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Regular Cryptic Super r-Ample Semigroups

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Abstract. By utilizing the $(*,\sim)$-Green's relations and quasi-strong semilattice of semigroups, we show that a super r-ample semigroup is a regular cryptic super r-ample semigroup if and only if it is a quasi-strong semilattice of completely $\mathscr{J}^{*,\sim}$-simple semigroups. This result can be regarded as a generalized result of M. Petrich on normal cryptogroups and regular cryptogroups within the class of regular semigroups to the class of r-ample semigroups.

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1. Introduction

It is known that the Green's relations are useful in the study of the structure of the regular semigroups, for example, see [1, 3, 21, 22]. In particular, it was shown by Clifford in the early fifty of last century that a semigroup is a completely regular semigroup if and only if it is a semilattice of completely simple semigroups [1].

By a completely regular semigroup, we mean a semigroup in which every its $\mathscr{H}$-class contains an idempotent of the semigroup. By using the above well known semilattice decomposition theorem of completely regular semigroups, Petrich proved that a completely regular semigroup is a normal cryptogroup, that is, a completely regular semigroup whose Green's relation $\mathscr{H}$ forms a normal band congruence if and only if it is a strong semilattice of completely simple semigroups,see [17]. The well known Clifford theorem of completely regular semigroup was also later generalized by Fountain in [3] and Ren *et al.* in [18]. In particular, Fountain proved that if each of the $\mathscr{H}^*$-class of a superabundant semigroup S contains an idempotent of S, then the semigroup S is a superabundant semigroup if and only if S is a semilattice of completely $\mathscr{J}^*$-simple semigroups.

In recent years, the structure of superabundant semigroups were widely investigated by Ren and Shum in [19–20] by using some kind of generalized Green's relations on a given semigroup, respectively.

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To study the structure of abundant semigroups, we often use the so called Green $*$-relations defined on the semigroup. A special kind of abundant semigroup has been studied by Guo [9] by using some other generalized Green reaktions. The Green $*$-relations were first defined by Pastijn in [14] in 1975 and were adopted by Fountain in [3]. Let S be a semigroup. Then, we define the following set of relations by the following equalities:

$$\mathscr{L}^* = \{(a,b) \in S \times S : (\forall x, y \in S^1) ax = ay \Leftrightarrow bx = by\},$$
$$\mathscr{R}^* = \{(a,b) \in S \times S : (\forall x, y \in S^1) xa = ya \Leftrightarrow xb = yb\},$$
$$\mathscr{H}^* = \mathscr{L}^* \cap \mathscr{R}^*, \ \mathscr{D}^* = \mathscr{L}^* \vee \mathscr{R}^*,$$
$$\mathscr{J}^* = \{(a,b) \in S \times S : J^*(a) = J^*(b)\},$$

where $J^*(a)$ is used to denote the smallest ideal containing the element a which is saturated by $\mathscr{L}^*$ and $\mathscr{R}^*$, that is, $J^*(a)$ is a union of some $\mathscr{L}^*$-classes and also a union of some $\mathscr{R}^*$-classes. It is noted that the following definition $\widetilde{\mathscr{R}}$ on a semigroup S was defined by Lawson in [12–13] as in the followings:

$$a\widetilde{\mathscr{R}}b \Leftrightarrow (\forall e \in E(S)) ea = a \leftrightarrow eb = b,$$

where $E(S)$ is the idempotents set of S. It is easy to see that $\mathscr{R}^* \subseteq \widetilde{\mathscr{R}}$ and for the regular elements a, b of a semigroup S, we have $a\mathscr{R}b$ if and only $a\widetilde{\mathscr{R}}b$. The term "*ample semigroups*" in the literature was first stated by Gomes and Gould in [7, 8]. In fact, an *ample semigroup* means exactly the same as the so called *type A semigroup* studied by Lawson in [12]. In particular, a type A (ample) semigroup is a special case of an adequate semigroup considered by Fountain in [4]. Thus, we always regard an adequate semigroup is a special abundant semigroup.

