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Multilinear Singular Integral Operators on Triebel-Lizorkin and Lebesgue Spaces

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Abstract. The boundedness for the multilinear operators associated to some singular integral operators with non-smooth kernels on Triebel-Lizorkin and Lebesgue spaces is obtained.

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1. Introduction

As the development of singular integral operators, their commutators and multilinear operators have been well studied (see [1–4]). From [8,10], we know that the commutators and multilinear operators generated by the singular integral operators and the Lipschitz functions are bounded on the Triebel-Lizorkin and Lebesgue spaces. The purpose of this paper is to introduce some multilinear operators associated to certain singular integral operators with non-smooth kernels and prove the boundedness properties for the multilinear operators on the Triebel-Lizorkin and Lebesgue spaces.

2. Preliminaries and theorems

In this paper, we will study a class of multilinear operators associated to some singular integral operators with non-smooth kernels as follows.

Definition 2.1. *A family of operators $D_t, t > 0$ is said to be an "approximations to the identity" if, for every $t > 0$, D_t can be represented by the kernel $a_t(x,y)$ in the following sense:*

$$D_t(f)(x) = \int_{\mathbb{R}^n} a_t(x,y) f(y) dy$$

for every $f \in L^p(\mathbb{R}^n)$ with $p \geq 1$ and every $x \in \mathbb{R}^n$, and $a_t(x,y)$ satisfies:

$$|a_t(x,y)| \leq h_t(x,y) = Ct^{-n/2} s(|x-y|^2/t),$$

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where s is a positive, bounded and decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\varepsilon} s(r^2) = 0$$

for some $\varepsilon > 0$.

Let m_j be positive integers($j = 1,\cdots,l$), $m_1 + \cdots + m_l = m$ and b_j be functions on $\mathbb{R}^n$ with $b_j \in C^{m_j}$ for $1 \le j \le l$. Set

$$R_{m_j+1}(b_j;x,y) = b_j(x) - \sum_{|\alpha|\le m} \frac{1}{\alpha!} D^\alpha b_j(y)(x-y)^\alpha.$$

We have

$$\begin{aligned} R_{m_j+1}(b_j;x,y) &= b_j(x) - \sum_{|\alpha|\le m-1} \frac{1}{\alpha!} D^\alpha b_j(y)(x-y)^\alpha - \sum_{|\alpha|=m} \frac{1}{\alpha!} D^\alpha b_j(y)(x-y)^\alpha \\ &= R_{m_j}(b_j;x,y) - \sum_{|\alpha|=m} \frac{1}{\alpha!} D^\alpha b_j(y)(x-y)^\alpha. \end{aligned}$$

Definition 2.2. *A linear operator T is called the singular integral operators with non-smooth kernels if T is bounded on* $L^2(\mathbb{R}^n)$ *and associated with a kernel* $K(x,y)$ *such that*

$$T(f)(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy$$

for every continuous function f with compact support, and for almost all x not in the support of f. Moreover, the following conditions hold:

(1) *There exists an "approximations to the identity"* $\{B_t, t > 0\}$ *such that* TB_t *has associated kernel* $k_t(x,y)$ *and there exist* $c_1, c_2 > 0$ *so that*

$$\int_{|x-y|>c_1 t^{1/2}} |K(x,y) - k_t(x,y)| dx \le c_2 \ \text{ for all } \ y \in \mathbb{R}^n.$$

(2) *There exists an "approximations to the identity"* $\{A_t, t > 0\}$ *such that* $A_t T$ *has associated kernel* $K_t(x,y)$ *which satisfies*

$$|K_t(x,y)| \le c_4 t^{-n/2} \ \text{ if } \ |x-y| \le c_3 t^{1/2},$$

and

$$|K(x,y) - K_t(x,y)| \le c_4 t^{\delta/2} |x-y|^{-n-\delta} \ \text{ if } \ |x-y| \ge c_3 t^{1/2},$$

for some $c_3, c_4 > 0, \delta > 0$.

Let m_j be positive integers($j = 1,\cdots,l$), $m_1 + \cdots + m_l = m$ and b_j be functions on $\mathbb{R}^n$ with $b_j \in C^{m_j}$ for $1 \le j \le l$. The multilinear operator associated to T is defined by

$$T^b(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(b_j;x,y)}{|x-y|^m} K(x,y) f(y) dy.$$

Note that when $m = 0$, T^b is just the multilinear commutator of T and b (see [5,10–11]), while when $m > 0$, it is non-trivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [2–4]). The purpose of this paper is to study the boundedness properties for the multilinear operator T^b on Triebel-Lizorkin and Lebesgue spaces. In Section 4, some applications of Theorems in this paper are given.

