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A New Iterative Method for Generalized Equilibrium and Fixed Point Problems of Nonexpansive Mappings

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Abstract. In this paper, a new iterative method for finding a common element of the set of solutions of a generalized equilibrium problem and the set of common fixed points of infinitely many nonexpansive mappings in Hilbert spaces, is introduced. For this method, a strong convergence theorem is given. This improves and extends some recent results.

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1. Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. A mapping S of C into itself is called nonexpansive, if $\|Sx - Sy\| \leq \|x - y\|$, for all $x, y \in C$. Also, a contraction on C is a self-mapping S of C such that $\|S(x) - S(y)\| \leq k\|x - y\|$, for all $x, y \in C$, where $k \in (0,1)$ is a constant. Moreover, $F(S)$ denotes the fixed points set of S. Let $\phi : C \times C \to \mathbb{R}$ be a bifunction of $C \times C$ into $\mathbb{R}$. We recall an equilibrium problem as follows: The equilibrium problem for $\phi : C \times C \to \mathbb{R}$ is to find $u \in C$ such that

$$\phi(u, v) \geq 0, \quad \text{for all } v \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(\phi)$. Set $\phi(u, v) = \langle Tu, v - u\rangle$, for all $u, v \in C$, where $T : C \to H$. Then, $w \in EP(\phi)$ if and only if $\langle Tw, v - w\rangle \geq 0$, for all $v \in C$, that is, w is a solution of the variational inequality.

Combettes and Hirstoaga [4] introduced an iterative scheme for finding the best approximation to the initial data when $EP(\phi)$ is nonempty and proved a strong convergence theorem. The equilibrium problem (1.1) includes, as special cases, numerous problems in physics, optimization and economics. Some authors (such as [6, 7, 10, 11, 14, 15]) have proposed some useful methods for solving the equilibrium problem (1.1). We describe some of them as follows:

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In 2007, Plubtieng and Punpaeng [11] introduced an iterative scheme for finding a common element of the set of the solutions (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space as follows:

$$(1.2)\qquad \begin{cases} \phi(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geq 0, & \text{for all } y\in H,\\ x_{n+1}=\alpha_n\gamma f(x_n)+(I-\alpha_nA)Su_n, & n\geq 1. \end{cases}$$

where $\phi : H\times H \to \mathbb{R}$ is a bifunction, A is a strongly positive bounded linear operator on H, S is a nonexpansive mapping of H into itself such that $F(S)\bigcap EP(\phi)\neq\emptyset$, f is a contraction, $\gamma>0$ is some constant, $\{\alpha_n\}\subset[0,1]$ and $\{r_n\}\subset(0,\infty)$. Also, they proved the strong convergence of $\{x_n\}$, defined by (1.2) and showed $\lim_{n\to\infty}x_n$ is the unique solution of a certain variational inequality.

Jung [7] introduced the following composite iterative scheme by the viscosity approximation method for finding a common point of the set of solutions of (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space:

$$(1.3)\qquad \begin{cases} \phi(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geq 0, & \text{for all } y\in C,\\ y_n=\alpha_n f(x_n)+(1-\alpha_n)Su_n, & \\ x_{n+1}=(1-\beta_n)y_n+\beta_nSy_n, & n\geq 1, \end{cases}$$

where $\phi : C\times C\to\mathbb{R}$ is a bifunction, S is a nonexpansive mapping of C into itself such that $F(S)\bigcap EP(\phi)\neq\emptyset$, f is a contraction, $\{\alpha_n\},\{\beta_n\}\subset[0,1]$ and $\{r_n\}\subset(0,\infty)$. He proved the sequence $\{x_n\}$, generated by (1.3), converges strongly to a point in $F(S)\bigcap EP(\phi)$ provided $\{\alpha_n\},\{\beta_n\}$ and $\{r_n\}$ satisfy

($C1$) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$ and $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$;
($C2$) $0<\liminf_{n\to\infty}r_n$ and $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$;
($C3$) $\lim_{n\to\infty}\beta_n=0$ and $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$.

Jung [6] studied the following composite iterative scheme:

$$(1.4)\qquad \begin{cases} \phi(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geq 0, & \text{for all } y\in H,\\ y_n=\alpha_n\gamma f(x_n)+(I-\alpha_nA)Su_n, & \\ x_{n+1}=(1-\beta_n)y_n+\beta_nSy_n, & n\geq 1, \end{cases}$$

where $\phi : H\times H\to\mathbb{R}$ is a bifunction, A is a strongly positive bounded linear operator on H, S is a nonexpansive mapping of H into itself such that $F(S)\bigcap EP(\phi)\neq\emptyset$, f is a contraction, $\gamma>0$ is some constant, $\{\alpha_n\},\{\beta_n\}\subset[0,1]$ and $\{r_n\}\subset(0,\infty)$. He proved, the sequence $\{x_n\}$, generated by (1.4), converges strongly to a point in $F(S)\bigcap EP(\phi)$ under the conditions ($C1$), ($C2$) and ($C3$).

Wang *et al.* [15] introduced the following composite iterative scheme:

$$(1.5)\qquad \begin{cases} \phi(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geq 0, & \text{for all } y\in H,\\ y_n=\alpha_n\gamma f(x_n)+(I-\alpha_nA)S_nu_n, & \\ x_{n+1}=(1-\beta_n)y_n+\beta_nS_ny_n, & n\geq 1, \end{cases}$$

where $\phi : H\times H\to\mathbb{R}$ is a bifunction, A is a strongly positive bounded linear operator on H, $\{S_n\}$ is a countable family of nonexpansive mappings of H into itself such that $\bigcap_{n=1}^{\infty}F(S_n)\bigcap EP(\phi)\neq\emptyset$, f is a contraction, $\gamma>0$ is some constant, $x_1\in H$, $\{\alpha_n\},\{\beta_n\}$ $\subset[0,1]$ and $\{r_n\}\subset(0,\infty)$. They proved, under any of the following conditions:

(H_1) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$;

(H_2) $\alpha_n \in (0,1]$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \alpha_n/\alpha_{n+1} = 1$;
(H_3) $|\alpha_{n+1} - \alpha_n| < o(\alpha_{n+1}) + \sigma_n$ and $\sum_{n=1}^{\infty} \sigma_n < \infty$,

on the sequence $\{\alpha_n\}$, $\{x_n\}$ (generated by (1.5)) converges strongly to a point in $\bigcap_{n=1}^{\infty} F(S_n) \bigcap EP(\phi) \neq \emptyset$. Recently, Razani and Yazdi [14] study the convergence of a new version of composite iterative scheme (1.5).

