduction

It is well-known that Horseshoe Lemma is a basic tool in the theory of homological algebra, which provides a method to construct a projective resolution for the middle term via the ones of the first and the third terms of a given short exact sequence. But what happens if we replace the projective resolutions by the minimal projective resolutions? See some easy examples first:

- (1) Let $A = \mathbb{k}[x]$, a graded polynomial algebra, $M = A/(x^2)$, $K = A/(x)[-1]$ and $N = \mathbb{k}$, a fixed field. Now under a routine computation, we can get the following corresponding minimal projective resolutions:

$$0 \longrightarrow A[-2] \longrightarrow A[-1] \longrightarrow K \longrightarrow 0,$$

$$0 \longrightarrow A[-2] \longrightarrow A \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow A[-1] \longrightarrow A \longrightarrow N \longrightarrow 0.$$

Now it is clear that we have $A[-2] \not\cong A[-2] \oplus A[-1]$ and $A \not\cong A \oplus A[-1]$ as graded A -modules, and the exact sequence

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0,$$

where $[]$ denotes the shift functor given by $(M[n])_t = M_{n+t}$ for any \mathbb{Z} -graded module M and $n, t \in \mathbb{Z}$.

- (2) Let R be a semiperfect Noetherian ring with identity (over which every finitely generated left module has a finitely generated projective cover), M a finitely generated R -module and $\text{Rad}(M)$ the radical of M . Set $K = \text{Rad}(M)$ and $N = M/\text{Rad}(M)$. Obviously, we have the following short exact sequence

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0.$$

Note that R is semiperfect, thus all the finitely generated R -modules possess projective covers.

Let $P_0 \longrightarrow K \longrightarrow 0$, $L_0 \longrightarrow M \longrightarrow 0$ and $Q_0 \longrightarrow N \longrightarrow 0$ be the corresponding projective covers. Then $L_0 \cong Q_0$ as R -modules since $N = M/\text{Rad}(M)$. Therefore, we have $L_0 \not\cong P_0 \oplus Q_0$ as R -modules since $P_0 \neq 0$.

- (3) Let A be a δ -Koszul algebra [6] and $\mathcal{K}^\delta(A)$ be the category of δ -Koszul modules. Let

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence in $\mathcal{K}^\delta(A)$, and $\mathcal{P}_* \longrightarrow K \longrightarrow 0$, $\mathcal{L}_* \longrightarrow M \longrightarrow 0$ and $\mathcal{Q}_* \longrightarrow N \longrightarrow 0$ be the corresponding minimal graded projective resolutions. Then by [11, Theorem 2.6] (also see the figure below), we have the following commutative diagram with exact rows and columns and $L_n \cong P_n \oplus Q_n$ as graded A -modules for all $n \geq 0$.

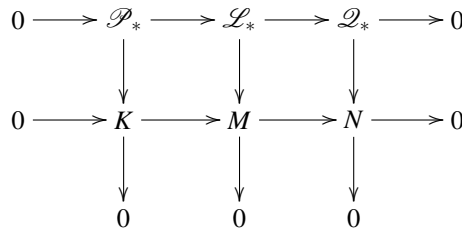


Figure 1. Minimal Horseshoe Lemma diagram.

From the above examples, we can see clearly that if we replace projective resolutions by minimal projective resolutions in the Horseshoe Lemma, the conclusion is inconclusive. For the convenience of narrating, we state the so-called “minimal Horseshoe Lemma” now. Roughly speaking, minimal Horseshoe Lemma is the “minimal” version and a special case of the classic Horseshoe Lemma, which can be stated as follows:

- Let R be any ring with identity and $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of R -modules. Then for any given diagram with \mathcal{P}_* and \mathcal{Q}_* being minimal projective resolutions of K and N , respectively. Then we can complete Figure 2 into Figure 1 such that the rows and columns in Figure 1 are all exact and $\mathcal{L}_* \longrightarrow M \longrightarrow 0$ is also a minimal projective resolution.

Therefore, it is interesting and meaningful to find conditions for the minimal Horseshoe Lemma to be true. In 2008, Wang and Li studied the conditions for the minimal Horseshoe Lemma to be true in the graded case and gave some sufficient conditions. Moreover, they

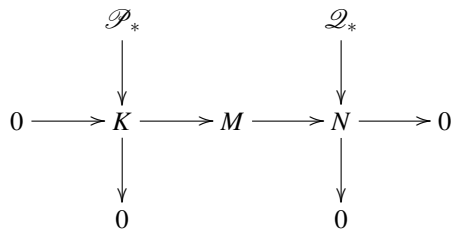


Figure 2. Pre-Minimal Horseshoe Lemma diagram.

said “Though we have found some sufficient conditions for the minimal Horseshoe Lemma to be held, an interesting but difficult question is how to find some necessary conditions”. In fact, [11, Theorem 2.6] has provided a necessary and sufficient condition for the minimal Horseshoe Lemma to be true via δ -Koszul modules in the graded case:

- (1) [11, Theorem 2.6,] Let A be a standard graded algebra and

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence with M, N being δ -Koszul modules. Then K is a δ -Koszul module if and only if the minimal Horseshoe Lemma holds, here we refer to Section 2 (or [11] and [6]) for the notions of standard graded algebra and δ -Koszul module.