For the class of abundant semigroups and their special subclasses, the reader is referred to Shum in [21], [22] and [23]. We notice that it was shown by Lawson that the type A semigroups constitute a class of semigroups that includes the full subsemigroups of inverse semigroups and cancellative monoids. In particular, Lawson called any two elements in a semigroup S $\mathscr{L}^*$-related if they are $\mathscr{L}$-related in some oversemigroup. He then proved that S is a type A semigroup if and only if each element of S is $\mathscr{L}$-related to an idempotent of S and is also R-related to an idempotent of S. By considering the modified generalized Green's relations on a semigroup S, Fu, Kong and Shum have recently established a structure theorem of G-semilattice decomposition of a super r-ample semigroup in [6].

In this paper, a new definition of a Green generalized relation, namely, the $(*, \sim)$-Green's relations is introduced on a semigroup S which is a combination of the well known Green-$*$ relation and the $\widetilde{\mathscr{R}}$ Green relation on S. For more information of the $(*, \sim)$-Green's relations, the reader is referred to [6]. We now introduce below the concept of a quasi-strong semilattice of semigroups and describe the semilattice decomposition of cryptic super r-ample semigroups. By using this quasi-strong semilattice decomposition of semigroups,we are able to show that a super r-ample semigroup S is a regular cryptic super r-ample semigroup if and only if S is a quasi-strong semilattice of completely $\mathscr{J}^{*,\sim}$-simple semigroups. Thus, our result generalizes and enriches the main results given by Fu, Kong and Shum in [6]. Moreover, some of the well known results obtained by Clifford, Petrich and Fountain (see 1, 3, 17) on completely regular semigroups and abundant semigroups, as mentioned in the first paragraph, are extended and generalized.

We state below the newly defined $(*,\sim)$-Green relations on a semigroup S. These $(*,\sim)$-Green relations can be regarded as a sort of generalized Green's relations on a semigroup S.

$$\mathscr{L}^{*,\sim}=\mathscr{L}^*, \mathscr{R}^{*,\sim}=\widetilde{R}$$
$$\mathscr{H}^{*,\sim}=\mathscr{L}^{*,\sim}\cap\mathscr{R}^{*,\sim},\ \mathscr{D}^{*,\sim}=\mathscr{L}^{*,\sim}\vee\mathscr{R}^{*,\sim}$$
$$\mathscr{J}^{*,\sim}=\{(a,b)\in S\times S: J^{*,\sim}(a)=J^{*,\sim}(b)\},$$

where $J^{*,\sim}(a)$ is the smallest ideal containing a saturated by $\mathscr{L}^{*,\sim}$ and $\mathscr{R}^{*,\sim}$. It can be easily seen that the $\mathscr{L}^{*,\sim}$ relation on S is a right congruence on S while the $(*,\sim)$-Green relation $\mathscr{R}^{*,\sim}$ is only an equivalence on S.

One can easily observe that there is at most one idempotent of S contained in each of the $\mathscr{H}^{*,\sim}$-class of S. If $e\in H_a^{*,\sim}\cap E(S)$ for some $a\in S$, then we write e as x^0, for any $x\in H_a^{*,\sim}$. It is clear that for any $x\in H_a^{*,\sim}$ with $a\in S$, we have $x=xx^0=x^0x$.

