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Γ-Semihyperrings: Approximations and Rough Ideals

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Abstract. The notion of rough sets was introduced by Z. Pawlak in 1982. The concept of Γ-semihyperring is a generalization of semihyperring, Γ-semiring and semiring. In this paper, we study the notion of a rough (rough prime) ideal in a Γ-semihyperring. Also, we discuss the relation between the upper and lower rough ideals and the upper and lower approximation of their homomorphism images.

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1. Preliminaries and basic definition

The process of analyzing data under uncertainty is a main goal for many real life problems. The present century is distinguished by the tendency of using the available data in the process of decision making. The real data derived from actual experiments needs a special treatment to get information more close to reality. Pawlak [16], introduced the rough set theory, which is an excellent tool to handle a granularity of data. In rough set theory, given an equivalence relation on a universe, we can define a pair of rough approximations which provide a lower bound and an upper bound for each subset of the universe set. Biswas and Nanda [3] defined the notion of rough subgroup. Kuroki [14], introduced the notion of a rough ideal in a semigroup, studied approximations of a subset in a semigroup and discussed the product structures of rough ideals. In [15], Kuroki and Wang provided some propositions in an investigation of the properties of lower and upper approximations with respect to normal subgroups. Davvaz [8, 11], examined a relationship between rough sets and ring theory and introduced the notions of rough ideal and rough subring with respect to an ideal of a ring. In [9], Davvaz and Mahdavipour considered a module over a ring as a universal set and introduced the notion of rough submodule with respect to a submodule of the module. In [13], Kazanci and Davvaz further introduced the notions of rough prime (primary) ideal and rough fuzzy prime (primary) ideal in a ring and presented some properties of such ideals.

Algebraic hyperstructures represent a natural extension of classical algebraic structures

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and they were introduced by the French mathematician F. Marty. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Since then, hundreds of papers and several books have been written on this topic, see [4, 5, 10, 18]. A recent book on hyperstructures [5] points out on their applications in fuzzy and rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. Another book [10] is devoted especially to the study of hyperring theory. Several kinds of hyperrings are introduced and analyzed. The volume ends with an outline of applications in chemistry and physics, analyzing several special kinds of hyperstructures: *e*-hyperstructures and transposition hypergroups, also see [7, 17, 19, 20]. The theory of suitable modified hyperstructures can serve as a mathematical background in the field of quantum communication systems.

Let H be a non-empty set and $\circ : H \times H \longrightarrow \mathscr{P}^*(H)$ be a *hyperoperation*, where $\mathscr{P}^*(H)$ is the family of all non-empty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid*. For any two non-empty subsets A and B of H and $x \in H$, we define $A \circ B = \bigcup_{a\in A, b\in B} a \circ b$, $A \circ \{x\} = A \circ x$ and $\{x\} \circ A = x \circ A$. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all a, b, c in H we have $(a \circ b) \circ c = a \circ (b \circ c)$. In addition, if for every $a \in H$, $a \circ H = H = H \circ a$, then $(H, \circ)$ is called a *hypergroup*. A non-empty subset K of a semihypergroup $(H, \circ)$ is called a *sub-semihypergroup* if it is a semihypergroup. In other words, a non-empty subset K of a semihypergroup $(H, \circ)$ is a sub-semihypergroup if $K \circ K \subseteq K$. In [1, 2, 12], Davvaz *et al.* studied the notion of a Γ-semihypergroup as a generalization of a semihypergroup. Many classical notions of semigroups and semihypergroups have been extended to Γ-semihypergroups and a lot of results on Γ-semihypergroups are obtained.

In [1] Davvaz *et al.* introduced the notion of roughness in a Γ-semihypergroup and discuss rough sets with respect to an idempotent regular relation in a quotient Γ-semihypergroup and upper approximation with respect to Green equivalence relations. In this paper, first, we consider the notion of a Γ-*semihyperring* as a generalization of semiring, a generalization of a semihyperring and a generalization of a Γ-semiring. Then, we introduce the notion of a *rough sub* Γ-*semihyperring (ideal)* of a Γ-semihyperring and give some properties of lower and upper approximations in a Γ-semihyperring. In addition, we introduce the concept of rough sets in a quotient Γ-semihyperring.

Let R be a commutative semihypergroup and Γ be a commutative group. Then, R is called a Γ-*semihyperring* if there exists a map $R \times \Gamma \times R \longrightarrow \mathscr{P}^*(R)$ (the image of (a, α, b) is denoted by $a\alpha b$ for $a, b \in R$ and $\alpha \in \Gamma$) satisfying the following conditions:

(i) $a\alpha(b+c) = a\alpha b + a\alpha \mathrm{c}$,
(ii) $(a+b)\alpha c = a\alpha c + b\alpha c$,
(iii) $a(\alpha+\beta)c = a\alpha c + a\beta c$,
(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$.

In the above definition if R is a semigroup, then R is called a *multiplicative* Γ-*semihyperring*. A Γ-semihyperring R is called *commutative* if $x\alpha y = y\alpha x$ for every $x, y \in R$ and $\alpha \in \Gamma$. We say that R is a Γ-semihyperring with zero, if there exists $0 \in R$ such that $a \in a + 0$ and $0 \in 0\alpha a$, $0 \in a\alpha 0$ for all $a \in R$ and $\alpha \in \Gamma$. Let A and B be two non-empty subsets of Γ-semihyperring R. We define

$$A\Gamma B = \{x \mid x \in a\alpha b\ a \in A,\ b \in B,\ \alpha \in \Gamma\}.$$

A non-empty subset R_1 of Γ-semihyperring R is called a *sub* Γ*-semihyperring* if it is closed with respect to the hyperaddition and hypermultiplication. In other words, a non-empty subset R_1 of Γ-semihyperring R is a sub Γ-semihyperring if $R_1+R_1 \subseteq R_1$ and $R_1\Gamma R_1 \subseteq R_1$.

Example 1.1. Let $(R,+,\circ)$ be a semihyperring such that $x\circ y = x\circ y + x\circ y$ and Γ be a commutative group. We define $x\alpha y = x\circ y$ for every $x,y\in R$ and $\alpha\in\Gamma$. Then, R is a Γ-semihyperring.

Example 1.2. Let $(R,+,\circ)$ be a semiring and $(\Gamma,+)$ be a subgroup of $(R,+)$. We define $x\alpha y = x\circ\alpha\circ y$ for every $x,y\in R$ and $\alpha\in\Gamma$. Then, R is a Γ-semihyperring.

Example 1.3. Let $R=\mathbb{Z}_4$ and $\Gamma=\{\overline{0},\overline{2}\}$. Then, R is a multiplicative Γ-semihyperring with the following hyperoperation:

$$x\alpha y = \{\overline{0},\overline{2}\},$$

where $x,y\in R$ and $\alpha\in\Gamma$.

Example 1.4. Let R be a ring, $\{A_g\}_{g\in R}$ be a family of disjoint non-empty sets and $(\Gamma,+)$ be a subgroup of $(R,+)$. Then, $S=\bigcup_{g\in R} A_g$ is a Γ-semihyperring with the following hyperoperations:

$$x\oplus y = A_{g_1+g_2},\ x\alpha y = A_{g_1\alpha g_2},$$

where $x\in A_{g_1}$ and $y\in A_{g_2}$.

Definition 1.1. *A non-empty subset I of Γ-semihyperring R is a right (left) ideal of R if I is a subhypergroup of $(R,+)$ and $I\Gamma R\subseteq I$ ($R\Gamma I\subseteq I$), and is an ideal of R if it is both a right and a left ideal.*

Let X be a non-empty subset of Γ-semihyperring R. By the term left ideal $\langle X\rangle_l$ (respectively, right ideal $\langle X\rangle_r$) of R generated by X, we mean the intersection of all left ideals (respectively, right ideals) of R contains X.