Definition 2.3. *Let $0<\beta<1$ and $1\le p<\infty$. The Triebel-Lizorkin space associated with the "approximations to the identity" $\{A_t, t>0\}$ is defined by*

$$\dot{F}^{\beta,\infty}_{p,A}(\mathbb{R}^n)=\{f\in L^1_{loc}(\mathbb{R}^n):||f||_{\dot{F}^{\beta,\infty}_{p,A}}<\infty\},$$

where

$$||f||_{\dot{F}^{\beta,\infty}_{p,A}}=\left|\left|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+\beta/n}}\int_Q|f(x)-A_{t_Q}(f)(x)|dx\right|\right|_{L^p},$$

and the supremum is taken over all cubes Q of $\mathbb{R}^n$ with sides parallel to the axes, $t_Q=l(Q)^2$ and $l(Q)$ denotes the side length of Q.

Now, let us introduce some notations. Throughout this paper, Q will denote a cube of $\mathbb{R}^n$ with sides parallel to the axes. For a locally integrable function f, the sharp function of f is defined by

$$f^\#(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)-f_Q|dy,$$

where, and in what follows, $f_Q=|Q|^{-1}\int_Q f(x)dx$. It is well-known that (see [7,12])

$$f^\#(x)\approx\sup_{Q\ni x}\inf_{c\in Q}\frac{1}{|Q|}\int_Q|f(y)-c|dy.$$

Let

$$M(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)|dy.$$

For $\eta>0$, let $M_\eta(f)(x)=M(|f|^\eta)^{1/\eta}(x)$.

For $1\le p<\infty$ and $0\le\delta<n$, let

$$M_{\delta,p}(f)(x)=\sup_{Q\ni x}\left(\frac{1}{|Q|^{1-p\delta/n}}\int_Q|f(y)|^p dy\right)^{1/p}.$$

For $\beta>0$, the Lipschitz space $Lip_\beta(\mathbb{R}^n)$ is the space of functions f such that (see [10])

$$||f||_{Lip_\beta}=\sup_{x,h\in\mathbb{R}^n,\ h\neq 0}|f(x+h)-f(x)|/|h|^\beta<\infty.$$

The sharp maximal function $M_A(f)$ associated with the "approximations to the identity" $\{A_t, t>0\}$ is defined by(see [6,9])

$$M^\#_A(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)-A_{t_Q}(f)(y)|dy,$$

where $t_Q=(l(Q))^2$ and $l(Q)$ denotes the side length of Q.

Now we can state our theorem as following.

Theorem 2.1. *Suppose T is a singular integral operator with non-smooth kernel as Definition 2.2. Let $0<\beta<\min(1,\varepsilon,\delta/l)$, $\{A_t, t>0\}$ be the "approximations to the identity" and $D^\alpha b_j\in Lip_\beta(\mathbb{R}^n)$ for all α with $|\alpha|=m_j$ and $j=1,\cdots,l$. Then T^b is bounded from $L^p(\mathbb{R}^n)$ to $\dot{F}^{l\beta,\infty}_{p,A}(\mathbb{R}^n)$ for any $1<p<\infty$.*

Theorem 2.2. *Suppose T is a singular integral operator with non-smooth kernel as Definition 2.2. Let $0<\beta<1$ and $D^\alpha b_j\in Lip_\beta(\mathbb{R}^n)$ for all α with $|\alpha|=m_j$ and $j=1,\cdots,l$. Then T^b is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for any $1<p<n/l\beta$ and $1/p-1/q=l\beta/n$.*

3. Proof of theorems

To prove the theorem, we need the following lemmas.

Lemma 3.1. (see [1]) *Suppose that* $0 \le \eta < n$, $1 \le r < p < n/\eta$ *and* $1/q = 1/p - \eta/n$. *Then*

$$||M_{\eta,r}(f)||_{L^q} \le C||f||_{L^p}.$$

Lemma 3.2. (see [4]) *Let A be a function on* $\mathbb{R}^n$ *and* $D^\alpha A \in L^q(\mathbb{R}^n)$ *for* $|\alpha| = m$ *and some* $q > n$. *Then*

$$|R_m(A;x,y)| \le C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x,y)$ *is the cube centered at x and having side length* $5\sqrt{n}|x-y|$.

Lemma 3.3. [6,9] *For any* $\gamma > 0$, *there exists a constant* $C > 0$ *independent of* γ *such that*

$$|\{x \in \mathbb{R}^n : M(f)(x) > D\lambda, M_A^\#(f)(x) \le \gamma\lambda\}| \le C\gamma|\{x \in \mathbb{R}^n : M(f)(x) > \lambda\}|$$

for $\lambda > 0$, *where D is a fixed constant which only depends on n. So that*

$$||M(f)||_{L^p} \le C||M_A^\#(f)||_{L^p}$$

for every $f \in L^p(\mathbb{R}^n), 1 < p < \infty$.