If a semigroup S is regular, then every $\mathscr{L}$-class of S contains at least one idempotent and so does every $\mathscr{R}$-class of S. If S is a completely regular semigroup, then every $\mathscr{H}$-class of S contains an idempotent of S. In this case, we see immediately that every $\mathscr{H}$-class is a group. According to Fountain in [3]. A semigroup is an *abundant* semigroup if every $\mathscr{L}^*$- and $\mathscr{R}^*$-class of the semigroup S contains an idempotent of S. Thus, one can immediately see that $\mathscr{L}^*=\mathscr{L}$ on all the regular elements of a semigroup S. Hence,it is trivial that every regular semigroup is obviously an abundant semigroup. We now call such a semigroup S *superabundant* if each of its $\mathscr{H}^*$-class contains an idempotent, in such a semigroup S, we can easily see that every $\mathscr{H}^*$-class of S is a cancellative monoid which is a generalization of a completely regular semigroup in the class of abundant semigroups. We now call such a semigroup r-ample if every $\mathscr{L}^{*,\sim}$-class and every $\mathscr{R}^{*,\sim}$-class of the semigroup S contains an idempotent of S. Certainly, an abundant semigroup is an r-ample semigroup [6], however, the converse of the above statement is not true, mfor example, see [8]. We now call a semigroup S super r-ample if each of the $\mathscr{H}^{*,\sim}$-class of S contains an idempotent of S, in such a semigroup, every $\mathscr{H}^{*,\sim}$-class is a left cancellative monoid which is a generalization of the completely regular semigroups and the superabundant semigroups in the class of r-ample semigroups.

Recall that a *regular band(normal band)* is a band that satisfies the identity $axya=axaya(axya=ayxa)$ and a semigroup is called a *cryptic* semigroup if each of its $(*,\sim)$-Green's relation $\mathscr{H}^{*,\sim}$ is a congruence on S in Clifford and Preston [1].

We now call a *regular(normal) cryptic super r-ample semigroup* a super r-ample semigroup if its $(*,\sim)$-Green's relation $\mathscr{H}^{*,\sim}$ is a regular(normal) band congruence.

For further notations and terminology such as the strong semilattice decomposition of semigroups, the reader is referred to [2, 3, 6]. For some concepts that have appeared in the literature, we occasionally use the alternative though equivalent definitions.

2. Preliminaries

Since a completely simple semigroup S is a $\mathscr{J}$-simple completely regular semigroup with the Green's relation $\mathscr{H}$ a congruence on S. We call a super r-ample semigroup S a completely $\mathscr{J}^{*,\sim}$-simple semigroup if it is $\mathscr{J}^{*,\sim}$-simple and the $(*,\sim)$-Green relation $\mathscr{H}^{*,\sim}$ is a congruence on S.

The following lemma is a crucial lemma for the super r-ample semigroups.

Lemma 2.1. *Let S be a super r-ample semigroup. Then $\mathscr{H}^{*,\sim}$ is a congruence on S if and only if for any $a,b \in S$, $(ab)^0 = (a^0b^0)^0$.*

Proof. Necessity. For any $a,b \in S$, $a\mathscr{H}^{*,\sim}a^0$ and $b\mathscr{H}^{*,\sim}b^0$. Since $\mathscr{H}^{*,\sim}$ is a congruence on S,$ab\mathscr{H}^{*,\sim}a^0b^0$ and so $ab\mathscr{H}^{*,\sim}(ab)^0$.

Sufficiency. Since $\mathscr{H}^{*,\sim}$ is an equivalence on the semigroup S, we only need prove that $\mathscr{H}^{*,\sim}$ is compatible. For this purpose, let $(a,b) \in \mathscr{H}^{*,\sim}$ and $c \in S$. Then, we have $(ca)^0 = (c^0a^0)^0 = (c^0b^0)^0 = (cb)^0$ and so $\mathscr{H}^{*,\sim}$ is left compatible with the semigroup multiplication.