Proposition 1.1. *Let R be a Γ-semihyperring with zero and X be a non-empty subset of R. Then,*

(1) $$\langle X\rangle_l = \left\{ t \mid t\in \sum_{i=1}^{n} n_i x_i + \sum_{j=1}^{m} r_j\alpha_j x_j\ , n_i, m, n\in\mathbb{N}, r_j\in R, \alpha_j\in\Gamma, x_j\in X \right\},$$

(2) $$\langle X\rangle_r = \left\{ t \mid t\in \sum_{i=1}^{n} n_i x_i + \sum_{j=1}^{m} x_j\alpha_j r_j\ , n_i, m, n\in\mathbb{N}, x_j\in X, r_j\in R, \alpha_j\in\Gamma \right\},$$

(3) $$\langle X\rangle = \left\{ t \mid t\in \sum_{i=1}^{n_1} m_i x_i + \sum_{j=1}^{n_2} r_j\alpha_j x_j + \sum_{r=1}^{n_3} y_r\beta_r r_r + \sum_{t=1}^{n_4} r_t\alpha_t x_t\beta_t s_t, n_i\in\mathbb{N}, x_i\in X, r_j\in R, \right\}.$$

Proof. The proof is straightforward. ∎

Definition 1.2. *Let R be a Γ-semihyperring and P a proper ideal of R. Then, P is called prime if for every $x,y\in R$, $x\Gamma R\Gamma y\subseteq P$ implies $x\in P$ or $y\in P$.*

Proposition 1.2. *Let R be a Γ-semihyperring with zero and P be an ideal of R. Then, the following statement are equivalent:*

(i) *P is prime,*
(ii) *$I\Gamma J \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$ where I and J are two ideals of R.*

Proof. It is straightforward. ∎

Example 1.5. Let S be a Γ-semihyperring in Example 1.4 and $\Gamma \subseteq R$ be a subgroup of $(R,+)$. If P is a prime ideal of R such that $\Gamma \cap P = \emptyset$, then $S_P = \bigcup_{g\in P} A_g$ is a prime ideal of S.

Let A and B be two non-empty subsets of R and ρ a relation on R. We define $(A,B) \in \overline{\rho}$ if for every $a \in A$ there exists $b \in B$ such that $(a,b) \in \rho$ and for every $c \in B$ there exists $d \in A$ such that $(d,c) \in \rho$ and $(A,B) \in \overline{\overline{\rho}}$ if for every $a \in A$ and $b \in B$, $(a,b) \in \rho$. Let R be a Γ-semihyperring. An equivalence relation ρ on R is called *regular* if for every $x \in R$ and $\alpha \in \Gamma$, we have

$$(a,b) \in \rho \text{ implies } (a+x)\overline{\rho}(b+x), (a\alpha x)\overline{\rho}(b\alpha x) \text{ and } (x\alpha a)\overline{\rho}(x\alpha b).$$

The relation ρ is called *strongly regular* if for every $x \in R$ and $\alpha \in \Gamma$, we have

$$(a,b) \in \rho \text{ implies } (a+x)\overline{\overline{\rho}}(b+x), (a\alpha x)\overline{\overline{\rho}}(b\alpha x) \text{ and } (x\alpha a)\overline{\overline{\rho}}(x\alpha b).$$

Let A be a non-empty subset of R. We define $\rho(A) = \{\rho(a) \mid a \in A\}$. Relation ρ is called *semi-complete*, if for every $x,y \in R$ and $\alpha \in \Gamma$, $\rho(x\alpha y) \subseteq \rho(x)\alpha\rho(y)$ and $\rho(x+y) \subseteq \rho(x)+\rho(y)$.

Lemma 1.1. *Let R be a Γ-semihyperring and ρ be a regular relation on R. If $a,b \in R$ and $\alpha \in \Gamma$, then $\rho(a)\alpha\rho(b) \subseteq \rho(a\alpha b)$ and $\rho(a)+\rho(b) \subseteq \rho(a+b)$.*

Proof. Let $x \in \rho(a)\alpha\rho(b)$. There exist $x_1 \in \rho(a)$ and $x_2 \in \rho(b)$ such that $x \in x_1\alpha x_2$. Since ρ is a regular relation, we have $(x_1\alpha x_2)\overline{\rho}(a\alpha b)$. So, there exists $y \in a\alpha b$ such that $(x,y) \in \rho$. Therefore, $x \in \rho(a\alpha b)$. In the same way, $\rho(a)+\rho(b) \subseteq \rho(a+b)$. ∎

Proposition 1.3. *Let ρ be a regular relation on a Γ-semihyperring R. If $a,b \in R$ and $\alpha \in \Gamma$, then*
(i) $\rho(a+b) = \{\rho(c) \mid c \in \rho(a)+\rho(b)\}$,
(ii) $\rho(a\alpha b) = \{\rho(c) \mid c \in \rho(a)\alpha\rho(b)\}$.

Proof. Let $c \in \rho(a)+\rho(b)$. Then, there exist $x_1 \in \rho(a)$ and $x_2 \in \rho(b)$ such that $c \in x_1+x_2$. Since ρ is a regular relation, there exists $x \in a+b$ such that $(x,c) \in \rho$. Hence $\rho(a+b) = \{\rho(c) \mid c \in \rho(a)+\rho(b)\}$.
(ii) The proof is similar to (i). ∎

Let I be a non-empty subset of a Γ-semihyperring R. We say that I is a 2-ideal of R if it satisfy the following condition:
(i) $I+R \subseteq I$ and $R+I \subseteq I$,
(ii) $I\alpha R \subseteq I$ and $R\alpha I \subseteq I$ for every $\alpha \in \Gamma$.

A 2-ideal I of R generates a regular relation as follows:

$$x\rho y \Longleftrightarrow x = y \text{ or } x,y \in I.$$

Let (R_1,Γ_1) and (R_2,Γ_2) be two Γ_1- and Γ_2-semihyperrings, respectively and $f : \Gamma_1 \longrightarrow \Gamma_2$ be a map. Then, $\psi : R_1 \longrightarrow R_2$ is called a (Γ_1,Γ_2)-homomorphism (shortly, a homomorphism) if for every $x,y \in R$ and $\alpha \in \Gamma$
(i) $\psi(x+y) = \{\psi(t) \mid t \in x+y\} \subseteq \psi(x)+\psi(y)$,

(ii) $\psi(x\alpha y) = \{\psi(t) \mid t \in x\alpha y\} \subseteq \psi(x)f(\alpha)\psi(y)$,
(iii) $f(x+y) = f(x) + f(y)$.

In the above definition if $\psi(x+y) = \psi(x) + \psi(y)$ and $\psi(x\alpha y) = \psi(x)f(\alpha)\psi(y)$, then ψ is called a *strong homomorphism*. The set $ker\psi = \{(a,b) \in R_1 \times R_2 \mid \psi(a) = \psi(b)\}$ is called the *kernel* of ψ.