In this paper, we prove a strong convergence theorem, concerning a new iterative scheme, for finding a common element of the set of solutions of a generalized equilibrium problem and the set of common fixed points of a countable family of nonexpansive mappings in a Hilbert space. In order to do this, we recall some definitions as follows:
A generalized equilibrium problem is to find $z \in C$ such that

$$\phi(z,y) + \langle Az, y-z\rangle \geq 0, \quad \text{for all } y \in C, \tag{1.6}$$

where $\phi : C \times C \to \mathbb{R}$ is a bifunction and $A : C \to H$ is a monotone map. The set of such $z \in C$ is denoted by EP, i.e.,

$$EP = \{z \in C : \phi(z,y) + \langle Az, y-z\rangle \geq 0, \text{ for all } y \in C\}.$$

In the case of $A \equiv 0$, EP is denoted by $EP(\phi)$. Numerous problems in physics, variational inequalities, optimization, minimax problems, the Nash equilibrium problem in noncooperative games and economics reduce to finding a solution of (1.6) (see [8], for instance).

A mapping $A : C \to H$ is called α-inverse-strongly monotone [3], if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \quad \text{for all } x, y \in C.$$

Remark 1.1. If $A : C \to H$ is α-inverse-strongly monotone map, then it is $1/\alpha$-Lipschitzian mapping.

Let B be a bounded operator on C. B is strongly positive; that is, there exists a constant $\overline{\gamma} > 0$ such that $\langle Bx, x\rangle \geq \overline{\gamma}\|x\|^2$, for all $x \in C$. A typical problem is that of minimizing a quadratic function over the set of the fixed points of nonexpansive mapping on a real Hilbert space:

$$\min_{x \in F(S)} \frac{1}{2}\langle Bx, x\rangle - \langle x, b\rangle,$$

where b is a given point in H.

Remark 1.2. Iterative method for nonexpansive mappings have been applied to solve convex minimization problems (see [12, 13]).

In this paper, a new iterative method (motivated by the above results) is introduced as follows:

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle + \langle Ax_n, y - u_n\rangle \geq 0, & \text{for all } y \in C, \\ y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n B) S_n u_n, & \\ x_{n+1} = (1-\beta_n) y_n + \beta_n S_n y_n, & n \geq 1, \end{cases} \tag{1.7}$$

where $\phi : C \times C \to \mathbb{R}$ is a bifunction, A is an α-inverse-strongly monotone, B **is** a strongly positive bounded linear operator on C; $\{S_n\}$ is a countable family of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \bigcap EP \neq \emptyset$; f is a contraction, $x_1 \in C$, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset [a,b] \subset (0, 2\alpha)$. Then, under any of three conditions (H_1), (H_2) and (H_3) on the sequence $\{\alpha_n\}$, the sequence $\{x_n\}$, generated by (1.7), converges strongly to a point in $\bigcap_{n=1}^{\infty} F(S_n) \bigcap EP$.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and the norm $\|.\|$. Weak and strong convergence are denoted by notation $\rightharpoonup$ and $\to$, respectively. In a real Hilbert space H,

$$\|\lambda x+(1-\lambda)y\|^2=\lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)\|x-y\|^2,$$

for all $x,y\in H$ and $\lambda\in\mathbb{R}$. Let C be a nonempty closed convex subset of H. Then, for any $x\in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x-P_C(x)\|\le\|x-y\|,\quad \text{for all } y\in C.$$

P_C is called the metric projection of H onto C. It is known P_C is nonexpansive. Further, for $x\in H$ and $z\in C$,

$$z=P_C(x)\ \Leftrightarrow\ \langle x-z,z-y\rangle\ge 0,\quad \text{for all } y\in C.$$

Now, we collect some lemmas which will be used in the main result.

Lemma 2.1. *Let H be a real Hilbert space. Then for all $x,y\in H$,*

(I) $\|x+y\|^2\le\|x\|^2+2\langle y,x+y\rangle$;
(II) $\|x+y\|^2\ge\|x\|^2+2\langle y,x\rangle$.

Lemma 2.2. [5] *Let H be a real Hilbert space, C a closed convex subset of H and $T:C\to C$ a nonexpansive mapping with $F(T)\ne\emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I-T)x_n\}$ converges to y, then $(I-T)x=y$.*

Lemma 2.3. [2] *Let C be a nonempty closed convex subset of H and $\phi:C\times C\to\mathbb{R}$ a bifunction satisfying the following conditions:*

(A_1) $\phi(x,x)=0$ *for all $x\in C$;*
(A_2) ϕ *is monotone, i.e., $\phi(x,y)+\phi(y,x)\le 0$ for all $x,y\in C$;*
(A_3) *for each $x,y,z\in C$,*

$$\lim_{t\downarrow 0}\phi(tz+(1-t)x,y)\le\phi(x,y);$$

(A_4) *for each $x\in C$, $y\mapsto\phi(x,y)$ is convex and weakly lower semicontinuous.*

Let $r>0$ and $x\in H$. Then, there exists $z\in C$ such that

$$\phi(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\ge 0,\quad \textit{for all } y\in C.$$

Lemma 2.4. [4] *Assume $\phi:C\times C\to\mathbb{R}$ satisfies (A_1)-(A_4). For $r>0$ and $x\in H$, define a mapping $T_r:H\to C$ as follows:*

$$T_rx=\{z\in C:\phi(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\ge 0,\quad \textit{for all } y\in C\},$$

for all $x\in H$. Then, the following hold:

(I) *T_r is single-valued;*
(II) *T_r is firmly nonexpansive, i.e., for any $x,y\in H$,*

$$\|T_rx-T_ry\|^2\le\langle T_rx-T_ry,x-y\rangle;$$

(III) *$F(T_r)=EP(\phi)$;*
(IV) *$EP(\phi)$ is closed and convex.*

Lemma 2.5. [9] *Assume B is a strongly positive bounded linear operator on a Hilbert space H with coefficient $\overline{\gamma}>0$ and $0<\rho\le\|B\|^{-1}$. Then $\|I-\rho B\|\le 1-\rho\overline{\gamma}$.*

Lemma 2.6. [1] *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n v_n + \mu_n,$$

where $\{\gamma_n\}$ *is a sequence in* $[0,1]$, $\{\mu_n\}$ *is a sequence of nonnegative real numbers and* $\{v_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(I) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(II) $\limsup_{n\to\infty} v_n \le 0$;
(III) $\sum_{n=1}^{\infty} \mu_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7. [1] *Let* C *be a nonempty closed convex subset of* H. *Suppose*

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in C\} < \infty.$$

Then, for each $y \in C$, $\{T_n y\}$ *converges strongly to some point of* C. *Moreover, let* T *be a mapping of* C *into itself defined by* $Ty = \lim_{n\to\infty} T_n y$, *for all* $y \in C$. *Then* $\lim_{n\to\infty} \sup\{\|Tz - T_n z\| : z \in C\} = 0$.

3. Main result

In this section, we prove a strong convergence theorem, concerning the iterative scheme (1.7), for finding a common element of the set of solutions of the generalized equilibrium problem (1.6) and the set of common fixed points of a countable family of nonexpansive mappings in a Hilbert space. Before this, three lemmas are proved as follows:

Lemma 3.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Assume* f *is a contraction of* C *into itself with coefficient* k, B *is a strongly positive bounded linear operator on* C *with coefficient* $\overline{\gamma} > 0$ *such that* $0 < \gamma < \frac{\overline{\gamma}}{k}$ *and* $\|B\| \le 1$. *Then* $P_C(I - B + \gamma f)$ *is a contraction.*

Proof. Let $Q = P_C$. Then

$$\begin{aligned}\|Q(I-B+\gamma f)(x) - Q(I-B+\gamma f)(y)\| &\le \|(I-B+\gamma f)(x) - (I-B+\gamma f)(y)\| \\ &\le \|(I-B)(x) - (I-B)(y)\| + \gamma\|f(x) - f(y)\| \\ &\le (1-\overline{\gamma})\|x-y\| + \gamma k\|x-y\| \\ &= (1-(\overline{\gamma} - \gamma k))\|x-y\|,\end{aligned}$$

for all $x, y \in C$. Therefore, $Q(I - B + \gamma f)$ is a contraction of C into itself. ∎

Lemma 3.2. *Suppose* C *is a nonempty closed convex subset of a real Hilbert space* H, A *is an* α*-inverse-strongly monotone on* C *and* $0 < r < 2\alpha$. *Then* $I - rA$ *is nonexpansive.*

Proof. For $x, y \in C$,

$$\begin{aligned}\|(I-rA)x - (I-rA)y\|^2 &= \|x-y-r(Ax-Ay)\|^2 \\ &= \|x-y\|^2 - 2r\langle x-y, Ax-Ay\rangle + r^2\|Ax-Ay\|^2 \\ &\le \|x-y\|^2 - 2\alpha r\|Ax-Ay\|^2 + r^2\|Ax-Ay\|^2 \\ &= \|x-y\|^2 + r(r-2\alpha)\|Ax-Ay\|^2 \le \|x-y\|^2.\end{aligned}$$

Thus $I - rA$ is nonexpansive. ∎

Lemma 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\phi : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $(A_1)-(A_4)$ (of Lemma 2.3) and A be an α-inverse-strongly monotone map. Suppose $\{x_n\}$ is a bounded sequence in C and $\{r_n\} \subset [a,b] \subset (0,2\alpha)$ is a real sequence. If $u_n = T_{r_n}(x_n - r_n A x_n)$, then*

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + |r_n - r_{n+1}| M_1,$$

where $M_1 = \sup\{\|Ax_n\| + 1/a\|u_{n+1} - k_{n+1}\| : n \in \mathbb{N}\}$.

Proof. Let $p \in EP$. Then $\phi(p,y) + \langle Ap, y-p\rangle \geq 0$, for all $y \in C$. So

$$\phi(p,y) + \frac{1}{r_n}\langle p - (p - r_n Ap), y - p\rangle \geq 0,$$

for all $y \in C$. Therefore, by Lemma 3.2,

$$\|u_n - p\| = \|T_{r_n}(I - r_n A)x_n - T_{r_n}(I - r_n A)p\| \leq \|x_n - p\|, \quad n \geq 1. \tag{3.1}$$

Therefore, $\{u_n\}$ is a bounded sequence. Set $k_n = x_n - r_n A x_n$, we have $u_n = T_{r_n} k_n$ and $u_{n+1} = T_{r_{n+1}} k_{n+1}$. So

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - k_n\rangle \geq 0, \quad \text{for all } y \in C, \tag{3.2}$$

and

$$\phi(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - k_{n+1}\rangle \geq 0, \quad \text{for all } y \in C. \tag{3.3}$$

Set $y = u_{n+1}$ in (3.2) and $y = u_n$ in (3.3), then by adding these two last inequalities and using condition (A_2), we have

$$\left\langle u_{n+1} - u_n, \frac{u_n - k_n}{r_n} - \frac{u_{n+1} - k_{n+1}}{r_{n+1}} \right\rangle \geq 0,$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - k_n - \frac{r_n}{r_{n+1}}(u_{n+1} - k_{n+1}) \right\rangle \geq 0.$$