As direct corollaries, we can obtain necessary and sufficient conditions for the minimal Horseshoe Lemma to be true via Koszul (see [16]), d -Koszul (see [3], [7] and [20]) and piecewise-Koszul (see [12]) objects and so on since all of them are special δ -Koszul objects. Recently, Green and Martínez-Villa generalized Koszul objects to the nongraded case and introduced quasi-Koszul objects (see [1]); He, Ye and Si generalized d -Koszul objects to the nongraded case and introduced quasi- d -Koszul objects (see [8] and [17]) and the author of the present paper generalized piecewise-Koszul objects to the nongraded case and introduced quasi-piecewise-Koszul objects (see [10] and [13]). Motivated by the above, now one can ask a natural question: Can we give some conditions for the minimal Horseshoe Lemma to be true via these “quasi-Koszul-type” objects?

The main purpose of this paper is to give an answer to the above question and we prove the following result:

Theorem 1.1. *Let R be an augmented Noetherian semiperfect algebra with Jacobson radical J and*

$$\xi : 0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

be a short exact sequence in the category of quasi- δ -Koszul modules. Then $JK = K \cap JM$ if and only if the minimal Horseshoe Lemma holds with respect to ξ .

As an immediate corollary of Theorem 1.1, we obtain the following results:

Corollary 1.1. *Let R be an augmented Noetherian semiperfect algebra with Jacobson radical J and*

$$\xi : 0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

be a short exact sequence in the category \mathcal{C} . Then the following statements are true:

- (1) If \mathcal{C} denotes the category of quasi-Koszul modules, then $JK = K \cap JM$ if and only if the minimal Horseshoe Lemma holds with respect to ξ .
- (2) If \mathcal{C} denotes the category of quasi- d -Koszul modules, then $JK = K \cap JM$ if and only if the minimal Horseshoe Lemma holds with respect to ξ .
- (3) If \mathcal{C} denotes the category of quasi-piecewise-Koszul modules, then $JK = K \cap JM$ if and only if the minimal Horseshoe Lemma holds with respect to ξ .

Remark 1.1. In Corollary 1.1, (1) and (2) show that [18, Theorem 2.8] and [14, Theorem 3.1] are in fact necessary and sufficient conditions; and (3) has been appeared and proved directly in [13].

With the help of minimal Horseshoe Lemma, one can obtain some surprising results which may be wrong in general:

Theorem 1.2. Let R be an augmented Noetherian semiperfect algebra with Jacobson radical J and

$$\xi : 0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

be a short exact sequence in the category of finitely generated R -modules. If the minimal Horseshoe Lemma holds for ξ , then we have the following statements:

- (1) M is projective if and only if K and N are both projective;
- (2) $\text{pd}(M) = \max\{\text{pd}(K), \text{pd}(N)\}$.

As mentioned above, the notion of *quasi-Koszul module* was introduced by Green and Martínez-Villa in 1996 (see [1]). Moreover, they studied the extension closure of the category of quasi-Koszul modules and got the following result:

- Let R be a Noetherian semiperfect algebra with Jacobson radical J and

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence of finitely generated R -modules with $JK = K \cap JM$. If K and N are quasi-Koszul modules, then so is M .

Motivated by the above, a naive but interesting question is: If M and N are quasi-Koszul, then is K quasi-Koszul or if K and M are quasi-Koszul, then is N quasi-Koszul? Green and Martínez-Villa did not discuss these in [1]. With the help of minimal Horseshoe Lemma, we get the following assertions:

Theorem 1.3. Let R be an augmented Noetherian semiperfect algebra with Jacobson radical J and

$$\xi : 0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

be a short exact sequence in the category of finitely generated R -modules with the minimal Horseshoe Lemma holding for ξ . Then we have the following statements:

- (1) If M is a quasi-Koszul module, then so is K ;
- (2) If we have $J^2\Omega^i(K) = \Omega^i(K) \cap J^2\Omega^i(M)$ for all $i \geq 0$, then N is a quasi-Koszul module provided that K and M are quasi-Koszul modules.

In a word, the main purposes of this paper are to find some equivalent conditions and applications for minimal Horseshoe Lemma. More precisely, in Section 2, as preknowledge, we will give the definition of quasi- δ -Koszul modules. In Section 3, we will prove Theorem 1.1. Section 4 mainly focus on the applications of minimal Horseshoe Lemma and we will prove Theorems 1.2 and 1.3.

2. Quasi- δ -Koszul modules

In this section, $A = \bigoplus_{i \geq 0} A_i$ denotes a standard graded algebra, i.e., A satisfies (a) $A_0 = \mathbb{k} \times \cdots \times \mathbb{k}$, a finite product of the ground field \mathbb{k} ; (b) $A_i \cdot A_j = A_{i+j}$ for all $0 \leq i, j < \infty$; and (c) $\dim_{\mathbb{k}} A_i < \infty$ for all $i \geq 0$. Clearly, the graded Jacobson radical of a standard graded algebra A is obvious $\bigoplus_{i \geq 1} A_i$, which is usually denoted by J .

From [1], we know that standard graded algebras can be realized by finite quivers:

Proposition 2.1. *Let A be a standard graded algebra. Then there exists a finite quiver Γ and a graded ideal I in $\mathbb{k}\Gamma$ with $I \subset \sum_{n \geq 2} (\mathbb{k}\Gamma)_n$ such that $A \cong \mathbb{k}\Gamma/I$ as graded algebras.*

Definition 2.1. *Let A be a standard graded \mathbb{k} -algebra and $M = \bigoplus_{i \geq 0} M_i$ a finitely generated graded A -module. We call M a δ -Koszul module provided that M admits a minimal graded projective resolution*

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

such that each P_n is generated in degree $\delta(n)$ for all $n \geq 0$, where $\delta : \mathbb{N} \rightarrow \mathbb{N}$ is a set function and \mathbb{N} denotes the set of natural numbers.