Lemma 3.4. [9] *Let* $1 < p < \infty$ *and T a singular integral operator with non-smooth kernels as Definition 2.2. Then T is bounded on* $L^p(\mathbb{R}^n)$.

Lemma 3.5. *Let* $\{A_t, t > 0\}$ *be an "approximations to the identity",* $0 < \beta < 1$ *and* $b \in Lip_\beta(\mathbb{R}^n)$. *Then for every* $f \in L^p(\mathbb{R}^n)$ *and* $\tilde{x} \in \mathbb{R}^n$, *we have*

(a) $\sup_{Q \ni \tilde{x}} 1/|Q| \int_Q |A_{t_Q}((b-b_Q)f)(x)|dx \le C||b||_{Lip_\beta} M_{\beta,1}(f)(\tilde{x})$;

(b) $\sup_{Q \ni \tilde{x}} 1/(|Q|^{1+\beta/n}) \int_Q |A_{t_Q}((b-b_Q)f)(x)|dx \le C||b||_{Lip_\beta} M(f)(\tilde{x})$ *if* $\beta < \varepsilon$;

(c) $\sup_{Q \ni \tilde{x}} 1/|Q| \int_Q |A_{t_Q}(f)(x)|dx \le CM(f)(\tilde{x})$.

Proof. (a).Write

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |A_{t_Q}((b-b_Q)f)(x)|dx &\le \frac{1}{|Q|}\int_Q \int_{R^n} h_{t_Q}(x,y)|(b(y)-b_Q)f(y)|dydx \\
&= \frac{1}{|Q|}\int_Q \int_{2Q} h_{t_Q}(x,y)|(b(y)-b_Q)f(y)|dydx \\
&\quad + \sum_{k=1}^\infty \frac{1}{|Q|}\int_Q \int_{2^{k+1}Q\setminus 2^kQ} h_{t_Q}(x,y)|(b(y)-b_Q)f(y)|dydx \\
&= I_1 + I_2.
\end{aligned}$$

We have, by the Hölder's inequality,

$$\begin{aligned}
I_1 &\le \frac{C}{|Q||2Q|}\int_Q\int_{2Q} |(b(y)-b_Q)f(y)|dydx \le C\frac{1}{|2Q|}\int_{2Q}|(b(y)-b_Q)f(y)|dy \\
&\le C||b||_{Lip_\beta}|Q|^{\frac{\beta}{n}} \frac{1}{|2Q|}\int_{2Q}|f(y)|dy \le C||b||_{Lip_\beta}\frac{1}{|2Q|^{1-\frac{\beta}{n}}}\int_{2Q}|f(y)|dy \\
&\le C||b||_{Lip_\beta} M_{\beta,1}(f)(\tilde{x}).
\end{aligned}$$

For I_2, notice for $x \in Q$ and $y \in 2^{k+1}Q \backslash 2^k Q$, then $|x-y| \geq 2^{k-1} t_Q^{1/2}$ and $h_{t_Q}(x,y) \leq C t_Q^{-n/2} s(2^{2(k-1)})$. Thus

$$\begin{aligned} I_2 &\leq C \sum_{k=1}^{\infty} 2^{kn} s(2^{2(k-1)}) \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |(b(y)-b_Q) f(y)| dy \\ &\leq C \sum_{k=1}^{\infty} 2^{kn} s(2^{2(k-1)}) ||b||_{Lip_\beta} |2^{k+1}Q|^{\frac{\beta}{n}} \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(y)| dy \\ &\leq C \sum_{k=1}^{\infty} 2^{(k-1)n} s(2^{2(k-1)}) ||b||_{Lip_\beta} \frac{1}{|2^{k+1}Q|^{1-\frac{\beta}{n}}} \int_{2^{k+1}Q} |f(y)| dy \\ &\leq C \sum_{k=1}^{\infty} 2^{kn} s(2^{2(k-1)}) ||b||_{Lip_\beta} M_{\beta,1}(f)(x) \leq C ||b||_{Lip_\beta} M_{\beta,1}(f)(\tilde{x}), \end{aligned}$$

where the last inequality follows from

$$\sum_{k=1}^{\infty} 2^{kn} s(2^{2(k-1)}) \leq C \sum_{k=1}^{\infty} 2^{(k-1)n} s(2^{2(k-1)}) \leq C \sum_{k=1}^{\infty} 2^{-k\varepsilon} < \infty$$

for $\varepsilon > 0$.