Similarly, $\mathscr{H}^{*,\sim}$ is also right compatible with the semigroup multiplication. Thus, $\mathscr{H}^{*,\sim}$ is indeed a congruence on S. ∎

Lemma 2.2. *If e,f are $\mathscr{D}^{*,\sim}$-related idempotents of a super r-ample semigroup, then $e\mathscr{D}f$.*

Proof. Since $e\mathscr{D}^{*,\sim}f$, there are elements $a_1,\cdots,a_k$ of S such that $e\mathscr{L}^{*,\sim}a_1\mathscr{R}^{*,\sim}a_2\cdots a_k\mathscr{L}^{*,\sim}f$. Since S is super r-ample, $e\mathscr{L}^{*,\sim}a_1^0\mathscr{R}^{*,\sim}a_2^0\cdots a_k^0\mathscr{R}^{*,\sim}f$. Thus, $e\mathscr{D}f$ since for regular elements, we have $\mathscr{R} = \mathscr{R}^* = \widetilde{\mathscr{R}}$ and $\mathscr{L} = \mathscr{L}^*$. ∎

Corollary 2.1. *If S is a super r-ample semigroup, then*

$$\mathscr{D}^{*,\sim} = \mathscr{L}^{*,\sim} \circ \mathscr{R}^{*,\sim} = \mathscr{R}^{*,\sim} \circ \mathscr{L}^{*,\sim}.$$

Proof. If $a,b \in S$ and $a\mathscr{D}^{*,\sim}b$, then by Lemma 2.2, we have $a^0\mathscr{D}b^0$. Thus, there exist elements c,d in S with $a^0\mathscr{L}c\mathscr{R}b^0$ and $a^0\mathscr{R}d\mathscr{L}b^0$. Now,it can be easily verified that $a\mathscr{L}^{*,\sim}c\mathscr{R}^{*,\sim}b$ and $a\mathscr{R}^{*,\sim}d\mathscr{L}^{*,\sim}b$ and the result follows. ∎

The relationship of the idempotents of a super r-ample semigroup is described in the following lemma.

Lemma 2.3. *Let e,f be idempotents in a super r-ample semigroup S. If $e\mathscr{J}f$, then $e\mathscr{D}f$.*

Proof. Since $SeS = SfS$, there exist elements x,y,s,t in S such that $f = set, e = xfy$. Let $h = (fy)^0$ and $k = (se)^0$. Then $hfy = fy = ffy$ so that $h = h^2 = fh$, and $sek = se = see$ so that $k = k^2 = ke$. It hence follows that hf, ek are idempotents with $hf\mathscr{R}h$ and $ek\mathscr{L}k$. Hence, $ehf\mathscr{R}eh$ and $ekf\mathscr{L}kf$. Now, we deduce that $eh = xfyh = xfy = e$ and $kf = kset = set = f$ so that $e\mathscr{R}ef\mathscr{L}f$.This leads to $e\mathscr{D}f$. ∎

Proposition 2.1. *If a is an element of a cryptic super r-ample semigroup S, then $J^{*,\sim}(a) = Sa^0S$.*

Proof. Certainly, $a^0 \in J^{*,\sim}(a)$ so that $Sa^0S \subseteq J^{*,\sim}(a)$. We now show that the ideal Sa^0S is indeed an ideal saturated by $\mathscr{L}^{*,\sim}$ and $\mathscr{R}^{*,\sim}$. Since $a = aa^0 \in Sa^0S$, the result follows. Let $b = xa^0y \in Sa^0S(x,y \in S)$ and $k = (a^0y)^0$. Then, we have $a^0a^0y = ka^0y$ so that $a^0(a^0y)^0 = k^2 = k$. Also since $\mathscr{H}^{*,\sim}$ is a congruence on S, $xa^0y\mathscr{H}^{*,\sim}xk$. Now, if $h = (xk)^0 = (xa^0y)^0$, Then $xkh = xkk$ so that $h = h^2 = hk = ha^0k \in Sa^0S$. Hence, if $c \in L_b^{*,\sim}, d \in R_b^{*,\sim}$ then $c = ch, d = hd \in Sa^0S$. This shows that Sa^0S is an ideal saturated by $\mathscr{L}^{*,\sim}$ and $\mathscr{R}^{*,\sim}$, as required. ∎

Proposition 2.2. *In a completely $\mathscr{J}^{*,\sim}$-simple semigroup S, we have $\mathscr{J}^{*,\sim} = \mathscr{D}^{*,\sim}$.*