In particular, the standard graded algebra A will be called a δ -Koszul algebra if the trivial A -module A_0 is a δ -Koszul module.

Remark 2.1.

- (1) The set function δ is in fact strictly increasing.
- (2) The notion of δ -Koszul algebra in this paper is different from its original definition [6] and we don't request its Yoneda algebra to be finitely generated.

Example 2.1.

- (1) Koszul algebras/modules (see [16]) are δ -Koszul algebras/modules, where the set function $\delta(i) = i$ for all $i \geq 0$;
- (2) d -Koszul algebras/modules (see [3] and [7]) are δ -Koszul algebras/modules, where the set function

$$\delta(n) = \frac{(n-r)d}{2} + r \quad \text{if } n \equiv r \pmod{2}.$$

- (3) Piecewise-Koszul algebras/modules (see [12]) are δ -Koszul algebras/modules, where the set function

$$\delta(n) = \frac{(n-r)d}{p} + r \quad \text{if } n \equiv r \pmod{p}.$$

and $d \geq p \geq 2$ are given integers.

The following theorem generalizes [7, Proposition 3.1].

Theorem 2.1. *Let $A = \mathbb{k}\Gamma/I$ be a standard graded algebra and*

$$\cdots \longrightarrow P_n \xrightarrow{d_n} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A_0 \longrightarrow 0$$

a minimal graded projective resolution of the trivial A -module A_0 . Then the following statements are equivalent:

- (1) A is a δ -Koszul algebra;
- (2) for all $n \geq 0$, $\ker d_n \subseteq J^{\delta(n+1)-\delta(n)} P_n$ and $J \ker d_n = \ker d_n \cap J^{\delta(n+1)-\delta(n)+1} P_n$;

- (3) for any fixed $n \geq 1$ and $1 \leq i \leq n$, $P_i = \bigoplus_{l \geq 1} Ae_{i_l}[-\delta(i)]$, the component of $d_i(e_{i_l})$ in some Ae_{i-1_m} is in $A_{\delta(i)-\delta(i-1)}$, $\ker d_n \subseteq J^{\delta(n+1)-\delta(n)}P_n$ and $J\ker d_n = \ker d_n \cap J^{\delta(n+1)-\delta(n)+1}P_n$.

Proof. (1) \Rightarrow (2) Suppose that A is a δ -Koszul algebra. Then for all $n \geq 0$, P_n is generated in degree $\delta(n)$. Note that $d_{n+1}(P_{n+1}) = \ker d_n$, which implies that $\ker d_n$ is generated in degree $\delta(n+1)$. But recall that P_n is generated in degree $\delta(n)$, hence the elements of degree $\delta(n+1)$ of P_n are in $J^{\delta(n+1)-\delta(n)}P_n$. Thus for all $n \geq 0$, $\ker d_n \subseteq J^{\delta(n+1)-\delta(n)}P_n$. Now it is clear that $J\ker d_n \subseteq \ker d_n \cap J^{\delta(n+1)-\delta(n)+1}P_n$. Now let $x \in \ker d_n \cap J^{\delta(n+1)-\delta(n)+1}P_n$ be a homogeneous element of degree i . It is easy to see that $i \geq \delta(n+1) + 1$. If x is not in $J\ker d_n$, then x is a generator of $\ker d_n$, which implies that $\ker d_n$ is generated in degree larger than $\delta(n+1) + 1$ since the degree of x is larger than $\delta(n+1) + 1$, which contradicts to that $\ker d_n$ is generated in degree $\delta(n+1)$. Therefore, $x \in J\ker d_n$ and $J\ker d_n \supseteq \ker d_n \cap J^{\delta(n+1)-\delta(n)+1}P_n$. Thus we are done.

(2) \Rightarrow (1) First we claim that for all $n \geq 0$, $(P_n)_j = 0$ for all $j < \delta(n)$. Do it by induction on n . First we prove that $(P_0)_j = 0$ for $j < \delta(0) = 0$. If not, since P_0 is a finitely generated graded module, there exists a smallest $j_0 < \delta(0)$ such that $(P_0)_{j_0} \neq 0$. Let $x \neq 0$ be a homogeneous element of P_0 of degree j_0 . Then $d_0(x) = 0$ since $d_0(x) \in (A_0)_{j_0}$ and $A_0 = (A_0)_0$, which implies that $x \in \ker d_0 \subset JP_0$, which contradicts the choice of j_0 . Now suppose that $(P_{n-1})_j = 0$ for all $j < \delta(n-1)$. Similarly, assume that there exists a smallest $j'_0 < \delta(n)$ such that $(P_n)_{j'_0} \neq 0$. Let $x \neq 0$ be a homogeneous element of P_n of degree j'_0 . Note that $d_n(x) \in \text{Im}d_n = \ker d_{n-1} \subseteq J^{\delta(n)-\delta(n-1)}P_{n-1}$, we have $d_n(x) = 0$ since $J^{\delta(n)-\delta(n-1)}P_{n-1}$ is supported in $\{i \mid i \geq \delta(n)\}$. Therefore, $x \in \ker d_n \subseteq J^{\delta(n+1)-\delta(n)}P_n$, which contradicts the choice of j'_0 .