This implies

$$\begin{aligned}\|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, k_{n+1} - k_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - k_{n+1}) \right\rangle \\ &\leq \|u_{n+1} - u_n\| \left\{ \|k_{n+1} - k_n\| + \frac{1}{a}|r_n - r_{n+1}|\|u_{n+1} - k_{n+1}\| \right\}.\end{aligned}$$

Therefore

$$\begin{aligned}&\|u_{n+1} - u_n\| \\ &\leq \|k_{n+1} - k_n\| + \frac{1}{a}|r_n - r_{n+1}|\|u_{n+1} - k_{n+1}\| \\ &= \|x_{n+1} - r_{n+1}Ax_{n+1} - (x_n - r_n A x_n)\| + \frac{1}{a}|r_n - r_{n+1}|\|u_{n+1} - k_{n+1}\| \\ &\leq \|x_{n+1} - r_{n+1}Ax_{n+1} - (x_n - r_{n+1}Ax_n)\| + |r_n - r_{n+1}|\|Ax_n\| + \frac{1}{a}|r_n - r_{n+1}|\|u_{n+1} - k_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + |r_n - r_{n+1}| M_1,\end{aligned}$$

where $M_1 = \sup\{\|Ax_n\| + 1/a\|u_{n+1} - k_{n+1}\| : n \in \mathbb{N}\}$. ∎

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\phi : C\times C \to \mathbb{R}$ be a bifunction satisfying the conditions $(A_1)-(A_4)$ (of Lemma 2.3) and $\{S_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C satisfying $F := \bigcap_{n=1}^{\infty} F(S_n)\bigcap EP \neq \emptyset$. Let f be a contraction of C into itself with coefficient k, B be a strongly positive bounded linear operator on C with coefficient $\overline{\gamma} > 0$ such that $\|B\| \le 1$ and A be an α-inverse-strongly monotone on C. Assume $0 < \gamma < \overline{\gamma}/k$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$ and $\{r_n\} \subset [a,b] \subset (0,2\alpha)$ is a real sequence satisfying the following conditions:*

(B_1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(B_2) $\lim_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty$;
(B_3) $\sum_{n=1}^{\infty}|r_{n+1}-r_n| < \infty$.

Suppose $\sum_{n=1}^{\infty}\sup\{\|S_{n+1}z - S_n z\| : z \in K\} < \infty$ for any bounded subset K of C. Let S be a mapping of C into itself defined by $Sz = \lim_{n\to\infty} S_n z$ for all $z \in C$ and $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$. If any of three conditions $(H_1)-(H_3)$ satisfies, then the sequences $\{x_n\}$ and $\{u_n\}$ defined by (1.7) converge strongly to $q \in F$, where $q = P_{\bigcap_{n=1}^{\infty} F(S_n)\bigcap EP}(I-B+\gamma f)(q)$, which solves the following variational inequality:

$$\langle (B-\gamma f)q, q-x\rangle \le 0, \quad \textit{for all } x \in F.$$

Proof. Let $Q = P_{\bigcap_{n=1}^{\infty} F(S_n)\bigcap EP}$. $Q(I-B+\gamma f)$ is a contraction of C into itself by Lemma 3.1. So, there exists a unique element $q \in C$ such that $q = Q(I-B+\gamma f)(q) = P_{\bigcap_{n=1}^{\infty} F(S_n)\bigcap EP}(I-B+\gamma f)(q)$. By using conditions (B_1) and (B_2), we may assume, without loss of generality, $\alpha_n \le (1-\beta_n)\|B\|^{-1}$. Since B is strongly positive bounded linear operator on C,

$$\|B\| = \sup\{|\langle Bx,x\rangle| : x \in C, \|x\| = 1\}.$$

Observe

$$\langle ((1-\beta_n)I - \alpha_n B)x,x\rangle = 1-\beta_n - \alpha_n\langle Bx,x\rangle \ge 1-\beta_n-\alpha_n\|B\| \ge 0.$$

Thus $(1-\beta_n)I-\alpha_n B$ is positive, and

$$\begin{aligned}\|(1-\beta_n)I-\alpha_n B\| &= \sup\{\langle((1-\beta_n)I-\alpha_n B)x,x\rangle : x\in C, \|x\|=1\}\\ &= \sup\{1-\beta_n-\alpha_n\langle Bx,x\rangle : x \in C, \|x\|=1\} \le 1-\beta_n-\alpha_n\overline{\gamma}.\end{aligned}$$

We proceed with the following steps:
Step 1. First, we claim, $\{x_n\}$ and $\{u_n\}$ are bounded. Let $p \in F$. From the definition of T_r, $u_n = T_{r_n}(I-r_nA)x_n$. Then, from (1.7), (3.1) and Lemma 2.5,

$$\begin{aligned}\|x_{n+1}-p\| &= \|(1-\beta_n)(y_n-p)+\beta_n(S_ny_n-p)\|\\ &\le \|y_n-p\| = \|\alpha_n(\gamma f(x_n)-Bp)+(I-\alpha_nB)(S_nu_n-p)\|\\ &\le (1-\alpha_n\overline{\gamma})\|x_n-p\| + \|\alpha_n\gamma(f(x_n)-f(p))\| + \alpha_n\|\gamma f(p)-Bp\|\\ &\le (1-\alpha_n\overline{\gamma})\|x_n-p\| + \alpha_n\gamma k\|x_n-p\| + \alpha_n\|\gamma f(p)-Bp\|\\ &\le (1-\alpha_n(\overline{\gamma}-\gamma k))\|x_n-p\| + \alpha_n(\overline{\gamma}-\gamma k)\frac{\|\gamma f(p)-Bp\|}{\overline{\gamma}-\gamma k}\\ &\le \max\left\{\|x_n-p\|, \frac{\|\gamma f(p)-Bp\|}{\overline{\gamma}-\gamma k}\right\}.\end{aligned}$$

By induction,

$$\|x_n-p\| \le \max\left\{\|x_1-p\|, \frac{1}{\overline{\gamma}-\gamma k}\|\gamma f(p)-Bp\|\right\}, \quad \text{for all } n \ge 1.$$

Hence, $\{x_n\}$ is bounded, so are $\{u_n\}$, $\{y_n\}$, $\{f(x_n)\}$, $\{BS_nu_n\}$ and $\{S_ny_n\}$. Without loss of generality, we may assume $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{f(x_n)\}$, $\{BS_nu_n\}$, $\{S_ny_n\} \subset K$, where K is a bounded subset of C.