Proposition 1.4. *Let ψ be a strong homomorphism. Then,* $\ker\psi$ *is a regular relation on R.*

Proof. It is easy to see that $ker\psi$ is an equivalence relation on R. Let $a,b,x \in R$, $\alpha \in \Gamma$ and $(a,b) \in ker\psi$. Thus, $\psi(a) = \psi(b)$. Now, we have

$$\psi(a+x) = \psi(a) + \psi(x) = \psi(b) + \psi(x) = \psi(b+x),$$
$$\psi(a\alpha x) = \psi(a)f(\alpha)\psi(x) = \psi(b)f(\alpha)\psi(x) = \psi(b\alpha x).$$

Hence for every $z_1 \in a+x$ there exist $z_2 \in b+x$ such that $(z_1,z_2) \in \ker\psi$ and for every $z_3 \in a\alpha x$ there exists $z_4 \in b\alpha x$ such that $(z_3,z_4) \in \ker\psi$. Hence $(a+x)\overline{\ker\psi}(b+x)$ and $(a\alpha x)\overline{\ker\psi}(b\alpha x)$. In the same way, $(x\alpha a)\overline{ker\psi}(x\alpha b)$ which implies that $ker\psi$ is a regular relation. ∎

Example 1.6. Let $S_1 = \bigcup_{n\in\mathbb{Z}} A_n$ and $S_2 = \bigcup_{n\in\mathbb{Z}} B_n$ be the $\mathbb{Z}$-semihyperrings in Example 1.4 and $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ be the identity map, where $A_n = (n,n+1)$ and $B_n = (2n,2n+2)$ are open intervals in $\mathbb{R}$. Then, $\psi : S_1 \longrightarrow S_2$ defined by $\psi(x) = 2x$ is a strong homomorphism.

In the following example ψ is not a strong homomorphism but $ker\psi$ is a regular relation on R.

Example 1.7. In Example 1.4, suppose that $R = \{A_n \mid n \in \mathbb{Z}\}$ where $A_n = [n,n+1)$. Then, R is a $\mathbb{Z}$-semihyperring. For every $x \in R$ there exists $n \in \mathbb{N}$ such that $x \in A_n$. Now, consider $\psi : R \longrightarrow R$ defined by $\psi(x) = n$ where $x \in A_n$ and $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ is the identity map. Then, ψ is a homomorphism.

It is easy to see that in the above example $ker\psi$ is a strong regular relation.

2. Lower and upper approximations

Let R be a Γ-semihyperring. Recall that an equivalence relation ρ on R is a reflexive, symmetric, and transitive binary relation on R. If ρ is an equivalence relation on R then the equivalence class of $x \in R$ is the set $\{y \in R \mid (x,y) \in \rho\}$. We write it as $\rho(x)$. Let A be a non-empty subset of R. Then, the sets

$$\overline{\mathrm{Apr}}_\rho(A) = \{x \mid \rho(x) \cap A \neq \emptyset\} \text{ and } \underline{\mathrm{Apr}}_\rho(A) = \{x \mid \rho(x) \subseteq A\},$$

are called, respectively, the *lower* and *upper approximations* of the set A with respect to ρ. $Apr_\rho(A) = (\overline{\mathrm{Apr}}_\rho(A), \underline{\mathrm{Apr}}_\rho(A))$ is called a *rough set* with respect to ρ. A non-empty subset A of R is called an *upper (lower) rough ideal (sub Γ-semihyperring)* of R if $\overline{\mathrm{Apr}}_\rho(A)$ $(\underline{\mathrm{Apr}}_\rho(A))$ is an ideal (sub Γ-semihyperring)
of R and is called a *rough ideal (sub Γ-semihyperring)* if $\overline{\mathrm{Apr}}_\rho(A)$ and $\underline{\mathrm{Apr}}_\rho(A)$ are ideals (sub Γ-semihyperring) of R .

Theorem 2.1. *Let (ψ, f) be a homomorphism of Γ-semihyperring R_1 to a Γ_2-semihyperring R_2, ρ be an equivalence relation on R_2 and A be a non-empty set of R_1. Then,*

(i) *If ψ is a strong homomorphism and ρ a regular (strongly regular) relation on R_2, then $\Theta = \{(a,b) \in R_1 \times R_1 \mid (\psi(a), \psi(b)) \in \rho\}$ is a regular (strongly regular) relation on R_1.*

(ii) *If ψ is an epimorphism, then $(\psi, f)(\overline{\text{Apr}}_\Theta(A)) = \overline{\text{Apr}}_\rho \psi(A)$.*

(iii) *If ψ is an epimorphism, then $(\psi, f)(\underline{\text{Apr}}_\Theta(A)) \subseteq \underline{\text{Apr}}_\rho \psi(A)$.*

Proof. (i) Let $(a,b) \in \rho$, $x \in R_1$ and $\alpha \in \Gamma$. Then, $(\psi(a), \psi(b)) \in \rho$. Then, $(\psi(a) + \psi(x), \psi(b) + \psi(x)) \in \overline{\rho}$ and $(\psi(a) f(\alpha) \psi(x), \psi(b) f(\alpha) \psi(x)) \in \overline{\rho}$ which implies that $(\psi(a + x), \psi(b + x)) \in \overline{\rho}$ and $(\psi(a\alpha x), \psi(b\alpha x)) \in \overline{\rho}$. Then, $(a+x, b+x) \in \overline{\Theta}$ and $(a\alpha x, b\alpha x) \in \overline{\Theta}$. In the same way one can see that if ρ is a strongly regular relation, then Θ is a strongly regular relation.

(ii) Suppose that y is an element of $(\psi, f)(\overline{\text{Apr}}_\Theta(A))$. Then, there exists $x \in \overline{\text{Apr}}_\Theta(A)$ such that $(\psi, f)(x) = \psi(x) = y$. So, $\Theta(x) \cap A \neq \emptyset$ and there exists $a \in \Theta(x) \cap A$. Then, $\psi(a) \in \psi(A)$ and $(\psi(a), \psi(x)) \in \rho$. So, $\rho(\psi(x)) \cap \psi(A) \neq \emptyset$ which implies that $y = \psi(x) \in \overline{\text{Apr}}_\rho(\psi(A))$. Therefore, we conclude that $(\psi, f)(\overline{\text{Apr}}_\Theta(A)) \subseteq \overline{\text{Apr}}_\rho \psi(A)$. Conversely, let $y \in \overline{\text{Apr}}_\rho \psi(A)$. Then, there exists $x \in R_1$ such that $\psi(x) = y$. Hence $\rho(\psi(x)) \cap \psi(A) \neq \emptyset$. So, there exists $a \in A$ and $\psi(a) \in \rho(\psi(x))$. Now, by definition of Θ we have $(a, x) \in \Theta$. Thus, $\Theta(x) \cap A \neq \emptyset$. Hence $y = \psi(x) \in \psi(\overline{\text{Apr}}_\Theta(A))$. This completes the proof.

(iii) The proof is similar to (ii). ∎

Proposition 2.1. *Let R_1 and R_2 be two Γ_1 and Γ_2-semihyperring respectively and ψ be a homomorphism from R_1 to R_2. If A is a non-empty subset of R_1, then $\psi(\overline{\text{Apr}}_{ker\psi}(A)) = \psi(A)$.*

Proof. Since $A \subseteq \overline{\text{Apr}}_{ker\psi}(A)$, it follows that $\psi(A) \subseteq \psi(\overline{\text{Apr}}_{ker\psi}(A))$. Conversely, suppose that $y \in \psi(\overline{\text{Apr}}_{ker\psi}(A))$. Then, there exists $x \in \overline{\text{Apr}}_{ker\psi}(A)$ such that $\psi(x) = y$, so there exists an element $a \in A$ such that $\psi(x) = \psi(a)$. Therefore, $y = \psi(x) = \psi(a) \in \psi(A)$. This completes the proof. ∎

Example 2.1. Let R be the Γ-semihyperring in Example 1.7. If $A = \{1,2\}$, then $\overline{\text{Apr}}_{ker\psi}(A) = [1,2) \cup [2,3)$ and $\underline{\text{Apr}}_{ker\psi}(A) = \emptyset$. Hence $\psi(\overline{\text{Apr}}_{ker\psi}(A)) = \psi(A) = \{1,2\}$ and $\psi(\underline{\text{Apr}}_{ker\psi}(A)) \subseteq \psi(A)$. If $B = [0,1)$, then $\underline{\text{Apr}}_{ker\psi}(B) = \overline{\text{Apr}}_{ker\psi}(B) = [0,1)$.