Proof. Suppose that $a,b \in S$ with $a\mathscr{J}^{*,\sim}b$. Then, by Proposition 2.1, we have $Sa^0S = Sb^0S$. By Lemma 2.3, $a^0\mathscr{D}b^0$ and so $a\mathscr{H}^{*,\sim}a^0\mathscr{D}b^0\mathscr{H}^{*,\sim}b$,which implies that $a\mathscr{D}^{*,\sim}b$ and hence $\mathscr{J}^{*,\sim} \subseteq \mathscr{D}^{*,\sim}$. Conversely, let $a,b \in S$ with $a\mathscr{D}^{*,\sim}b$. By Corollary 2.1, there exists $c \in S$

such that $a\mathscr{L}^{*,\sim}c\mathscr{R}^{*,\sim}b$. Thus $a^0\mathscr{L}c^0\mathscr{R}b^0$ and so $Sa^0S = Sc^0S = Sb^0S$. By Proposition 2.1, $(a,b) \in \mathscr{J}^{*,\sim}$ and hence $\mathscr{D}^{*,\sim} \subseteq \mathscr{J}^{*,\sim}$. Now, we have $\mathscr{J}^{*,\sim} = \mathscr{D}^{*,\sim}$. ∎

The following lemma is a crucial lemma of a completely $\mathscr{J}^{*,\sim}$-simple semigroup.

Lemma 2.4. *The idempotents of a completely $\mathscr{J}^{*,\sim}$-simple S are primitive idempotents.*

Proof. Let e, f be idempotents of the semigroup S with $e \le f$. Since S is a completely $\mathscr{J}^{*,\sim}$-simple semigroup, it follows from Proposition 2.1 that $f \in SeS$. Now, by the first part of Exercise 3 in [1, §8.4], there exists an idempotent g of S such that $f\mathscr{D}g$ and $g \le e$. Let $a \in S$ be such that $f\mathscr{L}a\mathscr{R}g$. Then, $f\mathscr{L}a^0\mathscr{R}g$ and since $g \le f$, we have

$$a^0 = ga^0(gf)a^0 = g(fa^0) = gf = g.$$

Now, we have $g \le f$ and $g\mathscr{L}f$ so that $f = fg = g$. But $g \le e$ so that $e = f$. This shows that and all idempotent of S are primitive. ∎

Lemma 2.5. *In a completely $\mathscr{J}^{*,\sim}$-simple semigroup S, the regular elements of S generate a completely simple subsemigroup.*

Proof. Let a, b be regular elements of S. Since S consists of a single $\mathscr{D}^{*,\sim}$-class(by Proposition 2.2), it follows from Corollary 2.1 that there exists an element $c \in S$ with $a\mathscr{L}^{*,\sim}c\mathscr{R}^{*,\sim}b$. Hence, we have $a\mathscr{L}^{*,\sim}c^0\mathscr{R}^{*,\sim}b$. Thus, $c^0b = b$ and $a\mathscr{L}c^0$ since a is regular. Now, we see that $ab\mathscr{L}b$ and the regularity of ab follows from that of b. The property of completely simple of the subsemigroup generated by the regular elements follows from Proposition 2.2, Lemma 2.2 and Corollary 2.1 easily. ∎

Theorem 2.1. *Let S be a cryptic super r-ample semigroup. Then S is a semi-lattice Y of completely $\mathscr{J}^{*,\sim}$-simple semigroups $S_\alpha (\alpha \in Y)$ such that for $\alpha \in Y$ and $a \in S_\alpha$, $\mathscr{L}_a^{*,\sim}(S) = \mathscr{L}_a^{*,\sim}(S_\alpha)$, $\mathscr{R}_a^{*,\sim}(S) = \mathscr{R}_a^{*,\sim}(S_\alpha)$.*