Step 2. We claim, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$. Since K is bounded, $\{S_ny_n - y_n\}, \{f(x_n)\}$, $\{BS_nu_n\}$ are bounded. Set

$$M = \sup\{\|S_ny_n - y_n\|, \|f(x_n)\|, \|BS_nu_n\| : n \in \mathbb{N}\}.$$

By the definition of $\{x_n\}$,

$$\begin{aligned}
&\|x_{n+2}-x_{n+1}\| \\
&= \|(1-\beta_{n+1})y_{n+1} + \beta_{n+1}S_{n+1}y_{n+1} - (1-\beta_n)y_n - \beta_nS_ny_n\| \\
&= \|(1-\beta_{n+1})y_{n+1} + \beta_{n+1}S_{n+1}y_{n+1} - (1-\beta_{n+1})y_n - \beta_nS_ny_n + (1-\beta_{n+1})y_n \\
&\quad - (1-\beta_n)y_n - \beta_{n+1}S_ny_n + \beta_{n+1}S_ny_n\| \\
&= \|(1-\beta_{n+1})(y_{n+1}-y_n) + \beta_{n+1}(S_{n+1}y_{n+1} - S_ny_n) + (\beta_{n+1}-\beta_n)(S_ny_n - y_n)\| \\
&\le (1-\beta_{n+1})\|y_{n+1}-y_n\| + \beta_{n+1}\|S_{n+1}y_{n+1} - S_ny_n\| + |\beta_{n+1}-\beta_n|M \\
&\le (1-\beta_{n+1})\|y_{n+1}-y_n\| + \beta_{n+1}\|S_{n+1}y_{n+1} - S_ny_{n+1}\| + \beta_{n+1}\|y_{n+1}-y_n\| \\
&\quad + |\beta_{n+1}-\beta_n|M \\
&\le \|y_{n+1}-y_n\| + \|S_{n+1}y_{n+1} - S_ny_{n+1}\| + |\beta_{n+1}-\beta_n|M,
\end{aligned} \tag{3.4}$$

for all $n \in \mathbb{N}$. From (1.7),

$$\begin{aligned}
&\|y_{n+1}-y_n\| \\
&= \|\alpha_{n+1}\gamma f(x_{n+1}) + (I-\alpha_{n+1}B)S_{n+1}u_{n+1} - \alpha_n\gamma f(x_n) - (I-\alpha_nB)S_nu_n\| \\
&= \|(I-\alpha_{n+1}B)(S_{n+1}u_{n+1} - S_nu_n) - (\alpha_{n+1}-\alpha_n)BS_nu_n + \alpha_{n+1}\gamma(f(x_{n+1}) - f(x_n)) \\
&\quad + (\alpha_{n+1}-\alpha_n)\gamma f(x_n)\| \\
&\le (1-\alpha_{n+1}\overline{\gamma})\|S_{n+1}u_{n+1} - S_nu_n\| + |\alpha_{n+1}-\alpha_n|\|BS_nu_n\| + \alpha_{n+1}\gamma k\|x_{n+1}-x_n\| \\
&\quad + |\alpha_{n+1}-\alpha_n|\gamma\|f(x_n)\| \\
&\le (1-\alpha_{n+1}\overline{\gamma})(\|S_{n+1}u_{n+1} - S_{n+1}u_n\| + \|S_{n+1}u_n - S_nu_n\|) + |\alpha_{n+1}-\alpha_n|M \\
&\quad + \alpha_{n+1}\gamma k\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|\gamma M \\
&\le (1-\alpha_{n+1}\overline{\gamma})\|u_{n+1}-u_n\| + M(1+\gamma)|\alpha_{n+1}-\alpha_n| + \alpha_{n+1}\gamma k\|x_{n+1}-x_n\| \\
&\quad + \|S_{n+1}u_n - S_nu_n\|,
\end{aligned} \tag{3.5}$$

for all $n \in N$. On the other hand, $u_n = T_{r_n}(x_n - r_nAx_n)$ (by Lemma 2.4). From Lemma 3.3,

$$\|u_{n+1}-u_n\| \le \|x_{n+1}-x_n\| + |r_n - r_{n+1}|M_1, \tag{3.6}$$

where $M_1 = \sup\{\|Ax_n\| + 1/a\|u_{n+1} - k_{n+1}\| : n \in \mathbb{N}\}$. Substituting (3.6) in (3.5), we have

$$\begin{aligned}
&\|y_{n+1}-y_n\| \\
&\le (1-\alpha_{n+1}\overline{\gamma})\{\|x_{n+1}-x_n\| + |r_n - r_{n+1}|M_1\} + M(1+\gamma)|\alpha_{n+1}-\alpha_n| \\
&\quad + \alpha_{n+1}\gamma k\|x_{n+1}-x_n\| + \|S_{n+1}u_n - S_nu_n\| \\
&\le [1-\alpha_{n+1}(\overline{\gamma}-\gamma k)]\|x_{n+1}-x_n\| + |r_n - r_{n+1}|M_1 + M(1+\gamma)|\alpha_{n+1}-\alpha_n| \\
&\quad + \|S_{n+1}u_n - S_nu_n\| \\
&\le [1-\alpha_{n+1}(\overline{\gamma}-\gamma k)]\|x_{n+1}-x_n\| + M_2(|r_n - r_{n+1}| + |\alpha_{n+1}-\alpha_n|)
\end{aligned} \tag{3.7}$$

$$+\|S_{n+1}u_n - S_nu_n\|,$$

where $M_2 = \max\{M_1, M(1+\gamma)\}$. Substituting (3.7) in (3.4), we have

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| \le [1-\alpha_{n+1}(\overline{\gamma}-\gamma k)]\|x_{n+1}-x_n\| + M_2(|r_n-r_{n+1}| \\ + |\alpha_{n+1}-\alpha_n| + |\beta_{n+1}-\beta_n|) + 2\sup\{\|S_{n+1}z - S_nz\| : z \in K\}.\end{aligned} \tag{3.8}$$