(b).

$$\begin{aligned} &\frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |A_{t_Q}((b-b_Q)f)(x)| dx \\ &\leq \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q \int_{2Q} h_{t_Q}(x,y) |(b(y)-b_Q) f(y)| dy dx \\ &\quad + \sum_{k=1}^{\infty} \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q \int_{2^{k+1}Q \backslash 2^k Q} h_{t_Q}(x,y) |(b(y)-b_Q) f(y)| dy dx \\ &\leq C ||b||_{Lip_\beta} \frac{1}{|2Q|} \int_{2Q} |f(y)| dy + C \sum_{k=1}^{\infty} 2^{(n+\beta)k} s(2^{2(k-1)}) ||b||_{Lip_\beta} \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(y)| dy \\ &\leq C ||b||_{Lip_\beta} M(f)(\tilde{x}), \end{aligned}$$

where the last inequality follows from

$$\sum_{k=1}^{\infty} 2^{(n+\beta)k} s(2^{2(k-1)}) \leq C \sum_{k=1}^{\infty} 2^{\beta k} 2^{(k-1)n} s(2^{2(k-1)}) \leq C \sum_{k=1}^{\infty} 2^{k(\beta-\varepsilon)} < \infty$$

for $\beta < \varepsilon$. This completes the proof. ∎

Proof of Theorem 2.1. Following [9], we first prove the sharp estimate for T^b as following:

$$\frac{1}{|Q|^{1+l\beta/n}} \int_Q |T^b(f)(x) - A_{t_Q} T^b(f)(x)| dx \leq C M_r(f)(\tilde{x})$$

for any cube Q and $1 < r < p < \infty$. Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n} Q$ and $\tilde{b}_j(x) = b_j(x) - \sum_{|\alpha|=m} 1/(\alpha!)(D^\alpha b_j)_{\tilde{Q}} x^\alpha$, then, by [4], $R_m(b_j;x,y) = R_m(\tilde{b}_j;x,y)$ and $D^\alpha \tilde{b}_j = D^\alpha b_j - (D^\alpha b_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{\mathbb{R}^n \backslash \tilde{Q}}$,

$$T^b(f)(x) = \int_{\mathbb{R}^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^m} K(x,y) f(y) dy$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,y)}{|x-y|^m} K(x,y) f_1(y)dy \\
&\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{\mathbb{R}^n} \frac{R_{m_2}(\tilde{b}_2;x,y)(x-y)^{\alpha_1} D^{\alpha_1}\tilde{b}_1(y)}{|x-y|^m} K(x,y) f_1(y)dy \\
&\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{\mathbb{R}^n} \frac{R_{m_1}(\tilde{b}_1;x,y)(x-y)^{\alpha_2} D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m} K(x,y) f_1(y)dy \\
&\quad + \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{\mathbb{R}^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1(y) D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m} K(x,y) f_1(y)dy \\
&\quad + \int_{\mathbb{R}^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^m} K(x,y) f_2(y)dy \\
&= T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m} f_1\right) - T\left(\sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1} D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m} f_1\right) \\
&\quad - T\left(\sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2} D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\right) \\
&\quad + T\left(\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \frac{(x-\cdot)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\right) \\
&\quad + T\left(\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m} f_2\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&A_{t_Q} T^b(f)(x) \\
&= \int_{\mathbb{R}^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,y)}{|x-y|^m} K_{t_Q}(x,y) f_1(y)dy \\
&\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{\mathbb{R}^n} \frac{R_{m_2}(\tilde{b}_2;x,y)(x-y)^{\alpha_1} D^{\alpha_1}\tilde{b}_1(y)}{|x-y|^m} K_{t_Q}(x,y) f_1(y)dy \\
&\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{\mathbb{R}^n} \frac{R_{m_1}(\tilde{b}_1;x,y)(x-y)^{\alpha_2} D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m} K_{t_Q}(x,y) f_1(y)dy \\
&\quad + \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{\mathbb{R}^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1(y) D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m} K_{t_Q}(x,y) f_1(y)dy \\
&\quad + \int_{\mathbb{R}^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^m} K_{t_Q}(x,y) f_2(y)dy \\
&= A_{t_Q} T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m} f_1\right) - A_{t_Q} T\left(\sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1} D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m} f_1\right) \\
&\quad - A_{t_Q} T\left(\sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2} D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\right)
\end{aligned}
$$

$$+A_{t_Q}T\left(\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\frac{(x-\cdot)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right)$$
$$+A_{t_Q}T\left(\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_2\right),$$