Proof. If $a \in S$, then $a\mathscr{H}^{*,\sim}a^2$ so that by Proposition 2.1, $J^{*,\sim}(a) = J^{*,\sim}(a^2)$. Now for $a, b \in S$, $(ab)^2 \in SbaS$, and so we have

$$J^{*,\sim}(ab) = J^{*,\sim}((ab)^2) \subseteq J^{*,\sim}(ba)$$

and by symmetry, we get $J^{*,\sim}(ab) = J^{*,\sim}(ba)$. By Proposition 2.1, $J^{*,\sim}(a) = Sa^0S$, $J^{*,\sim}(b) = Sb^0S$ so that if $c \in J^{*,\sim}(a) \cap J^{*,\sim}(b)$, we have $c = xa^0y = zb^0t$ for some $x, y, z, t \in S$. Now, $c^2 = zb^0txa^0y \in Sb^0txa^0S \subseteq J^{*,\sim}(b^0txa^0)$ and $J^{*,\sim}(b^0txa^0) = J^{*,\sim}(a^0b^0tx)$ by the preceding paragraph. Hence, we deduce that $c^2 \in J^{*,\sim}(a^0b^0)$ and since $c\mathscr{H}^{*,\sim}c^2$, we also have $c \in J^{*,\sim}(a^0b^0)$. Since $a\mathscr{H}^{*,\sim}a^0$, $b\mathscr{H}^{*,\sim}b^0$ and $\mathscr{H}^{*,\sim}$ is a congruence on S and so $ab\mathscr{H}^{*,\sim}a^0b^0$. Hence, $c \in \mathscr{H}^{*,\sim}(ab)$. This shows that $J^{*,\sim}(a) \cap J^{*,\sim}(b) \subseteq J^{*,\sim}(ab)$ and since the opposite inclusion is clear, we have proved that $J^{*,\sim}(a) \cap J^{*,\sim}(b) = J^{*,\sim}(ab)$.

Now, it is clear that the set Y of all ideals $J^{*,\sim}(a) (a \in S)$ is a semilattice under set intersection and that the map $a \mapsto J^{*,\sim}(a)$ is a homomorphism from S onto Y. The inverse image of $J^{*,\sim}(a)$ is just the $\mathscr{J}^{*,\sim}$-class $J_a^{*,\sim}$ which is a subsemigroup of S. This proves that the semigroup S is a semilattice Y of the semigroups $J_a^{*,\sim}$.

Finally, we let a, b be elements of $\mathscr{J}^{*,\sim}$-class $J^{*,\sim}$ and suppose that $(a,b) \in \mathscr{L}^{*,\sim}(J^{*,\sim})$. Certainly $a^0, b^0 \in J^{*,\sim}$ so that we have $(a^0, b^0) \in \mathscr{L}^{*,\sim}(J^{*,\sim})$, that is, $a^0b^0 = a^0, b^0a^0 = b^0$, and $(a^0, b^0) \in \mathscr{L}^{*,\sim}(S)$. It follows that $(a,b) \in \mathscr{L}^{*,\sim}(S)$ and consequently, since $L_a^{*,\sim}(S) \subseteq J^{*,\sim}$, $L_a^{*,\sim}(S) = L_a^{*,\sim}(J^{*,\sim})$. By using similar arguments, we can similarly prove that $R_a^{*,\sim}(S) = R_a^{*,\sim}(J^{*,\sim})$.

From the last paragraph, we have $H_a^{*,\sim}(J^{*,\sim}) = H_a^{*,\sim}(S)$ so that $J^{*,\sim}$ is super r-ample. Furthermore, if $a,b \in J^{*,\sim}$, then by Proposition 2.2, $(a,b) \in \mathscr{D}^{*,\sim}(S)$ so that, by Corollary 2.1, there exists an element c in $L_a^{*,\sim}(S) \cap R_b^{*,\sim}(S)L_a^{*,\sim}(J^{*,\sim}) \cap R_b^{*,\sim}(J^{*,\sim})$. Thus, a,b are $\mathscr{D}^{*,\sim}$-related in $J^{*,\sim}$ so that $J^{*,\sim}$ is $\mathscr{J}^{*,\sim}$-simple. ∎

We now consider the cryptic super r-simple semigroups.