Now, we show that under any of three conditions $(H_1)-(H_3)$, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$ as follows:
Let (H_1) holds. Set $\mu_n = M_2(|r_n - r_{n+1}| + |\alpha_{n+1}-\alpha_n| + |\beta_{n+1}-\beta_n|) + 2\sup\{\|S_{n+1}z - S_nz\| : z \in K\}$, then

$$\begin{aligned}\sum_{n=1}^{\infty}\mu_n = M_2\sum_{n=1}^{\infty}(|r_n-r_{n+1}| + |\alpha_{n+1}-\alpha_n| + |\beta_{n+1}-\beta_n|) + 2\sum_{n=1}^{\infty}\sup\{\|S_{n+1}z - S_nz\| : z \in K\} \\ < \infty.\end{aligned}$$

Therefore, by Lemma 2.6, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.
If (H_2) holds, then from (3.8),

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| &\le [1-\alpha_{n+1}(\overline{\gamma}-\gamma k)]\|x_{n+1}-x_n\| + \alpha_{n+1}M_2\left|1-\frac{\alpha_n}{\alpha_{n+1}}\right| \\ &\quad + M_2(|r_{n+1}-r_n| + |\beta_{n+1}-\beta_n|) + 2\sup\{\|S_{n+1}z - S_nz\| : z \in K\}.\end{aligned}$$

Set $\mu_n = M_2(|r_{n+1}-r_n| + |\beta_{n+1}-\beta_n|) + 2\sup\{\|S_{n+1}z - S_nz\| : z \in K\}$, then

$$\sum_{n=1}^{\infty}\mu_n = M_2\sum_{n=1}^{\infty}(|r_{n+1}-r_n| + |\beta_{n+1}-\beta_n|) + 2\sum_{n=1}^{\infty}\sup\{\|S_{n+1}z - S_nz\| : z \in K\} < \infty.$$

Therefore, by Lemma 2.6, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.
If (H_3) holds, then from (3.8),

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| &\le [1-\alpha_{n+1}(\overline{\gamma}-\gamma k)]\|x_{n+1}-x_n\| + M_2 o(\alpha_{n+1}) \\ &\quad + M_2(\sigma_n + |r_{n+1}-r_n| + |\beta_{n+1}-\beta_n|) + 2\sup\{\|S_{n+1}z - S_nz\| : z \in K\}.\end{aligned}$$

Set $\mu_n = M_2(\sigma_n + |r_{n+1}-r_n| + |\beta_{n+1}-\beta_n|) + 2\sup\{\|S_{n+1}z - S_nz\| : z \in K\}$, then

$$\sum_{n=1}^{\infty}\mu_n = M_2\sum_{n=1}^{\infty}(\sigma_n + |r_{n+1}-r_n| + |\beta_{n+1}-\beta_n|) + 2\sum_{n=1}^{\infty}\sup\{\|S_{n+1}z - S_nz\| : z \in K\} < \infty.$$

Therefore, by Lemma 2.6, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

Step 3. We claim, $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Indeed, from (B_2) and (1.7),

$$\lim_{n\to\infty}\|x_{n+1}-y_n\| = \lim_{n\to\infty}\beta_n\|y_n - S_ny_n\| = 0.$$

So, from step 2 and $\|x_n - y_n\| \le \|x_{n+1}-x_n\| + \|x_{n+1}-y_n\|$, we get

$$\lim_{n\to\infty}\|x_n - y_n\| = 0.$$

Step 4. We claim $\lim_{n\to\infty}\|x_n - u_n\| = 0$, $\lim_{n\to\infty}\|y_n - u_n\| = 0$ and $\lim_{n\to\infty}\|Su_n - u_n\| = 0$. To this end, let $p \in F$. Then

$$\begin{aligned}\|u_n - p\|^2 &= \|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(p - r_nAp)\|^2 \le \|x_n - r_nAx_n - p + r_nAp\|^2 \\ &= \|x_n - p\|^2 + r_n^2\|Ax_n - Ap\|^2 - 2r_n\langle x_n - p, Ax_n - Ap\rangle \\ &\le \|x_n - p\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Ap\|^2.\end{aligned}$$

Therefore

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\|(1-\beta_n)(y_n-p)+\beta_n(S_ny_n-p)\|^2\\
&\leq \|y_n-p\|^2=\|\alpha_n\gamma f(x_n)+(I-\alpha_nB)S_nu_n-p\|^2\\
&=\|\alpha_n(\gamma f(x_n)-BS_nu_n)+S_nu_n-p\|^2\\
&=\alpha_n^2\|\gamma f(x_n)-BS_nu_n\|^2+\|S_nu_n-p\|^2+2\alpha_n\langle\gamma f(x_n)-BS_nu_n,S_nu_n-p\rangle\\
&\leq \alpha_n^2\|\gamma f(x_n)-BS_nu_n\|^2+\|u_n-p\|^2+2\alpha_n\langle\gamma f(x_n)-BS_nu_n,S_nu_n-p\rangle\\
&\leq \alpha_n^2\|\gamma f(x_n)-BS_nu_n\|^2+\|x_n-p\|^2+r_n(r_n-2\alpha)\|Ax_n-Ap\|^2\\
&\quad +2\alpha_n\|\gamma f(x_n)-BS_nu_n\|\|u_n-p\|.
\end{aligned}
\tag{3.9}
$$