thus

$$\frac{1}{|Q|^{1+2\beta/n}}\int_Q\left|T^b(f)(x)-A_{t_Q}T^b(f)(x)\right|dx$$
$$\leq\frac{1}{|Q|^{1+2\beta/n}}\int_Q\left|T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_1\right)\right|dx$$
$$+\frac{C}{|Q|^{1+2\beta/n}}\int_Q\left|T\left(\sum_{|\alpha_1|=m_1}\frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1}D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m}f_1\right)\right|dx$$
$$+\frac{C}{|Q|^{1+2\beta/n}}\int_Q\left|T\left(\sum_{|\alpha_2|=m_2}\frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2}D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right)\right|dx$$
$$+\frac{C}{|Q|^{1+2\beta/n}}\int_Q\left|T\left(\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{(x-\cdot)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right)\right|dx$$
$$+\frac{1}{|Q|^{1+2\beta/n}}\int_Q\left|A_{t_Q}T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_1\right)\right|dx$$
$$+\frac{C}{|Q|^{1+2\beta/n}}\int_Q\left|A_{t_Q}T\left(\sum_{|\alpha_1|=m_1}\frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1}D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m}f_1\right)\right|dx$$
$$+\frac{C}{|Q|^{1+2\beta/n}}\int_Q\left|A_{t_Q}T\left(\sum_{|\alpha_2|=m_2}\frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2}D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right)\right|dx$$
$$+\frac{C}{|Q|^{1+2\beta/n}}\int_Q\left|A_{t_Q}T\left(\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{(x-\cdot)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right)\right|dx$$
$$+\frac{1}{|Q|^{1+2\beta/n}}\int_Q\left|(T-A_{t_Q}T)\left(\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_2\right)\right|dx$$
$$:=I_0+I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8.$$

Now, let us estimate I_0, I_1, I_2, I_3 and I_4, respectively. First, by Lemma 3.2, we get, for $x\in Q$ and $y\in\tilde{Q}$,

$$|R_m(\tilde{b}_j;x,y)|\leq C|x-y|^m\sum_{|\alpha|=m}\sup_{x\in\tilde{Q}}|D^\alpha b_j(x)-(D^\alpha b_j)_{\tilde{Q}}|\leq C|x-y|^m|Q|^{\beta/n}\sum_{|\alpha|=m}||D^\alpha b_j||_{Lip_\beta},$$

thus, by the Hölder's inequality and L^r-boundedness of T, we obtain

$$I_0\leq\frac{C}{|Q|^{1+2\beta/n}}\int_Q\left|T\left(\frac{|Q|^{2\beta/n}|x-y|^{m_1+m_2}\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{Lip_\beta}\right)}{|x-y|^m}f_1\right)(x)\right|dx$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{Lip_\beta}\right)\frac{1}{|Q|}\int_Q|T(f_1)(x)|dx$$
$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{Lip_\beta}\right)\left(\frac{1}{|Q|}\int_{\mathbb{R}^n}|T(f_1)(x)|^r dx\right)^{1/r}$$
$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{Lip_\beta}\right)\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)|^r dx\right)^{1/r}$$
$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{Lip_\beta}\right)M_r(f)(\tilde{x});$$
$$I_1\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{Lip_\beta}\sum_{|\alpha_1|=m_1}\frac{1}{|Q|^{1+\beta/n}}\int_Q|T(D^{\alpha_1}\tilde{b}_1f_1)(x)|dx$$
$$\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{Lip_\beta}\sum_{|\alpha_1|=m_1}\frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_{\mathbb{R}^n}|T(D^{\alpha_1}\tilde{b}_1f_1)(x)|^r dx\right)^{1/r}$$
$$\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{Lip_\beta}\sum_{|\alpha_1|=m_1}\frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_{\mathbb{R}^n}|(D^{\alpha_1}b_1(x)-(D^{\alpha_1}b_1)_{\tilde{Q}})f_1(x)|^r dx\right)^{1/r}$$
$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{Lip_\beta}\right)\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)|^r dx\right)^{1/r}$$
$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{Lip_\beta}\right)M_r(f)(\tilde{x}).$$

For I_2, similar to the proof of I_2, we get

$$I_2\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{Lip_\beta}\right)M_r(f)(\tilde{x}).$$

Similarly, for I_3, we obtain

$$I_3\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}|Q|^{-2\beta/n}\left(\frac{1}{|Q|}\int_{\mathbb{R}^n}|T(D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2f_1)(x)|^r dx\right)^{1/r}$$
$$\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}|Q|^{-2\beta/n}\left(\frac{1}{|Q|}\int_{\mathbb{R}^n}|D^{\alpha_1}\tilde{b}_1(x)D^{\alpha_2}\tilde{b}_2(x)f_1(x)|^r dx\right)^{1/r}$$
$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{Lip_\beta}\right)\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)|^r dx\right)^{1/r}$$
$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{Lip_\beta}\right)M_r(f)(\tilde{x}).$$