Theorem 2.2. *Let ρ be a regular relation on a Γ-semihyperring R and A, B be two non-empty subsets of R. Then,*

(i) $\overline{\text{Apr}}_\rho(A) \Gamma \overline{\text{Apr}}_\rho(B) \subseteq \overline{\text{Apr}}_\rho(A \Gamma B)$.

(ii) *If ρ is semi-complete, then $\underline{\text{Apr}}_\rho(A) \Gamma \underline{\text{Apr}}_\rho(B) \subseteq \underline{\text{Apr}}_\rho(A \Gamma B)$.*

Proof. (i) Let $x \in \overline{\text{Apr}}_\rho(A) \Gamma \overline{\text{Apr}}_\rho(B)$. Then, $x \in x_1 \alpha x_2$ with $x_1 \in \overline{\text{Apr}}_\rho(A)$, $x_2 \in \overline{\text{Apr}}_\rho(B)$ and $\alpha \in \Gamma$. Hence there exist $a \in A$ and $b \in B$ such that $(a, x_1) \in \rho$ and $(b, x_2) \in \rho$. Since ρ is a regular relation, it follows that $a\alpha b \subseteq \rho(x_1)\alpha\rho(x_2) \subseteq \rho(x_1 \alpha x_2)$. So, $\rho(x_1 \alpha x_2) \cap A\Gamma B \neq \emptyset$ which implies that $\overline{\text{Apr}}_\rho(A) \Gamma \overline{\text{Apr}}_\rho(B) \subseteq \overline{\text{Apr}}_\rho(A \Gamma B)$.

(ii) Let $x \in \underline{\text{Apr}}_\rho(A) \Gamma \underline{\text{Apr}}_\rho(B)$. Then, $x \in x_1 \alpha x_2$ with $x_1 \in \underline{\text{Apr}}_\rho(A)$ and $x_2 \in \underline{\text{Apr}}_\rho(B)$ and $\alpha \in \Gamma$. It follows that $\rho(x_1) \subseteq A$ and $\rho(x_2) \subseteq B$. Since ρ is a semi-complete relation, we have

$$\rho(x) \in \rho(x_1 \alpha x_2) \subseteq \rho(x_1) \alpha \rho(x_2) \subseteq A \Gamma B,$$

which implies that $x \in \underline{\text{Apr}}_\rho(A \Gamma B)$. This completes the proof. ∎

Example 2.2. Let R be the Γ-semihyperring in Example 1.7. If $A=\{0\}$ and $B=[1,2]$ are subsets of R, then $A\Gamma B=A_0=[0,1)$. Also, $\underline{\mathrm{Apr}}_{ker\psi}(A)=\emptyset$ and $\underline{\mathrm{Apr}}_{ker\psi}(B)=[0,1)=A_0$ and so $\underline{\mathrm{Apr}}_{ker\psi}(A)\Gamma\underline{\mathrm{Apr}}_{ker\psi}(B)=\emptyset\Gamma A_0=\emptyset$. Then, $\underline{\mathrm{Apr}}_{ker\psi}(A)\Gamma\underline{Apr}_{ker\psi}(B)\subset\underline{Apr}_{ker\psi}(A\Gamma B)$.

Proposition 2.2. *Let ρ be a regular relation on a Γ-semihyperring R and A,B be two non-empty subsets of R. Then,*

(i) $\overline{\mathrm{Apr}}_\rho(A)+\overline{\mathrm{Apr}}_\rho(B)\subseteq\overline{\mathrm{Apr}}_\rho(A+B)$.

(ii) *If ρ is semi-complete relation, then* $\underline{\mathrm{Apr}}_\rho(A)+\underline{\mathrm{Apr}}_\rho(B)\subseteq\underline{\mathrm{Apr}}_\rho(A+B)$.

Proof. (i) Let $x\in\overline{\mathrm{Apr}}_\rho(A)+\overline{\mathrm{Apr}}_\rho(B)$. Then, $x\in x_1+x_2$ with $x_1\in\overline{\mathrm{Apr}}_\rho(A)$ and $x_2\in\overline{\mathrm{Apr}}_\rho(B)$. Hence there exist $a\in A$ and $b\in B$ such that $(x_1,a)\in\rho$ and $(x_2,b)\in\rho$. Since ρ is a regular relation, there exists $y\in a+b$ such that $(x,y)\in\rho$. Since $a+b\subseteq A+B$, it follows that $\rho(x)\cap(A+B)\neq\emptyset$. This implies that $\overline{\mathrm{Apr}}_\rho(A)+\overline{\mathrm{Apr}}_\rho(B)\subseteq\overline{\mathrm{Apr}}_\rho(A+B)$.

(ii) Let $x\in\underline{\mathrm{Apr}}_\rho(A)+\underline{\mathrm{Apr}}_\rho(B)$. Then, $x\in x_1+x_2$ with $x_1\in\underline{\mathrm{Apr}}_\rho(A)$ and $x_2\in\underline{\mathrm{Apr}}_\rho(B)$. It follows that $\rho(x_1)\subseteq A$ and $\rho(x_2)\subseteq B$. Since ρ is a semi-complete relation, we have $\rho(x)\in\rho(x_1+x_2)\subseteq\rho(x_1)+\rho(x_2)\subseteq A+B$, and so $x\in\underline{\mathrm{Apr}}_\rho(A+B)$. Hence $\underline{\mathrm{Apr}}_\rho(A)+\underline{\mathrm{Apr}}_\rho(B)\subseteq\underline{\mathrm{Apr}}_\rho(A+B)$. ∎

Example 2.3. Let R be the Γ-semihyperring in Example 1.7, $A=\{0\}$ and $B=\{1\}$. Then, $\overline{\mathrm{Apr}}_{ker\psi}(A)+\overline{\mathrm{Apr}}_{ker\psi}(B)=\overline{\mathrm{Apr}}_{\ker\psi}(A+B)=[1,2)$ and $\emptyset=\underline{\mathrm{Apr}}_{ker\psi}(A)+\underline{\mathrm{Apr}}_{ker\psi}(B)\subset\underline{\mathrm{Apr}}_{ker\psi}(A+B)=[1,2)$.

Proposition 2.3. *Let R be a Γ-semihyperring and ρ be a regular and semi-complete relation on R. If A,B are non-empty subsets of R, then*

(i) $\overline{\mathrm{Apr}}_\rho(A)+\overline{\mathrm{Apr}}_\rho(B)=\overline{\mathrm{Apr}}_\rho(A+B)$.

(ii) $\overline{\mathrm{Apr}}_\rho(A)\Gamma\overline{\mathrm{Apr}}_\rho(B)=\overline{\mathrm{Apr}}_\rho(A\Gamma B)$.

Proof. (i) Let $x\in\overline{\mathrm{Apr}}_\rho(A+B)$. Then, $\rho(x)\cap(A+B)\neq\emptyset$. Therefore, there exists $y\in\rho(x)\cap(A+B)$, and so for some $a\in A$ and $b\in B$, we have $y\in a+b$. Now, we have $x\in\rho(y)\subseteq\rho(a+b)\subseteq\rho(a)+\rho(b)$. Thus, there exist $x_1\in\rho(a)$ and $x_2\in\rho(b)$ such that $x\in x_1+x_2$. So, $\rho(x_1)\cap A\neq\emptyset$ and $\rho(x_2)\cap B\neq\emptyset$. Hence $x_1\in\overline{\mathrm{Apr}}_\rho(A)$ and $x_2\in\overline{\mathrm{Apr}}_\rho(B)$ which implies that $x\in\overline{\mathrm{Apr}}_\rho(A)+\overline{\mathrm{Apr}}_\rho(B)$. Now, by Proposition 2.2, we have $\overline{\mathrm{Apr}}_\rho(A)+\overline{\mathrm{Apr}}_\rho(B)=\overline{\mathrm{Apr}}_\rho(A+B)$.