This implies

$$
\begin{aligned}
&r_n(2\alpha-r_n)\|Ax_n-Ap\|^2\\
&\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n^2\|\gamma f(x_n)-BS_nu_n\|^2+2\alpha_n\|\gamma f(x_n)-BS_nu_n\|\|u_n-p\|\\
&\leq (\|x_n-p\|+\|x_{n+1}-p\|)\|x_{n+1}-x_n\|+\alpha_n^2\|\gamma f(x_n)-BS_nu_n\|^2\\
&\quad +2\alpha_n\|\gamma f(x_n)-BS_nu_n\|\|u_n-p\|.
\end{aligned}
$$

From $\lim_{n\to\infty}\alpha_n=0$, $r_n\in[a,b]\subset(0,2\alpha)$ and $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$,

$$
\lim_{n\to\infty}\|Ax_n-Ap\|=0. \tag{3.10}
$$

Also, from (II) in Lemma 2.4,

$$
\begin{aligned}
&\|u_n-p\|^2\\
&=\|T_{r_n}(x_n-r_nAx_n)-T_{r_n}(p-r_nAp)\|^2\leq\langle x_n-r_nAx_n-(p-r_nAp),u_n-p\rangle\\
&=\frac{1}{2}\{\|x_n-r_nAx_n-(p-r_nAp)\|^2+\|u_n-p\|^2-\|x_n-r_nAx_n-(p-r_nAp)-(u_n-p)\|^2\}\\
&\leq\frac{1}{2}\{\|x_n-p\|^2+\|u_n-p\|^2-\|x_n-u_n-r_n(Ax_n-Ap)\|^2\}\\
&=\frac{1}{2}\{\|x_n-p\|^2+\|u_n-p\|^2-\|x_n-u_n\|^2+2r_n\langle x_n-u_n,Ax_n-Ap\rangle-r_n^2\|Ax_n-Ap\|^2\}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
\|u_n-p\|^2&\leq\|x_n-p\|^2-\|x_n-u_n\|^2+2r_n\langle x_n-u_n,Ax_n-Ap\rangle-r_n^2\|Ax_n-Ap\|^2.\\
&\leq\|x_n-p\|^2-\|x_n-u_n\|^2+2r_n\langle x_n-u_n,Ax_n-Ap\rangle\\
&\leq\|x_n-p\|^2-\|x_n-u_n\|^2+2r_n\|x_n-u_n\|\|Ax_n-Ap\|.
\end{aligned}
$$

By the same argument in (3.9),

$$
\begin{aligned}
\|x_{n+1}-p\|^2&\leq\alpha_n^2\|\gamma f(x_n)-BS_nu_n\|^2+\|u_n-p\|^2+2\alpha_n\langle\gamma f(x_n)-BS_nu_n,S_nu_n-p\rangle\\
&\leq\alpha_n^2\|\gamma f(x_n)-BS_nu_n\|^2+\|x_n-p\|^2-\|x_n-u_n\|^2+2r_n\|x_n-u_n\|\|Ax_n-Ap\|\\
&\quad+2\alpha_n\|\gamma f(x_n)-BS_nu_n\|\|u_n-p\|.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|x_n-u_n\|^2&\leq\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n^2\|\gamma f(x_n)-BS_nu_n\|^2\\
&\quad+2r_n\|x_n-u_n\|\|Ax_n-Ap\|+2\alpha_n\|\gamma f(x_n)-BS_nu_n\|\|u_n-p\|
\end{aligned}
$$

$$\leq (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\| + \alpha_n^2 \|\gamma f(x_n) - BS_n u_n\|^2$$
$$+ 2r_n \|x_n - u_n\| \|Ax_n - Ap\| + 2\alpha_n \|\gamma f(x_n) - BS_n u_n\| \|u_n - p\|.$$

Then $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and (3.10) show

$$\lim_{n\to\infty} \|x_n - u_n\| = 0.$$

Moreover, from $\|y_n - u_n\| \leq \|y_n - x_n\| + \|x_n - u_n\|$ and step 3,

$$\lim_{n\to\infty} \|y_n - u_n\| = 0.$$

Since

$$\|S_n u_n - u_n\| \leq \|S_n u_n - y_n\| + \|y_n - u_n\| \leq \alpha_n \|\gamma f(x_n) - BS_n u_n\| + \|y_n - u_n\|,$$

we have $\lim_{n\to\infty} \|S_n u_n - u_n\| = 0$. Observe

$$\|Su_n - u_n\| \leq \|Su_n - S_n u_n\| + \|S_n u_n - u_n\| \leq \sup\{\|Sz - S_n z\| : z \in K\} + \|S_n u_n - u_n\|.$$

From Lemma 2.7, $\lim_{n\to\infty} \|Su_n - u_n\| = 0$.

Step 5. We claim, $\limsup_{n\to\infty} \langle \gamma f(q) - Bq, y_n - q\rangle \leq 0$, where $q = P_{\bigcap_{n=1}^{\infty} F(S_n) \bigcap EP}(I - B + \gamma f)(q)$.To show this, choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\limsup_{n\to\infty} \langle (B - \gamma f)q, q - u_n\rangle = \lim_{i\to\infty} \langle (B - \gamma f)q, q - u_{n_i}\rangle.$$

Since $\{u_{n_i}\}$ is bounded in C, without loss of generality, we assume $u_{n_i} \rightharpoonup z \in C$. From $\lim_{n\to\infty} \|Su_n - u_n\| = 0$ and Lemma 2.2, $z \in F(S)$. Now, we show $z \in EP$. By $u_n = T_{r_n}(x_n - r_n A x_n)$, one can write

$$\phi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \text{for all } y \in C.$$

From (A_2),

$$\langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \phi(y, u_n), \quad \text{for all } y \in C.$$

Replacing n by n_i, we have

$$\langle Ax_{n_i}, y - u_{n_i}\rangle + \frac{1}{r_{n_i}}\langle y - u_{n_i}, u_{n_i} - x_{n_i}\rangle \geq \phi(y, u_{n_i}), \quad \text{for all } y \in C. \tag{3.11}$$