Now we claim that for any $x \in (P_n)_i$ with $i > \delta(n)$, then $x \in J^s P_n$ for some $s > 0$. If we prove this claim, then it is clear that for all $n \geq 0$, P_n is generated in degree $\delta(n)$. In fact, we also prove this by induction on n . Note that A_0 is generated in degree 0, thus $d_0(x) \in JA_0 = J$, which implies that $x \in d_0^{-1}(J) = JP_0 + \ker d_0 \subseteq JP_0$. Therefore, P_0 is generated in degree 0. Suppose that for any $x \in (P_{n-1})_i$ with $i > \delta(n-1)$, then we have $x \in J^s P_{n-1}$ for some $s > 0$ and P_{n-1} is generated in degree $\delta(n-1)$. By the condition $J\ker d_{n-1} = \ker d_{n-1} \cap J^{\delta(n)-\delta(n-1)+1}P_{n-1}$, we have $\ker d_{n-1}$ is generated in degree $\delta(n)$, which implies that P_n is generated in degree $\delta(n)$ for all $n \geq 0$. Of course, for any $x \in (P_n)_i$ with $i > \delta(n)$, we have $x \in J^s P_n$ for some $s > 0$.

(1), (2) \Rightarrow (3) Suppose that A is a δ -Koszul algebra. Then for all $i \geq 0$, P_i is generated in degree $\delta(i)$. Thus all e_{i_l} are of degree $\delta(i)$, which implies that $d_i(e_{i_l}) \in (P_{i-1})_{\delta(i)}$. But P_{i-1} is generated in degree $\delta(i-1)$, hence $(P_{i-1})_{\delta(i)} \subseteq A_{\delta(i)-\delta(i-1)}(P_{i-1})_{\delta(i-1)}$. Now (3) is clear by (2).

(3) \Rightarrow (1) By an induction on n , it suffices to prove that P_0 is generated in degree $\delta(0)$ and $\ker d_0$ is generated in degree $\delta(1)$, which is similar to the proof of (2) \Rightarrow (1) and we omit the details. \blacksquare

Corollary 2.1. *Let A be a standard graded algebra, M a finitely 0-generated graded A -module and*

$$\cdots \longrightarrow P_n \xrightarrow{d_n} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

a minimal graded projective resolution of M . Then M is a δ -Koszul module if and only if for all $n \geq 0$, $\ker d_n \subseteq J^{\delta(n+1)-\delta(n)}P_n$ and $J\ker d_n = \ker d_n \cap J^{\delta(n+1)-\delta(n)+1}P_n$.

Motivated by Corollary 2.1, we get the following definition:

Definition 2.2. Let R be a Noetherian semiperfect algebra with Jacobson radical J and M a finitely generated R -module. Let

$$\dots \longrightarrow P_n \xrightarrow{d_n} \dots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

be a minimal projective resolution of M . Then we call M a quasi- δ -Koszul module if for all $n \geq 0$, we have $\ker d_n \subseteq J^{\delta(n+1)-\delta(n)}P_n$ and $J\ker d_n = \ker d_n \cap J^{\delta(n+1)-\delta(n)+1}P_n$, where $\delta : \mathbb{N} \rightarrow \mathbb{N}$ is a strictly increasing set function.

In particular, R is called a quasi- δ -Koszul algebra if R/J is a quasi- δ -Koszul module.

Let $\mathcal{Q}^\delta(R)$ denote the category of quasi- δ -Koszul modules.

Example 2.2. Quasi-Koszul algebras/modules (see [1]), quasi- d -Koszul algebras/modules (see [8]) and quasi-piecewise-Koszul algebras/modules (see [11]) are all special quasi- δ -Koszul algebras/modules.

3. Criteria for minimal Horseshoe Lemma

Throughout this section, R denotes an augmented Noetherian semiperfect algebra with Jacobson radical J and we will mainly concentrate on the proof of Theorem 1.1.

Lemma 3.1. Let $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of finitely generated R -modules. Then $JK = K \cap JM$ if and only if we have the following commutative diagram with exact rows and columns such that $P_0 \rightarrow K \rightarrow 0$, $L_0 \rightarrow M \rightarrow 0$ and $Q_0 \rightarrow N \rightarrow 0$

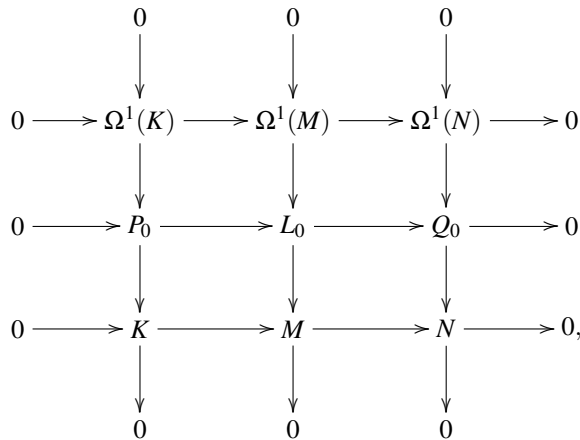


Figure 3. Commutative diagram with exact rows and columns.

are projective covers.

Proof. (\Rightarrow) By hypothesis, $JK = K \cap JM$, which implies the exact sequence

$$0 \longrightarrow JK \longrightarrow JM \longrightarrow JN \longrightarrow 0.$$

Now consider the following diagram with exact rows and columns by the ‘‘Snake-Lemma’’,

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & JK & \longrightarrow & JM & \longrightarrow & JN \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N \longrightarrow 0, \end{array}$$

Figure 4. Commutative diagram with exact rows and columns.

we obtain the following exact sequence

$$0 \longrightarrow K/JK \longrightarrow M/JM \longrightarrow N/JN \longrightarrow 0.$$

Note that for any finitely generated R -module X , $R \otimes_{R/J} X/JX \longrightarrow X \longrightarrow 0$ is a projective cover and if a module has projective covers then all projective covers are unique up to isomorphisms. Now set

$$P_0 := R \otimes_{R/J} K/JK, \quad L_0 := R \otimes_{R/J} M/JM \quad \text{and} \quad Q_0 := R \otimes_{R/J} N/JN,$$

we have the following exact sequence

$$0 \longrightarrow P_0 \longrightarrow L_0 \longrightarrow Q_0 \longrightarrow 0$$

since R/J is a semisimple algebra. Therefore, we have the following commutative diagram which implies the desired diagram (Figure 3) since the ‘‘ 3×3 -Lemma’’.