(ii) The proof is similar to (i) ∎

Theorem 2.3. *Let ρ be a regular relation on a Γ-semihyperring R. If R_1 is a sub Γ-semihyperring of R, then*

(i) *R_1, is an upper rough sub Γ-semihyperring.*

(ii) *If ρ is semi-complete relation, then R is a lower rough sub Γ-semihyperring.*

Proof. (i) By Theorem 2.2 and Proposition 2.2, $\overline{\mathrm{Apr}}_\rho(R_1)+\overline{\mathrm{Apr}}_\rho(R_1)\subseteq\overline{\mathrm{Apr}}_\rho(R_1)$ and $\overline{\mathrm{Apr}}_\rho(R_1)\Gamma\overline{\mathrm{Apr}}_\rho(R_1)\subseteq\overline{\mathrm{Apr}}_\rho(R_1)$. Hence $\overline{\mathrm{Apr}}_\rho(R_1)$ is a sub Γ-semihyperring.

(ii) By Propositions 2.2 and 2.2 the proof is easy. ∎

Proposition 2.4. *Let ρ be a regular relation on a Γ-semihyperring R. If I is an ideal of R, then*

(i) *I is an upper rough ideal.*

(ii) *If ρ is a semi-complete relation, then I is a lower rough ideal.*

Proof. The proof is straightforward. ∎

Example 2.4. Let R be the Γ-semihyperring in Example 1.7 and $I = \bigcup_{n\in 2\mathbb{Z}} A_n$. Then, I is an ideal of R and $\overline{\mathrm{Apr}}_{ker\psi}(I)$, $\underline{\mathrm{Apr}}_{ker\psi}(I)$ are ideals.

Proposition 2.4, shows that the notion of an upper rough ideal is an extended notion of a usual ideal of a ring. It is not difficult to see that the converse of Proposition 2.4 does not hold in general.

Example 2.5. Let R be the Γ-semihyperring in Example 1.7 and $A = \{0\}$. Then, $\overline{\mathrm{Apr}}_{ker\psi}(A)$ is an ideal of R but $A = \{0\}$ is not an ideal of R.

Corollary 2.1. *Let R be a Γ-semihyperring, ρ be a regular relation and I be a 2-ideal of R. Then, $\overline{\mathrm{Apr}}_\rho(I)$ is a 2-ideal of R and if ρ is a semi-complete regular relation, then $\underline{\mathrm{Apr}}_\rho(I)$ is a 2-ideal.*

Theorem 2.4. *Let ρ be a regular relation on R. If I and J are two right and left ideals of R, respectively, then*

$$\overline{\mathrm{Apr}}_\rho(I\Gamma J) \subseteq \overline{\mathrm{Apr}}_\rho(I) \cap \overline{\mathrm{Apr}}_\rho(J) \text{ and } \underline{\mathrm{Apr}}_\rho(I\Gamma J) \subseteq \underline{\mathrm{Apr}}_\rho(I) \cap \underline{\mathrm{Apr}}_\rho(J).$$

Proof. Let I and J are a right ideal and a left ideal of R, respectively. Then, $I\Gamma J \subseteq I\Gamma R \subseteq I$ and $I\Gamma J \subseteq R\Gamma J \subseteq J$. Hence $I\Gamma J \subseteq I \cap J$. Then, we have

$$\overline{\mathrm{Apr}}_\rho(I\Gamma J) \subseteq \overline{\mathrm{Apr}}_\rho(I \cap J) \subseteq \overline{\mathrm{Apr}}_\rho(I) \cap \overline{\mathrm{Apr}}_\rho(J)$$

and

$$\underline{\mathrm{Apr}}_\rho(I\Gamma J) \subseteq \underline{\mathrm{Apr}}_\rho(I) \cap \underline{\mathrm{Apr}}_\rho(J).$$ ∎

Example 2.6. Let R be the Γ-semihyperring in Example 1.7. We know that $I = \bigcup_{n\in 2\mathbb{Z}} A_n$ is an ideal of R and $\overline{\mathrm{Apr}}_{ker\psi}(I) = I$. But $\overline{\mathrm{Apr}}_{ker\psi}(I\Gamma I) \subset \overline{\mathrm{Apr}}_{ker\psi}(I)$.

In the following example we show that in Theorem 2.4, I and J must be ideals.

Example 2.7. Let R be the Γ-semihyperring in Example 1.7, $A = \{0\}$ and $B = \{1\}$. It easy to see that A and B are not ideals of R. We know $\underline{\mathrm{Apr}}_{ker\psi}(A) = \underline{\mathrm{Apr}}_{ker\psi}(B) = \emptyset$ and $\underline{\mathrm{Apr}}_{ker\psi}(A\Gamma B) = A$.

Proposition 2.5. *Let R be a commutative Γ-semihyperring with zero, ρ a semi-complete regular relation on R and P be a prime ideal of R. Then, $\underline{\mathrm{Apr}}_\rho P$ is a prime ideal of R or the empty set.*

Proof. By Proposition 2.4, $\underline{\mathrm{Apr}}_\rho(P)$ is an ideal of R. Suppose that $\underline{\mathrm{Apr}}_\rho(P)$ is not a prime ideal of R. There exist ideals A, B of R such that $A\Gamma B \subseteq \underline{\mathrm{Apr}}_\rho(P)$ and $A \nsubseteq \underline{\mathrm{Apr}}_\rho(P)$, and $B \nsubseteq \underline{\mathrm{Apr}}_\rho(P)$. Then, there exist $a \in A \setminus \underline{\mathrm{Apr}}_\rho(P)$ and $b \in B \setminus \underline{\mathrm{Apr}}_\rho(P)$. Hence there exist $x_1 \in \rho(a) \setminus P$ and $x_2 \in \rho(b) \setminus P$. Since ρ is a regular relation,

$$x_1\Gamma x_2 \subseteq \rho(a)\Gamma\rho(b) \subseteq \rho(a\Gamma b) \subseteq P.$$

This implies that $\langle x_1\rangle\Gamma\langle x_2\rangle \subseteq P$. Since P is a prime ideal of R, $x_1 \in P$ or $x_2 \in P$. It is a contradiction. Then, $\underline{\mathrm{Apr}}_\rho(P)$ is a prime ideal of R. ∎

Example 2.8. Let R be the Γ-semihyperring in Example 1.7. Then, $P = A_0$ is a prime ideal of R.

3. Approximation in multiplicative Γ-semihyperring

In this section, we assume that R is a multiplicative Γ-semihyperring with zero. Let I be an ideal of R. One can see that the following relation is a regular relation:

$$(x,y) \in \rho_I \Longleftrightarrow \exists\, a,b \in I \text{ such that, } x+a = y+b.$$

An ideal I of a Γ-semihyperring R is called *subtractive* if $a \in I,\ a+b \subseteq I$, implies that $b \in I$.