Set $y_t = ty + (1-t)z$, for all $t \in (0, 1]$ and $y \in C$. Then $y_t \in C$. So, from (3.11),

$$\begin{aligned}
&\langle y_t - u_{n_i}, Ay_t\rangle \\
&\geq \langle y_t - u_{n_i}, Ay_t\rangle - \langle Ax_{n_i}, y_t - u_{n_i}\rangle - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + \phi(y_t, u_{n_i}) \\
&= \langle y_t - u_{n_i}, Ay_t - Au_{n_i}\rangle + \langle y_t - u_{n_i}, Au_{n_i} - Ax_{n_i}\rangle - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + \phi(y_t, u_{n_i}).
\end{aligned}$$

Since $\lim_{i\to\infty} \|u_{n_i} - x_{n_i}\| = 0$, we have $\lim_{i\to\infty} \|Au_{n_i} - Ax_{n_i}\| = 0$. Further, from the monotonicity of A, $\langle y_t - u_{n_i}, Ay_t - Au_{n_i}\rangle \geq 0$. So, from (A_4),

$$\langle y_t - z, Ay_t\rangle \geq \phi(y_t, z), \tag{3.12}$$

as $i \to \infty$. From (A_1), (A_2) and (3.12),

$$0 = \phi(y_t, y_t) \leq t\phi(y_t, y) + (1-t)\phi(y_t, z) \leq t\phi(y_t, y) + (1-t)\langle y_t - z, Ay_t\rangle$$

$$= t\phi(y_t, y) + (1-t)t\langle y - z, Ay_t\rangle,$$

hence

$$0 \leq \phi(y_t, y) + (1-t)\langle y - z, Ay_t\rangle.$$

Letting $t \to 0$, we get

$$0 \leq \phi(z, y) + \langle y - z, Az\rangle, \quad \text{for all } y \in C.$$

This implies $z \in EP$. Since $q = P_{\bigcap_{n=1}^{\infty} F(S_n) \bigcap EP}(I - B + \gamma f)(q)$,

$$\begin{aligned}\limsup_{n\to\infty}\langle (B-\gamma f)(q), q - y_n\rangle &= \lim_{i\to\infty}\langle (B-\gamma f)(q), q - y_{n_i}\rangle = \lim_{i\to\infty}\langle (B-\gamma f)(q), q - u_{n_i}\rangle \\ &= \lim_{i\to\infty}\langle (B-\gamma f)(q), q - z\rangle \leq 0.\end{aligned}$$

Step 6. We claim $\{u_n\}$ and $\{x_n\}$ converges strongly to q. From (1.7),

$$\begin{aligned}&\|x_{n+1} - q\|^2 \\ &= \|(1-\beta_n)(y_n - q) + \beta_n(S_n y_n - q)\|^2 \\ &\leq \|y_n - q\|^2 = \|\alpha_n \gamma f(x_n) + (I - \alpha_n B)S_n u_n - q\|^2 \\ &= \|\alpha_n(\gamma f(x_n) - Bq) + (I - \alpha_n B)(S_n u_n - q)\|^2 \\ &\leq \|(I - \alpha_n B)(S_n u_n - q)\|^2 + 2\alpha_n\langle \gamma f(x_n) - Bq, y_n - q\rangle \\ &\leq (1 - \alpha_n\overline{\gamma})^2\|u_n - q\|^2 + 2\alpha_n\langle \gamma f(x_n) - \gamma f(q), y_n - q\rangle + 2\alpha_n\langle \gamma f(q) - Bq, y_n - q\rangle \\ &\leq (1 - \alpha_n\overline{\gamma})^2\|x_n - q\|^2 + 2\alpha_n\gamma k\|x_n - q\|(\|y_n - x_n\| + \|x_n - q\|) + 2\alpha_n\langle \gamma f(q) - Bq, y_n - q\rangle \\ &\leq (1 - 2\alpha_n(\overline{\gamma} - \gamma k))\|x_n - q\|^2 + (\alpha_n\overline{\gamma})^2\|x_n - q\|^2 + 2\alpha_n\gamma k\|x_n - q\|\|y_n - x_n\| \\ &\quad + 2\alpha_n\langle \gamma f(q) - Bq, y_n - q\rangle \\ &\leq (1 - 2\alpha_n(\overline{\gamma} - \gamma k))\|x_n - q\|^2 + 2\alpha_n(\overline{\gamma} - \gamma k)\Bigg\{\frac{(\alpha_n\overline{\gamma}^2)M_3^2}{2(\overline{\gamma} - \gamma k)} + \frac{\gamma k M_3}{\overline{\gamma} - \gamma k}\|y_n - x_n\| \\ &\quad + \frac{1}{\overline{\gamma} - \gamma k}\langle \gamma f(q) - Bq, y_n - q\rangle\Bigg\} \\ &= (1 - \delta_n)\|x_n - q\|^2 + \delta_n\theta_n,\end{aligned}$$

where $M_3 = \sup\{\|x_n - q\| : \ n \geq 1\}$, $\delta_n = 2\alpha_n(\overline{\gamma} - \gamma k)$ and $\theta_n = ((\alpha_n\overline{\gamma}^2)M_3^2)/(2(\overline{\gamma} - \gamma k)) + (\gamma k M_3)/(\overline{\gamma} - \gamma k)\|y_n - x_n\| + 1/(\overline{\gamma} - \gamma k)\langle \gamma f(q) - Bq, y_n - q\rangle$. It is easy to see that $\lim_{n\to\infty}\delta_n = 0$, $\sum_{n=1}^{\infty}\delta_n = \infty$ and $\limsup_{n\to\infty}\theta_n \leq 0$. Hence, by Lemma 2.6, $\{x_n\}$ converges strongly to q. Consequently, $\{u_n\}$ converges strongly to q. This completes the proof. ∎

Remark 3.1. Theorem 3.1 is a generalization of [15, Theorem 3.1]. To see this, Set $A = 0$ in Theorem 3.1, and assume $r_n \geq a > 0$ (it is not necessary to assume $\{r_n\} \subset [a, b] \subset (0, 2\alpha)$).

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