For I_4, I_5, I_6 and I_7, by Lemma 3.5 and similar to the proof of I_0, I_1, I_2 and I_3, we get

$$\begin{aligned}
I_4+I_5+I_6+I_7 \le{}& C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{Lip_\beta}\right)\frac{1}{|Q|}\int_Q |A_{t_Q}T(f_1)(x)|dx \\
&+C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{Lip_\beta}\sum_{|\alpha_1|=m_1}\frac{1}{|Q|^{1+\beta/n}}\int_Q |A_{t_Q}T(D^{\alpha_1}\tilde{b}_1 f_1)(x)|dx \\
&+C\sum_{|\alpha_1|=m_1}||D^{\alpha_1}b_1||_{Lip_\beta}\sum_{|\alpha_2|=m_2}\frac{1}{|Q|^{1+\beta/n}}\int_Q |A_{t_Q}T(D^{\alpha_2}\tilde{b}_2 f_1)(x)|dx \\
&+C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\frac{1}{|Q|^{1+2\beta/n}}\int_Q |A_{t_Q}T(D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2 f_1)(x)|dx \\
\le{}& C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{Lip_\beta}\right)M_r(f)(\tilde{x}).
\end{aligned}$$

For I_8, we write

$$\begin{aligned}
&(T-A_{t_Q}T)\left(\frac{\prod_{j=1}^{2}R_{m_j+1}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_2\right) \\
&=\int_{\mathbb{R}^n}\frac{\prod_{j=1}^{2}R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^m}(K(x,y)-K_{t_Q}(x,y))f_2(y)dy \\
&=\int_{\mathbb{R}^n}\frac{\prod_{j=1}^{2}R_{m_j}(\tilde{b}_j;x,y)}{|x-y|^m}(K(x,y)-K_{t_Q}(x,y))f_2(y)dy \\
&\quad-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{\mathbb{R}^n}\frac{D^{\alpha_1}\tilde{b}_1(y)(x-y)^{\alpha_1}R_{m_2}(\tilde{b}_2;x,y)}{|x-y|^m}(K(x,y)-K_{t_Q}(x,y))f_2(y)dy \\
&\quad-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{\mathbb{R}^n}\frac{D^{\alpha_2}\tilde{b}_2(y)(x-y)^{\alpha_2}R_{m_1}(\tilde{b}_1;x,y)}{|x-y|^m}(K(x,y)-K_{t_Q}(x,y))f_2(y)dy \\
&\quad+\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{\mathbb{R}^n}\frac{D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}(K(x,y)-K_{t_Q}(x,y))f_2(y)dy.
\end{aligned}$$

By Lemma 3.2 and the following inequality, for $b\in Lip_\beta(R^n)$,

$$|b(x)-b_Q|\le\frac{1}{|Q|}\int_Q ||b||_{Lip_\beta}|x-y|^\beta dy\le C||b||_{Lip_\beta}(|x-x_0|+\sqrt{n}d/2)^\beta,$$

we get

$$|R_{m_j}(\tilde{b}_j;x,y)|\le C\sum_{|\alpha|=m_j}||D^\alpha b_j||_{Lip_\beta}(|x-x_0|+d)^{m_j+\beta}.$$

Note that $|x-y|\sim|x_0-y|$ for $x\in Q$ and $y\in R^n\setminus\tilde{Q}$, so that, we obtain, by the conditions on K and K_{t_Q},

$$\left|(T-A_{t_Q}T)\left(\frac{\prod_{j=1}^{2}R_{m_j+1}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_2\right)\right|$$

$$\leq C\int_{\mathbb{R}^n\setminus\tilde{Q}}\frac{|x-x_0|^\delta}{|x_0-y|^{m+n+\delta}}\prod_{j=1}^{2}|R_{m_j}(\tilde{b}_j;x,y)||f(y)|dy$$

$$+C\sum_{|\alpha_1|=m_1}\int_{\mathbb{R}^n\setminus\tilde{Q}}\frac{|D^{\alpha_1}\tilde{b}_1(y)||x-x_0|^\delta}{|x_0-y|^{m_2+n+\delta}}|R_{m_2}(\tilde{b}_2;x,y)||f(y)|dy$$

$$+C\sum_{|\alpha_2|=m_2}\int_{\mathbb{R}^n\setminus\tilde{Q}}\frac{|D^{\alpha_2}\tilde{b}_2(y)||x-x_0|^\delta}{|x_0-y|^{m_1+n+\delta}}|R_{m_1}(\tilde{b}_1;x,y)||f(y)|dy$$

$$+C\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\int_{\mathbb{R}^n\setminus\tilde{Q}}\frac{|D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)||x-x_0|^\delta}{|x_0-y|^{n+\delta}}|f(y)|dy$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{Lip_\beta}\right)\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|^\delta}{|x_0-y|^{n+\delta-2\beta}}|f(y)|dy$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{Lip_\beta}\right)|Q|^{2\beta/n}\sum_{k=0}^{\infty}2^{k(2\beta-\delta)}\frac{1}{|2^{k+1}\tilde{Q}|}\int_{2^{k+1}\tilde{Q}}|f(y)|dy$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{Lip_\beta}\right)|Q|^{2\beta/n}M_r(f)(\tilde{x}),$$

thus

$$I_8\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{Lip_\beta}\right)M_r(f)(\tilde{x}).$$