Example 3.1. Let R be a ring and I, Γ two subgroups of $(R,+)$ such that $I\Gamma = \Gamma I = 0$. Then, R is a multiplicative Γ-semihyperring with respect to the following hyperoperation:

$$x \cdot \alpha \cdot y = x\alpha y + I.$$

Example 3.2. Let $R = \mathbb{Z}_6$, $\Gamma = \{\overline{0},\overline{3}\}$ and $I = \{\overline{0},\overline{2},\overline{4}\}$. Then, R is a multiplicative Γ-semihyperring with respect to the above hyperoperation.

Proposition 3.1. *Let R be a multiplicative Γ-semihyperring, J be a subtractive ideal of R and I an ideal of R contained in J. Then, $\overline{\mathrm{Apr}}_{\rho_I}(J) = J$ and if $\underline{\mathrm{Apr}}_{\rho_I}(J) \neq \emptyset$, then $\underline{\mathrm{Apr}}_{\rho_I}(J) = J$.*

Proof. Suppose that $x \in \overline{\mathrm{Apr}}_{\rho_I}(J)$ then there exist $y \in J$ and $x_1,x_2 \in I$ such that $y+x_1 = x+x_2$. Since $I \subseteq J$ and J is a subtractive ideal of R, $x \in J$. Hence $\overline{\mathrm{Apr}}_{\rho_I}(J) = J$. In the same way we can prove that $\underline{\mathrm{Apr}}_{\rho_I}(J) = J$. ∎

Theorem 3.1. *Let (R_1,Γ_1) and (R_2,Γ_2) be two multiplicative Γ_1 and Γ_2-semihyperrings and $(\psi,f) : (R_1,\Gamma_1) \longrightarrow (R_2,\Gamma_2)$ be a strong isomorphism. If I is an ideal of R_1 and A is a non-empty subset of R_1, then*

(i) $\overline{\mathrm{Apr}}_{\rho_{\psi(I)}}\psi(A) = \psi(\overline{\mathrm{Apr}}_{\rho_I}(A))$,

(ii) $\underline{\mathrm{Apr}}_{\rho_{\psi(I)}}\psi(A) = \psi(\underline{\mathrm{Apr}}_{\rho_I}(A))$.

Proof. (i) Suppose that y is an element of $\psi(\overline{\mathrm{Apr}}_{\rho_I}(A))$. Then, there exists $x \in \overline{\mathrm{Apr}}_{\rho_I}(A)$ such that $\psi(x) = y$. Hence $\rho_I(x) \cap A \neq \emptyset$ and so there exists $a \in \rho_I(x) \cap A$. Then, $\psi(a) \in \psi(A)$ and there exist $x_1,x_2 \in I$ such that $a+x_1 = x+x_2$. Hence $\psi(a)+\varphi(x_1) = y+\psi(x_2)$. Then, $y \in \overline{\mathrm{Apr}}_{\rho_{\psi(I)}}\psi(A)$.

Conversely, let $y \in \overline{\mathrm{Apr}}_{\rho_{\psi(I)}}\psi(A)$. Since ψ is onto, there exists $x \in R_1$ such that $y = \psi(x)$. Hence there exist $x_1,x_2 \in I$ such that $\psi(x)+\psi(x_1) = \psi(x_2)+\psi(a)$. Since ψ is one-one, $\psi(x+x_1) = \psi(x_2+a)$ implies that $x+x_1 = x_2+a$. Since $x_1,x_2 \in I$ and $a \in A$, then $x \in \overline{\mathrm{Apr}}_{\rho_I}(A)$. It means $\overline{\mathrm{Apr}}_{\rho_{\psi(I)}}\psi(A) \subseteq \psi(\overline{\mathrm{Apr}}_{\rho_I}(A))$.

(ii) Suppose that y is an element of $\psi(\underline{\mathrm{Apr}}_{\rho_I}(A))$. Then, there exists $x \in \underline{\mathrm{Apr}}_{\rho_I}(A)$ such that $y = \psi(x)$. We have $\rho_I(x) \subseteq A$. We show that $\psi(\rho_I(x)) = \rho_{\psi(I)}\psi(x)$. Let $z \in \psi(\rho_I(x))$. Then, there exists $t \in \rho_I(x)$ such that $z \in \psi(x)$. As $t \in \rho_I(x)$ there exist $x_1,x_2 \in I$ such that $t+x_1 = x+x_2$ and then $z+\psi(x_1) = \psi(x)+\psi(x_2)$ such that $z = \psi(t)$. Hence $\psi(\rho_I(x)) \subseteq \rho_{\psi(I)}\psi(x)$. Let $z_1 \in \rho_{\psi(I)}\psi(x)$. Then, there exist $x_1,x_2 \in I$ and $r \in R$ such that

$$z_1 = \psi(r) \text{ and } z_1+\psi(x_1) = \psi(x)+\psi(x_2).$$

Since ψ is an isomorphism, $r \in \rho_I(x)$. Therefore, $\psi(\rho_I(x)) = \rho_{\psi(I)}\psi(x)$. This implies that $\psi(\underline{\mathrm{Apr}}_{\rho_I}(A)) \subseteq \underline{\mathrm{Apr}}_{\rho_{\psi(I)}}\psi(A)$. Conversely, let $y \in \underline{\mathrm{Apr}}_{\rho_{\psi(I)}}\psi(A)$. Then, there exist $x_1, x_2 \in I$ such that $y + \psi(x_1) = \psi(x) + \psi(x_2)$. Since ψ is onto, then $y = \psi(t)$ for some $t \in R$. Since $\psi(t + x_1) = \psi(x + x_2)$, we have $t + x_1 = x + x_2$. Hence $t \in \rho_I(x)$. Then, $y = \varphi(t) \in \psi(\underline{\mathrm{Apr}}_{\rho_I}(A))$ and so $\underline{\mathrm{Apr}}_{\rho_{\psi(I)}}\psi(A) \subseteq \psi(\underline{\mathrm{Apr}}_{\rho_I}(A))$. ∎

Theorem 3.2. *Let (R_1,Γ_1) and (R_2,Γ_2) be two Γ_1 and Γ_2-semihyperrings respectively and $(\varphi, f) : (R_1,\Gamma_1) \longrightarrow (R_2,\Gamma_2)$ be a (Γ_1,Γ_2)-isomorphism. If I is an ideal of R_2 and A is a non-empty subset of (R_1,Γ_1), then*

$$(\varphi^{-1}, f^{-1})(\overline{\mathrm{Apr}}_{\rho_I}(\varphi(A)) = \overline{\mathrm{Apr}}_{\rho_{\varphi^{-1}(I)}}(A).$$

Proof. The proof is straightforward. ∎

Corollary 3.1. *Let (R_1,Γ_1) and (R_2,Γ_2) be two Γ_1 and Γ_2-semihyperrings respectively and (φ, f) be an isomorphism from (R_1,Γ_1) to (R_2,Γ_2). If I is an ideal of (R_2,Γ_2) and A is a non-empty subset of R_1, then*

(i) *$\overline{\mathrm{Apr}}_{\rho_{\varphi}^{-1}(I)}(A)$ is an ideal of (R_1,Γ_1) if and only if $\overline{\mathrm{Apr}}_{\rho_I}\varphi(A)$ is an ideal of (R_2,Γ_2).*

(ii) *$\overline{\mathrm{Apr}}_{\rho_{\varphi}^{-1}(I)}(A)$ is a prime ideal of (R_1,Γ_1) if and only if $\overline{\mathrm{Apr}}_{\rho_I}\varphi(A)$ is a prime ideal of (R_2,Γ_2).*