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \Omega^1(K) & \longrightarrow & \Omega^1(M) & \longrightarrow & \Omega^1(N) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & P_0 & \longrightarrow & L_0 & \longrightarrow & Q_0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N \longrightarrow 0, \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Figure 5. Commutative diagram with exact rows and columns.

(\Leftarrow) Suppose that we have Figure 3. We may assume that

$$P_0 := R \otimes_{R/J} K/JK, \quad L_0 := R \otimes_{R/J} M/JM \quad \text{and} \quad Q_0 := R \otimes_{R/J} N/JN$$

since the projective cover of a module is unique up to isomorphisms. From the middle row of Figure 3, we have the following exact sequence

$$0 \longrightarrow R \otimes_{R/J} K/JK \longrightarrow R \otimes_{R/J} M/JM \longrightarrow R \otimes_{R/J} N/JN \longrightarrow 0.$$

Thus, we have the following short exact sequence as R/J -modules

$$0 \longrightarrow K/JK \longrightarrow M/JM \longrightarrow N/JN \longrightarrow 0$$

since R/J is semisimple. Now consider the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K/JK & \longrightarrow & M/JM & \longrightarrow & N/JN & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Figure 6. Commutative diagram with exact rows and columns.

By the ‘‘Snake-Lemma’’ again, we have the exact sequence

$$0 \longrightarrow JK \longrightarrow JM \longrightarrow JN \longrightarrow 0,$$

which is equivalent to $JK = K \cap JM$. ■

Lemma 3.2. *Let $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ be a short exact sequence of finitely generated R -modules. Then $J\Omega^i(K) = \Omega^i(K) \cap J\Omega^i(M)$ for all $i \geq 0$ if and only if the minimal Horseshoe Lemma holds.*

Proof. (\Rightarrow) By Lemma 3.1, $J\Omega^i(K) = \Omega^i(K) \cap J\Omega^i(M)$ for all $i \geq 0$ if and only if for all $i \geq 0$, we have the following commutative diagram with exact rows and columns such that P_i, L_i and Q_i are projective covers of $\Omega^i(K), \Omega^i(M)$ and $\Omega^i(N)$, respectively. Now putting these commutative diagrams together, we obtain the commutative diagram (Figure 2), i.e., the minimal Horseshoe Lemma holds.

(\Leftarrow) Suppose that the minimal Horseshoe Lemma is true for the exact sequence

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0,$$

i.e., we have the commutative diagram (Figure 2). Then Figure 2 can be divided into a lot of commutative diagrams similar to Figure 7. Now by Lemma 3.1, we get the desired equations. ■

Lemma 3.3. *Let $\xi : 0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence in $\mathcal{Q}^\delta(R)$. Then the following statements are equivalent:*

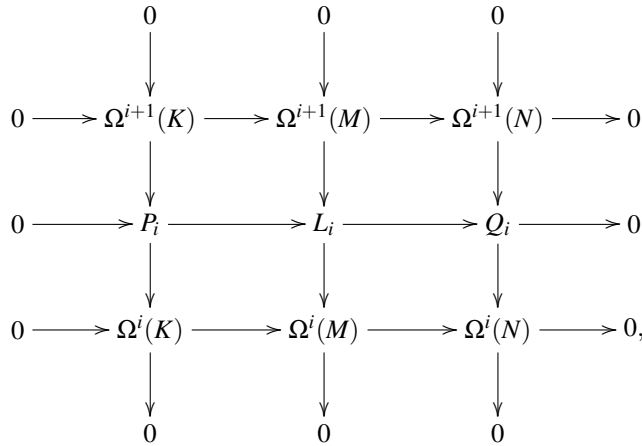


Figure 7. Commutative diagram with exact rows and columns.

- (1) $JK = K \cap JM$;
- (2) $0 \longrightarrow JK \longrightarrow JM \longrightarrow JN \longrightarrow 0$ is exact;
- (3) $0 \longrightarrow K/JK \longrightarrow M/JM \longrightarrow N/JN \longrightarrow 0$ is exact;
- (4) $R/J \otimes_R K \rightarrow R/J \otimes_R M$ is a monomorphism;
- (5) the minimal Horseshoe Lemma holds with respect to ξ .

Proof. (1) \Rightarrow (2) and (2) \Rightarrow (3) have been proved in the proof of Lemma 3.2.

(3) \Rightarrow (4) Consider the following commutative diagram: which implies that $R/J \otimes_R K \rightarrow$

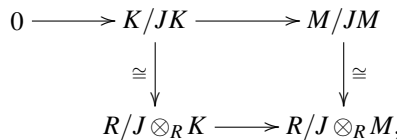


Figure 8. Commutative diagram with exact rows and columns.

$R/J \otimes_R M$ is a monomorphism.