We now put these estimates together and take the supremum over all Q such that $\tilde{x}\in Q$, then, by Lemma 3.1,

$$||T^b(f)||_{\dot{F}^{2\beta,\infty}_{p,A}}\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{Lip_\beta}\right)||M_r(f)||_{L^p}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{Lip_\beta}\right)||f||_{L^p}.$$

This completes the proof of Theorem 2.1. ∎

Proof of Theorem 2.2. By using the same argument as in proof of Theorem 2.1, we obtain

$$\frac{1}{|Q|}\int_Q|T^b(f)(x)-A_{t_Q}T^b(f)(x)|dx\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{Lip_\beta}\right)M_{2\beta,r}(f),$$

for any cube Q and $1<r<p<\infty$, thus, we get the sharp estimate of T^b as following

$$M_A^\#(T^b(f))\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{Lip_\beta}\right)M_{2\beta,r}(f).$$

By Lemma 3.3, $||M(g)||_{L^p}\leq C||M_A^\#(g)||_{L^p}$ for every $g\in L^p(\mathbb{R}^n)$ and $1<p<\infty$. Now, using Lemma 3.1 and 3.4, we get

$$||T^b(f)||_{L^q}\leq||M(T^b(f))||_{L^q}\leq C||M_A^\#(T^b(f))||_{L^q}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{Lip_\beta}\right)||M_{2\beta,r}(f)||_{L^q}\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{Lip_\beta}\right)||f||_{L^p}.$$

This completes the proof of the theorem. ∎

4. Applications

In this section we shall apply Theorems of the paper to the holomorphic functional calculus of linear elliptic operators. First, we review some definitions regarding the holomorphic functional calculus (see [6,9]). Given $0 \leq \theta < \pi$, define

$$S_\theta = \{z \in \mathbb{C} : |\arg(z)| \leq \theta\}\bigcup\{0\}$$

and its interior by S_θ^0. Set $\tilde{S}_\theta = S_\theta \setminus \{0\}$. An closed operator L on some Banach space E is said to be of type θ if its spectrum $\sigma(L) \subset S_\theta$ and for every $\nu \in (\theta,\pi]$, there exists a constant C_ν such that

$$|\eta|||(\eta I - L)^{-1}|| \leq C_\nu, \quad \eta \notin \tilde{S}_\theta.$$

For $\nu \in (0,\pi]$, let

$$H_\infty(S_\mu^0) = \{f : S_\theta^0 \to \mathbb{C} : \ f \text{ is holomorphic and } ||f||_{L^\infty} < \infty\},$$

where $||f||_{L^\infty} = \sup\{|f(z)| : z \in S_\mu^0\}$. Set

$$\Psi(S_\mu^0) = \left\{g \in H_\infty(S_\mu^0) : \exists s > 0, \exists c > 0 \text{ such that } |g(z)| \leq c\frac{|z|^s}{1+|z|^{2s}}\right\}.$$

If L is of type θ and $g \in H_\infty(S_\mu^0)$, we define $g(L) \in L(E)$ by

$$g(L) = -(2\pi i)^{-1}\int_\Gamma (\eta I - L)^{-1} g(\eta)d\eta,$$

where Γ is the contour $\{\xi = re^{\pm i\phi} : r \geq 0\}$ parameterized clockwise around S_θ with $\theta < \phi < \mu$. If, in addition, L is one-one and has dense range, then, for $f \in H_\infty(S_\mu^0)$,

$$f(L) = [h(L)]^{-1}(fh)(L),$$

where $h(z) = z(1+z)^{-2}$. L is said to have a bounded holomorphic functional calculus on the sector S_μ, if

$$||g(L)|| \leq N||g||_{L^\infty}$$

for some $N > 0$ and for all $g \in H_\infty(S_\mu^0)$.