4. Rough sets in a quotient Γ-semihyperring

Let ρ be a regular relation on a Γ-semihyperring R and $A \subseteq R$. Then, $R/\rho = \{\rho(a) \mid a \in R\}$ is a $\widehat{\Gamma}$-semihyperring where $\widehat{\Gamma} = \{\hat{\alpha} \mid \alpha \in \Gamma\}$. Let $\rho(a),\ \rho(b) \in R/\rho$. We define $\rho(a) \oplus \rho(b) = \rho(a+b)$ and $\rho(a)\hat{\alpha}\rho(b) = \rho(a\alpha b)$. One can see that R/ρ is a Γ-semihyperring. Let A be a non-empty subset of R. Then,

$$\overline{\overline{\mathrm{Apr}}}_{\rho}A = \{\rho(a) \mid \rho(a) \cap A \neq \varnothing\} \text{ and } \underline{\underline{\mathrm{Apr}}}_{\rho}A = \{\rho(a) \mid \rho(a) \subseteq A\}.$$

Example 4.1. Let R be a Γ-semihyperring in Example 1.7. If $A = \{0,1\}$ is a subset of R, then $\overline{\overline{\mathrm{Apr}}}_{\ker\psi}(A) = [0,1) \cup [1,2)$ and $\underline{\underline{\mathrm{Apr}}}_{\ker\psi}(A) = \emptyset$. If $A = [0,1]$ is a subset of R, then $\overline{\overline{\mathrm{Apr}}}_{\ker\psi}(A) = [0,1) \cup [1,2)$ and $\underline{\underline{\mathrm{Apr}}}_{\ker\psi}(A) = [0,1)$. If $A = [2,3)$ is a subset of R, then $\overline{\overline{\mathrm{Apr}}}_{\ker\psi}(A) = \underline{\underline{\mathrm{Apr}}}_{\ker\psi}(A)$.

Proposition 4.1. *Let ρ be a regular relation on Γ-semihyperring R. Then, the following statement are true.*

(i) *If I is an ideal of R, then $\overline{\overline{\mathrm{Apr}}}_{\rho}(I)$is an ideal of R/ρ,*

(ii) *Let ρ be a semi-complete regular relation and I be an ideal of R, then $\underline{\underline{\mathrm{Apr}}}_{\rho}(I)$ is an ideal of R/ρ.*

Proof. (i) Let $\rho(x_1),\ \rho(x_2) \in \overline{\overline{\mathrm{Apr}}}_{\rho}(I)$. Then, $\rho(x_1) \cap I \neq \emptyset$ and $\rho(x_2) \cap I \neq \emptyset$. This implies that $x_1, x_2 \in \overline{\mathrm{Apr}}_{\rho}(I)$. Since $\overline{\mathrm{Apr}}_{\rho}(I)$ is an ideal of R, $x_1 + x_2 \subseteq \overline{\mathrm{Apr}}_{\rho}(I)$. Then, for every $t \in x_1 + x_2$, $\rho(t) \cap I \neq \emptyset$. Hence $\rho(x_1) \oplus \rho(x_2) \subseteq \overline{\overline{\mathrm{Apr}}}_{\rho}I$. Let $\rho(x) \in \overline{\overline{\mathrm{Apr}}}_{\rho}(I)$, $\rho(y) \in R/\rho$ and $\alpha \in \Gamma$. Then, $\rho(x) \cap I \neq \emptyset$. This implies that $x \in \overline{\mathrm{Apr}}_{\rho}(I)$. Since $\overline{\mathrm{Apr}}_{\rho}(I)$ is an ideal of R, $x\alpha y \subseteq \overline{\mathrm{Apr}}_\rho(I)$. Then, $\rho(x\alpha y)\cap I \neq \emptyset$. Then, for every $t \in x\alpha y$, $\rho(t)\cap I \neq \emptyset$ which implies that $\rho(x)\hat{\alpha}\rho(y) \subseteq \overline{\overline{\mathrm{Apr}}}_\rho I$. Similarly, $\rho(y)\hat{\alpha}\rho(x) \subseteq \overline{\overline{\mathrm{Apr}}}_\rho I$. Then, $\overline{\overline{\mathrm{Apr}}}_\rho I$ is an ideal of R/ρ

(ii) Let $\rho(x_1), \rho(x_2) \in \underline{\underline{\mathrm{Apr}}}_\rho I$. Then, $\rho(x_1) \subseteq I$ and $\rho(x_2) \subseteq I$. This implies that $x_1,\ x_2 \in \underline{Apr_\rho I}$. Since $\underline{Apr_\rho I}$ is an ideal of R, $x_1+x_2 \subseteq \underline{Apr_\rho I}$. We have $\rho(x_1)\oplus\rho(x_2) = \rho(x_1+x_2) \subseteq I$. Hence $\rho(x_1)\oplus\rho(x_2) \subseteq \underline{\underline{\mathrm{Apr}}}_\rho I$. Let $\rho(x) \in \underline{\underline{\mathrm{Apr}}}_\rho(I)$, $\rho(y) \in R/\rho$ and $\alpha \in \Gamma$. This implies that $x \in \underline{\mathrm{Apr}}_\rho(I)$. Since $\underline{\mathrm{Apr}}_\rho(I)$ is an ideal of R, $x\alpha y \subseteq \underline{\mathrm{Apr}}_\rho(I)$. Then, $\rho(x\alpha y) = \{\rho(t) \mid t \in x\alpha y\} \subseteq \underline{\mathrm{Apr}}_\rho(I)$. We have $\rho(x)\hat{\alpha}\rho(y) = \rho(x\alpha y) \subseteq I$. Then, $\rho(x)\hat{\alpha}\rho(y) \subseteq \underline{\mathrm{Apr}}_\rho(I)$. Similarly, $\rho(y)\hat{\alpha}\rho(x) = \rho(y\alpha x) \subseteq \underline{\underline{\mathrm{Apr}}}_\rho(I) \subseteq I$. Then, $\underline{\underline{\mathrm{Apr}}}_\rho(I)$ is an ideal of R/ρ. ∎

Proposition 4.2. *Let R be a commutative Γ-semihyperring, ρ be a semi-complete regular relation and P is a prime ideal of R. Then, $\underline{\underline{\mathrm{Apr}}}_\rho(P)$ is a prime ideal of R/ρ.*

Proof. By Proposition 4.1, $\underline{\underline{\mathrm{Apr}}}_\rho(P)$ is an ideal of R/ρ. Suppose that $\underline{\underline{\mathrm{Apr}}}_\rho(P)$ is not prime ideal of R. Then, there exist $A,B \subseteq R/\rho$, such that A and B are ideals, $A\widehat{\Gamma}B \subseteq \underline{\underline{\mathrm{Apr}}}_\rho(P)$ and $A \not\subseteq \underline{\mathrm{Apr}}_\rho(P)$ and $B \not\subseteq \underline{\mathrm{Apr}}_\rho(P)$. Then, there exist $\rho(a) \in A\setminus\underline{\underline{\mathrm{Apr}}}_\rho(P)$ and $\rho(b) \in B\setminus \underline{\underline{\mathrm{Apr}}}_\rho(P)$. Since $\rho(a)\hat{\alpha}\rho(b) \subseteq A\widehat{\Gamma}B \subseteq \underline{\underline{\mathrm{Apr}}}_\rho(P)$ and $\underline{\mathrm{Apr}}_\rho(P)$ is a prime ideal, $a \in \underline{\mathrm{Apr}}_\rho(P)$ or $b \in \underline{\mathrm{Apr}}_\rho(P)$. It is contradiction, since $\rho(a) \not\subseteq P$ and $\rho(b) \not\subseteq P$. Then, $\underline{\underline{\mathrm{Apr}}}_\rho(P)$ is a prime ideal of R/ρ. ∎