(4) \Rightarrow (1) Consider the following commutative diagram with exact rows and columns: which implies that $JK = K \cap JM$ since the “Five-Lemma” and the the following commutative diagram (1) \Rightarrow (5) By Lemma 3.1, we have Figure 3 since $JK = K \cap JM$, thus we have the following commutative diagram with exact rows since K, M and N are quasi- δ -Koszul modules. Now applying the functor $R/J \otimes_R -$ to Figure 11, we have the following commutative diagram with exact rows where α_1 and γ_1 are monomorphisms since K, M are in $\mathcal{Q}^\delta(R)$ and (1) \Leftrightarrow (4), which implies that β_1 is also a monomorphism induced by the commutativity of the left square. By (1) \Leftrightarrow (4), we have $J\Omega^1(K) = \Omega^1(K) \cap J\Omega^1(M)$. By Lemma 3.1 again, we have Figure 7 in the case of $i = 1$, which implies the following commutative diagram with exact rows and columns since K, M and N are quasi- δ -Koszul modules.

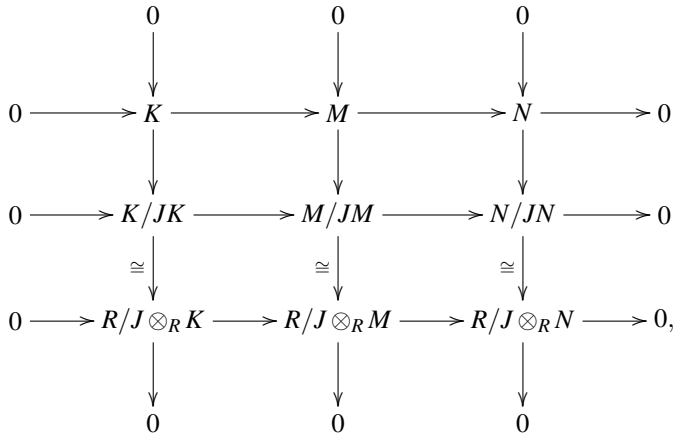


Figure 9. Commutative diagram with exact rows and columns.

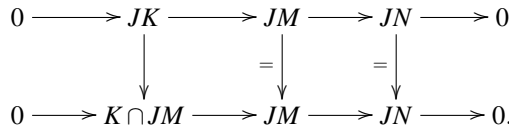


Figure 10. Commutative diagram with exact rows and columns.

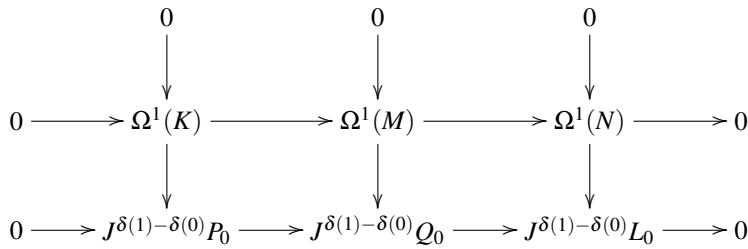


Figure 11. Commutative diagram with exact rows and columns.

Similar to the above, we have the following commutative diagram with exact rows and $J\Omega^2(K) = \Omega^2(K) \cap J\Omega^2(M)$.

Now repeating the above procedures, we have $J\Omega^n(K) = \Omega^n(K) \cap J\Omega^n(M)$ since the following commutative diagram with exact rows for all $n \geq 3$. Now by Lemma 3.2, we finish the proof of (1) \Rightarrow (5).

(5) \Rightarrow (1) By Lemma 3.2, (5) is equivalent to $J\Omega^i(K) = \Omega^i(K) \cap J\Omega^i(M)$ for all $i \geq 0$. In particular, let $i = 0$, we have $JK = K \cap JM$. ■

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & R/J \otimes_R \Omega^1(K) & \xrightarrow{\beta_1} & R/J \otimes_R \Omega^1(M) & \longrightarrow & R/J \otimes_R \Omega^1(N) & \longrightarrow 0 \\
 & \downarrow \alpha_1 & & \downarrow \gamma_1 & & \downarrow & \\
 0 \longrightarrow & R/J \otimes_R J^{\delta(1)-\delta(0)} P_0 & \longrightarrow & R/J \otimes_R J^{\delta(1)-\delta(0)} Q_0 & \longrightarrow & R/J \otimes_R J^{\delta(1)-\delta(0)} L_0 & \longrightarrow 0,
 \end{array}$$

Figure 12. Commutative diagram with exact rows and columns.

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \longrightarrow & \Omega^2(K) & \longrightarrow & \Omega^2(M) & \longrightarrow & \Omega^2(N) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \longrightarrow & J^{\delta(2)-\delta(1)} P_1 & \longrightarrow & J^{\delta(2)-\delta(1)} Q_1 & \longrightarrow & J^{\delta(2)-\delta(1)} L_1 & \longrightarrow 0
 \end{array}$$

Figure 13. Commutative diagram with exact rows and columns.

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & R/J \otimes_R \Omega^2(K) & \xrightarrow{\beta_2} & R/J \otimes_R \Omega^2(M) & \longrightarrow & R/J \otimes_R \Omega^2(N) & \longrightarrow 0 \\
 & \downarrow \alpha_2 & & \downarrow \gamma_2 & & \downarrow & \\
 0 \longrightarrow & R/J \otimes_R J^{\delta(2)-\delta(1)} P_1 & \longrightarrow & R/J \otimes_R J^{\delta(2)-\delta(1)} Q_1 & \longrightarrow & R/J \otimes_R J^{\delta(2)-\delta(1)} L_1 & \longrightarrow 0
 \end{array}$$

Figure 14. Commutative diagram with exact rows and columns.