Now, let L be a linear operator on $L^2(R^n)$ with $\theta < \pi/2$ so that $(-L)$ generates a holomorphic semigroup e^{-zL}, $0 \leq |\arg(z)| < \pi/2 - \theta$. Applying Theorem 6 of [9], we get

Theorem 4.1. *Let* $0 < \beta < 1$ *and* $1 \leq p < \infty$. *The Triebel-Lizorkin space associated with the "approximations to the identity"* $\{A_t, t > 0\}$ *is defined by*

$$\dot{F}_{p,A}^{\beta,\infty}(\mathbb{R}^n) = \{f \in L^1_{loc}(\mathbb{R}^n) : ||f||_{\dot{F}_{p,A}^{\beta,\infty}} < \infty\},$$

where

$$||f||_{\dot{F}_{p,A}^{\beta,\infty}} = \left|\left|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+\beta/n}}\int_Q |f(x) - A_{t_Q}(f)(x)|dx\right|\right|_{L^p},$$

Assume the following conditions are satisfied:

(i). *The holomorphic semigroup* e^{-zL}, $0 \le |\arg(z)| < \pi/2 - \theta$ *is represented by the kernels* $a_z(x,y)$ *which satisfy, for all* $\nu > \theta$*, an upper bound*

$$|a_z(x,y)| \le c_\nu h_{|z|}(x,y)$$

for $x,y \in \mathbb{R}^n$*, and* $0 \le |\arg(z)| < \pi/2 - \theta$*, where* $h_t(x,y) = Ct^{-n/2}s(|x-y|^2/t)$ *and* s *is a positive, bounded and decreasing function satisfying*

$$\lim_{r\to\infty} r^{n+\varepsilon}s(r^2) = 0.$$

(ii). *The operator* L *has a bounded holomorphic functional calculus in* $L^2(\mathbb{R}^n)$*, that is, for all* $\nu > \theta$ *and* $g \in H_\infty(S^0_\mu)$*, the operator* $g(L)$ *satisfies*

$$||g(L)(f)||_{L^2} \le c_\nu ||g||_{L^\infty}||f||_{L^2}.$$

Let $D^\alpha b_j \in Lip_\beta(\mathbb{R}^n)$ *for all* α *with* $|\alpha| = m_j$ *and* $j = 1,\cdots,l$.

(a). *If* $0 < \beta < \min(1,\varepsilon,\delta/l)$*, then the multilinear operator* $g(L)^b$ *associated to* $g(L)$ *is bounded from* $L^p(\mathbb{R}^n)$ *to* $\dot{F}^{l\beta,\infty}_{p,A}(\mathbb{R}^n)$ *for any* $1 < p < \infty$*;*

(b). *If* $0 < \beta < 1$*, then* $g(L)^b$ *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$ *for any* $1 < p < n/l\beta$ *and* $1/p - 1/q = l\beta/n$.

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References

[1] S. Chanillo, A note on commutators, *Indiana Univ. Math. J.* **31** (1982), no. 1, 7–16.

[2] J. Cohen, A sharp estimate for a multilinear singular integral in $\mathbf{R}^n$, *Indiana Univ. Math. J.* **30** (1981), no. 5, 693–702.

[3] J. Cohen and J. A. Gosselin, On multilinear singular integrals on $\mathbf{R}^n$, *Studia Math.* **72** (1982), no. 3, 199–223.

[4] J. Cohen and J. A. Gosselin, A BMO estimate for multilinear singular integrals, *Illinois J. Math.* **30** (1986), no. 3, 445–464.

[5] D. Deng and L. Yan, Commutators of singular integral operators with non-smooth kernels, *Acta Math. Sci. Ser. B Engl. Ed.* **25** (2005), no. 1, 137–144.

[6] X. T. Duong and A. MacIntosh, Singular integral operators with non-smooth kernels on irregular domains, *Rev. Mat. Iberoamericana* **15** (1999), no. 2, 233–265.

[7] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland Mathematics Studies, 116, North-Holland, Amsterdam, 1985.

[8] S. Janson, Mean oscillation and commutators of singular integral operators, *Ark. Mat.* **16** (1978), no. 2, 263–270.

[9] L. Liu, Continuity for multilinear integral operators on Besov spaces, *Commun. Korean Math. Soc.* **22** (2007), no. 2, 259–275.

[10] L. Liu, Boundedness for multilinear singular integral operators on Morrey spaces, *Bull. Malays. Math. Sci. Soc. (2)* **33** (2010), no. 1, 93–103.

[11] L. Liu, Weighted sharp inequality for vector-valued multilinear integral operator, *Bull. Malays. Math. Sci. Soc. (2)* **34** (2011), no. 2, 201–214.

[12] J. M. Martell, Sharp maximal functions associated with approximations of the identity in spaces of homogeneous type and applications, *Studia Math.* **161** (2004), no. 2, 113–145.

[13] M. Paluszyński, Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss, *Indiana Univ. Math. J.* **44** (1995), no. 1, 1–17.

[14] C. Pérez and R. Trujillo-González, Sharp weighted estimates for multilinear commutators, *J. London Math. Soc. (2)* **65** (2002), no. 3, 672–692.

[15] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Mathematical Series, 43, Princeton Univ. Press, Princeton, NJ, 1993.