Theorem 4.1. *Let (R_1,Γ_1) and (R_2,Γ_2) be multiplicative Γ_1 and Γ_2-semihyperring with zero respectively, $(\varphi,f) : (R_1,\Gamma_1) \longrightarrow (R_2,\Gamma_2)$ be a strong (Γ_1,Γ_2)-epimorphism and A be a sub Γ_1-semihyperring R_1. Then, (φ,f) induces a (Γ_1,Γ_2)-homomorphism betweens $\overline{\overline{\mathrm{Apr}}}_{\rho_I}(A)$ and $\overline{\overline{\mathrm{Apr}}}_{\rho_{\varphi(I)}}(A)$.*

Proof. One can see that $\varphi(A)$ is a sub Γ_2-semihyperring of R_2. We know that $\overline{\overline{\mathrm{Apr}}}_{\rho_I}(A)$ is a sub $\widehat{\Gamma}_1$-semihyperring of R_1/ρ_I and $\overline{\overline{\mathrm{Apr}}}_{\rho_{\varphi(I)}}\varphi(A)$ is a sub $\widehat{\Gamma}_2$-semihyperring of $R_2/\rho_{\varphi(I)}$. we define

$$\begin{aligned}&\overline{\varphi} : \overline{\overline{\mathrm{Apr}}}_\rho(A) \longrightarrow \overline{\overline{\mathrm{Apr}}}_{\rho_{\varphi(I)}}\varphi(A),\\ &\rho_I(x) \longmapsto \rho_{\varphi(I)}\varphi(x)\end{aligned}$$

and

$$\begin{aligned}&\overline{f} : \overline{\Gamma_1} \longrightarrow \overline{\Gamma_2}\\ &\hat{\alpha} \longrightarrow \widehat{f(\alpha)}\end{aligned}$$

We prove that $\overline{\varphi}$ is well-defined function. Suppose that $\rho_I(x) = \rho_I(y)$, then there exist $x_1,x_2 \in I$ such that $x+x_1 = y+y_1$. Since (φ,f) is a strong homomorphism, $\varphi(x)+\varphi(x_1) = \varphi(y)+\varphi(y_1)$. Therefore, $\rho_{\varphi(I)}\varphi(x) = \rho_{\varphi(I)}\varphi(y)$. Thus, $\overline{\varphi}$ is well-defined. Now we have

$$\begin{aligned}\overline{\varphi}(\rho_I(x)\oplus\rho_I(y)) &= \overline{\varphi}(\rho_I(x+y)) = \rho_{\varphi(I)}\varphi(x+y)\\ &= \rho_{\varphi(I)}(\varphi(x)+\varphi(y)) = \rho_{\varphi(I)}\varphi(x)\oplus\rho_{\varphi(I)}\varphi(y)\\ &= \overline{\varphi}(\rho_I(x))\oplus\overline{\varphi}(\rho_I(y)),\end{aligned}$$

and

$$\begin{aligned}\overline{\varphi}(\rho_I(x)\hat{\alpha}\rho_I(y)) &= \overline{\varphi}(\rho_I(x\alpha y)) = \overline{\varphi}\{\rho_I(t) \mid t \in x\alpha y\}\\ &= \{\rho_{\varphi(I)}\varphi(t) \mid t \in x\alpha y\} = \rho_{\varphi(I)}\varphi(x\alpha y)\\ &= \rho_{\varphi(I)}(\varphi(x)\widehat{f(\alpha)}\varphi(y)) = \rho_{\varphi(I)}(\varphi(x))\widehat{f(\alpha)}\rho_{\varphi(I)}(\varphi(y))\\ &= \overline{\varphi}(\rho_I(x))\widehat{f(\alpha)}\overline{\varphi}(\rho_I(y)).\end{aligned}$$

Hence $(\overline{\varphi},\overline{f})$ is a $(\widehat{\Gamma}_1,\widehat{\Gamma}_2)$-homomorphism. ∎

Corollary 4.1. *Let $(\varphi,f):(R_1,\Gamma_1)\longrightarrow(R_2,\Gamma_2)$ be a strong isomorphism. Then, in Theorem 4.3 induced homomorphism $(\overline{\varphi},\overline{f})$ is isomorphism.*

Proof. Suppose that $(\overline{\varphi},\overline{f})(\rho_I(x)) = (\overline{\varphi},\overline{f})(\rho_I(y))$. Then, there exist $x_1,x_2\in I$ such that $\varphi(x)+\varphi(x_1)=\varphi(y)+\varphi(x_2)$. Since (φ,f) is a strong isomorphism, $\varphi(x+x_1)=\varphi(y+x_2)$. Thus, $x+x_1=y+x_2$ which implies that $\rho_I(x)=\rho_I(y)$. Then, $(\overline{\varphi},\overline{f})$ is isomorphism. ∎

Proposition 4.3. *Let R be a Γ-semihyperring and equivalence relation ξ on R defined by the rule that $(a,b)\in\xi$ if and only if $a=b$ or $a\in R\Gamma b$, $b\in R\Gamma a$, $a\in R+b$ and $b\in R+a$. Assume that $\rho\subseteq\xi$ be a regular relation. Then, $(a,b)\in\xi$ if and only if $(\rho(a),\rho(b))\in\xi$ in R/ρ.*

Proof. Let $a,b\in R$ such that $(a,b)\in\xi$. Then, $a=b$ or $a\in R\Gamma b$, $b\in R\Gamma a$, $a\in R+b$ and $a\in R+b$. If $a=b$, then $\rho(a)=\rho(b)$. Hence $(\rho(a),\rho(b))\in\xi$. In the second case there exists $x_1,x_2,x_3,x_4\in R$ and $\alpha,\beta\in\Gamma$ such that $a\in x_1\alpha b$, $b\in x_2\beta a$, $a\in x_3+b$ and $b\in x_4+a$. Then, $\rho(a)\in\rho(x_1\alpha b)$, $\rho(b)\in\rho(x_2\beta a)$, $\rho(a)\in\rho(x_3+b)$ and $\rho(b)\in\rho(x_4+a)$. So, $\rho(a)\in\rho(x_1)\hat{\alpha}\rho(b)$, $\rho(b)\in\rho(x_2)\hat{\beta}\rho(a)$, $\rho(a)\in\rho(x_3)\oplus\rho(b)$ and $\rho(a)\in\rho(x_4)\oplus\rho(a)$. Hence, $(\rho(a),\rho(b))\in\xi$ in R/ρ.

Conversely, let $a,b\in R$ and $(\rho(a),\rho(b))\in\xi$. Then, $\rho(a)=\rho(b)$ or $\rho(a)\in R/\rho\widehat{\Gamma}\rho(b)$, $\rho(b)\in R/\rho\widehat{\Gamma}\rho(a)$, $\rho(a)\in R/\rho\oplus\rho(b)$ and $\rho(a)\in R/\rho\oplus\rho(b)$. If $\rho(a)=\rho(b)$, since $\rho\subseteq\xi$, then $(a,b)\in\xi$. In the second case, $\rho(a)\in\rho(R\Gamma b)$, $\rho(b)\in\rho(R\Gamma a)$, $\rho(a)\in\rho(R+b)$, $\rho(b)\in\rho(R+a)$. Then, $(a,b)\in\xi$. ∎

Corollary 4.2. *Let ρ be a regular relation on Γ-semihyperring R such that $\rho\subseteq\xi$ and A be a non-empty subset of R. Then, $\overline{\overline{\mathrm{Apr}}}_{\xi}(A)=\overline{\mathrm{Apr}}_{\xi_{R/\rho}}(\rho(A))$ where $(\rho(a),\rho(b))\in\xi_{R/\rho}$ if and only if $(\rho(a),\rho(b))\in\xi$ in R/ρ.*

Proof. The proof is straightforward by using Proposition 4.3. ∎

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