Now by Lemma 3.3 and note that $\{\text{Quasi-Koszul modules}\} \subseteq \{\text{Quasi-}d\text{-Koszul modules}\} \subseteq \{\text{Quasi-piecewise-Koszul modules}\} \subseteq \{\text{Quasi-}\delta\text{-Koszul modules}\}$, Theorem 1.1 and Corollary 1.1 are obvious.

4. Some applications of minimal Horseshoe Lemma

In this section, we will give some applications of minimal Horseshoe Lemma. More precisely, we will prove Theorems 1.2 and 1.3.

Lemma 4.1. *Let R be an augmented Noetherian semiperfect algebra with Jacobson radical J and $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ be a short exact sequence in the category of*

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & R/J \otimes_R \Omega^n(K) & \xrightarrow{\beta_n} & R/J \otimes_R \Omega^n(M) & \longrightarrow & R/J \otimes_R \Omega^n(N) \longrightarrow 0 \\
 & & \downarrow \alpha_n & & \downarrow \gamma_n & & \downarrow \\
 0 & \twoheadrightarrow & R/J \otimes_R J^{\delta(n)-\delta(n-1)} P_{n-1} & \twoheadrightarrow & R/J \otimes_R J^{\delta(n)-\delta(n-1)} Q_{n-1} & \twoheadrightarrow & R/J \otimes_R J^{\delta(n)-\delta(n-1)} L_{n-1} \twoheadrightarrow 0
 \end{array}$$

Figure 15. Commutative diagram with exact rows and columns.

finitely generated R -modules with $JK = K \cap JM$. Then M is projective if and only if K and N are both projective.

Proof. (\Rightarrow) By Lemma 3.1, we have Figure 3, which implies the following exact sequence

$$0 \longrightarrow \Omega^1(K) \longrightarrow \Omega^1(M) \longrightarrow \Omega^1(N) \longrightarrow 0.$$

By hypothesis, M is a projective R -modules, thus the projective cover of M is itself. Hence we have $\Omega^1(M) = 0$. Now combining the above exact sequence, we have $\Omega^1(N) = 0$, which implies that $Q_0 \cong N$ in Figure 3, thus N is a projective R -module.

(\Leftarrow) Assume that K and N are projective R -modules, repeating the same argument as in the proof of the necessity, we have $\Omega^1(K) = \Omega^1(N) = 0$ since K and N are projective R -modules, which implies that $\Omega^1(M) = 0$ and hence M is a projective R -module. \blacksquare

Lemma 4.2. *Let R be a Noetherian semiperfect algebra with Jacobson radical J and M a finitely generated R -module. Then the length of a minimal projective resolution of M , denoted by l , equals to the projective dimension of M , $pd(M)$.*

Proof. By hypothesis, M has a minimal projective resolution of length l , we have $pd(M) \leq l$ since a minimal projective resolution is in particular a projective resolution. But if there would be a minimal resolution of M of length strictly less than l , then we have $\text{Ext}_R^l(M, R/J) \cong \text{Tor}_l^R(R/J, M) = 0$, which is a contradiction. \blacksquare

Lemma 4.3. *Let R be an augmented Noetherian semiperfect algebra with Jacobson radical J and $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ be a short exact sequence in the category of finitely generated R -modules. If the minimal Horseshoe Lemma holds for ξ , then we have $pd(M) = \max\{pd(K), pd(N)\}$.*

Proof. By hypothesis the minimal Horseshoe Lemma holds, i.e., we have Figure 2. More precisely, we obtain that

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 \longrightarrow K \longrightarrow 0, \\
 \cdots & \longrightarrow & L_2 & \longrightarrow & L_1 & \longrightarrow & L_0 \longrightarrow M \longrightarrow 0
 \end{array}$$

and

$$\cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow N \longrightarrow 0$$

are minimal projective resolution of K , M and N , respectively, and $L_n = P_n \oplus Q_n$ for all $n \geq 0$.

If $\text{pd}(M) = \infty$, by Lemma 4.2, there exists an infinite minimal graded projective resolution of M

$$\cdots \longrightarrow L_n \longrightarrow \cdots \longrightarrow L_2 \longrightarrow L_1 \longrightarrow L_0 \longrightarrow M \longrightarrow 0.$$

Note that we have $L_n = P_n \oplus Q_n$ for all $n \geq 0$ and the minimal projective resolution of a module is unique up to isomorphisms. Thus at least one of the lengths of

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow K \longrightarrow 0$$

and

$$\cdots \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow N \longrightarrow 0$$

is infinite, which implies that $\text{pd}(M) = \max\{\text{pd}(K), \text{pd}(N)\}$.

If $\text{pd}(M) = n < \infty$, by Lemma 4.2, there exists a minimal projective resolution of M of length n :

$$0 \longrightarrow L_n \longrightarrow \cdots \longrightarrow L_2 \longrightarrow L_1 \longrightarrow L_0 \longrightarrow M \longrightarrow 0,$$

which implies that K and N possess the following minimal projective resolutions

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow K \longrightarrow 0,$$

$$0 \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow N \longrightarrow 0$$

such that at least one of P_n and L_n isn't zero, which implies that $\text{pd}(M) = \max\{\text{pd}(K), \text{pd}(N)\}$ by Lemma 4.2. ■

Now it is easy to see that Theorem 1.2 is immediate from Lemmas 4.1 and 4.3.

With the help of Theorem 1.1 and Lemma 3.3, we can prove Theorem 1.3 directly.

Proof. (1) By Theorem 1.1, we have Figure 7 for all $i \geq 0$, which implies the following commutative diagram with exact rows and columns for all $i \geq 0$: Now applying the additive

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega^{i+1}(K) & \longrightarrow & \Omega^{i+1}(M) & \longrightarrow & \Omega^{i+1}(N) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & JP_i & \longrightarrow & JL_i & \longrightarrow & JQ_i \longrightarrow 0. \end{array}$$

Figure 16. Commutative diagram with exact rows and columns.

right functor $R/J \otimes_R -$ to Figure 16, we get the following commutative diagram with exact rows and columns for all $i \geq 0$: where δ_{i+1} is a monomorphism for all $i \geq 0$ since the exact sequence

$$0 \longrightarrow JP_i \longrightarrow JL_i \longrightarrow JQ_i \longrightarrow 0$$

is split, and γ_{i+1} is a monomorphism for all $i \geq 0$ since M is a quasi-Koszul module and Lemma 3.3.

Now we claim that β_{i+1} is a monomorphism for all $i \geq 0$. In fact, by the hypothesis, the minimal Horseshoe Lemma holds for the given exact sequence ξ , by Lemma 3.2, we

$$\begin{array}{ccccccc}
 R/J \otimes_R \Omega^{i+1}(K) & \xrightarrow{\alpha_{i+1}} & R/J \otimes_R \Omega^{i+1}(M) & \longrightarrow & R/J \otimes_R \Omega^{i+1}(N) & \longrightarrow & 0 \\
 \beta_{i+1} \downarrow & & \gamma_{i+1} \downarrow & & \downarrow & & \\
 R/J \otimes_R JP_i & \xrightarrow{\delta_{i+1}} & R/J \otimes_R JL_i & \longrightarrow & R/J \otimes_R JQ_i & \longrightarrow & 0,
 \end{array}$$

Figure 17. Commutative diagram with exact rows and columns.

have $J\Omega^i(K) = \Omega^i(K) \cap J\Omega^i(M)$ for all $i \geq 0$. By Lemma 3.3, α_{i+1} is a monomorphism for all $i \geq 0$, which implies β_{i+1} is a monomorphism for all $i \geq 0$ since the left above square is commutative. By Lemma 3.3, we have $J\Omega^{i+1}(K) = \Omega^{i+1}(K) \cap J^2P_i$ for all $i \geq 0$, which imply that K is a quasi-Koszul module.

(2) Similarly, we have Figure 7 for all $i \geq 0$. Since the minimal Horseshoe Lemma is true for ξ , then by Lemma 3.2, we have $J\Omega^i(K) = \Omega^i(K) \cap J\Omega^i(M)$ for all $i \geq 0$. By Lemma 3.3, we have the following exact sequence

$$0 \longrightarrow J\Omega^i(K) \longrightarrow J\Omega^i(M) \longrightarrow J\Omega^i(N) \longrightarrow 0$$

for all $i \geq 0$.

Now note that all the columns are projective covers, which imply the following commutative diagram with exact rows and columns for all $i \geq 0$:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \Omega^{i+1}(K) & \longrightarrow & \Omega^{i+1}(M) & \longrightarrow & \Omega^{i+1}(N) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & JP_i & \longrightarrow & JL_i & \longrightarrow & JQ_i \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & J\Omega^i(K) & \longrightarrow & J\Omega^i(M) & \longrightarrow & J\Omega^i(N) \longrightarrow 0, \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Figure 18. Commutative diagram with exact rows and columns.

Now applying the additive right functor $R/J \otimes_R -$ to Figure 18, we get the following commutative diagram with exact rows and columns for all $i \geq 0$:

$$\begin{array}{ccccccc}
 R/J \otimes_R \Omega^{i+1}(K) & \xrightarrow{\varepsilon_{i+1}} & R/J \otimes_R \Omega^{i+1}(M) & \longrightarrow & R/J \otimes_R \Omega^{i+1}(N) & \longrightarrow & 0 \\
 \varepsilon_{i+1} \downarrow & & \zeta_{i+1} \downarrow & & \eta_{i+1} \downarrow & & \\
 R/J \otimes_R JP_i & \xrightarrow{\theta_{i+1}} & R/J \otimes_R JL_i & \longrightarrow & R/J \otimes_R JQ_i & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 R/J \otimes_R J\Omega^i(K) & \xrightarrow{\vartheta_i} & R/J \otimes_R J\Omega^i(M) & \longrightarrow & R/J \otimes_R J\Omega^i(N) & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & & 0 & & 0 & &
 \end{array}$$

Figure 19. Commutative diagram with exact rows and columns.

Similar to the analysis of (1), we have that ε_{i+1} , ζ_{i+1} and θ_{i+1} are monomorphisms for all $i \geq 0$. Note that

$$\begin{aligned}
 J\Omega^i(K) \cap J(J\Omega^i(M)) &= J\Omega^i(K) \cap J^2\Omega^i(M) = J\Omega^i(K) \cap J^2\Omega^i(M) \cap \Omega^i(K) \\
 &= J\Omega^i(K) \cap J^2\Omega^i(K) = J^2\Omega^i(K).
 \end{aligned}$$

By Lemma 3.3, we have that ϑ_i is a monomorphism for each $i \geq 0$. Now by “ 3×3 -Lemma” to Figure 19, we have that η_{i+1} is a monomorphism for each $i \geq 0$. By Lemma 3.3, we have $J\Omega^{i+1}(N) = \Omega^{i+1}(N) \cap J^2Q_i$ for all $i \geq 0$, thus N is a quasi-Koszul module. ■

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