Lemma 2.6. *Let $S = (Y; S_\alpha)$ be a cryptic super r-ample semigroup. Then, the following statements hold:*

(i) *Let $a \in S_\alpha$ and $\alpha \geq \beta$. Then there exists $b \in S_\beta$ with $a \geq b$;*
(ii) *Let $a,b,c \in S$, $b\mathscr{H}^{*,\sim}c$, and $a \geq b,c$. Then $b = c$;*
(iii) *Let $a \in E(S)$ and $b \in S$ be such that $a \geq b$. Then $b \in E(S)$.*

Proof. (i) Let $b \in S_\beta$, by Lemma 2.1, $a(aba)^0, (aba)^0a$ and $(aba)^0$ are in the same $\mathscr{H}^{*,\sim}$-class and so $a(aba)^0 = (aba)^0a(aba)^0 = (aba)^0a$. Let $b = a(aba)^0$. Then $b \in S_\beta$ and $a \geq b$.

(ii) By the definition of "$\geq$", there exist $e,f,g,h \in E(S)$ such that $b = ea = af$, $c = ga = ah$. From $eb = b$ and $b\mathscr{H}^{*,\sim}b^0$, we have $eb^0 = b^0$. Similarly, $c^0h = c^0$. Thus, $ec = ec^0c = eb^0c = b^0c = c$. Similarly, $bh = b$ so that $b = bh = eah = ec = c$, as required.

(iii) We have $b = ea = af$ for some $e,f \in E(S)$, and whence

$$b^2 = (ea)(af) = ea^2f = b.$$ ∎

Following Proposition 2.2, we can easily prove the following lemma.

Lemma 2.7. *Let ϕ be a homomorphism from a completely $\mathscr{J}^{*,\sim}$-simple semigroup S into another completely $\mathscr{J}^{*,\sim}$-simple semigroup T. Then $(a\phi)^0 = a^0\phi$.*

Let ϕ be a homomorphism between two completely $\mathscr{J}^{*,\sim}$-simple semigroups. Then the $(*,\sim)$-Green's relations $\mathscr{L}^{*,\sim}$, $\mathscr{R}^{*,\sim}$ are preserved, so that $\mathscr{D}^{*,\sim}$ is preserved. We call the homomorphism which preserves the $\mathscr{L}^{*,\sim}$ and $\mathscr{R}^{*,\sim}$ classes the *good* homomorphisms. By applying Proposition 2.2 and Lemma 2.6, we can easily show that the idempotents of a completely $\mathscr{J}^{*,\sim}$-simple semigroup are primitive.

3. Quasi-strong semilattice of semigroups

In this section, we introduce the quasi-strong semilattice of semigroups which is a generalization of strong semilattice of semigroups.

Definition 3.1. *Let $S = (Y; S_\alpha)$ be a semilattice Y decomposition of semigroup S into subsemigroups $S_\alpha (\alpha \in Y)$. Suppose that the following conditions hold in the semigroup S. Then the following properties hold:*

(C1) for any $\alpha,\beta \in Y$, there is a band congruence $\rho_{\alpha,\beta}$ on S_β with congruence classes $\{S_{d(\alpha,\beta)} : d(\alpha,\beta) \in D(\alpha,\beta)\}$, where $D(\alpha,\beta)$ is the index set and for $\alpha \in Y$, $\rho_{\alpha,\alpha}$ is the universal relation ω_{S_α};

(C2) for $\alpha \geq \beta$ on Y and any $d(\alpha,\beta) \in D(\alpha,\beta)$, there is a homomorphism $\phi_{d(\alpha,\beta)}$ from S_α into $S_{d(\alpha,\beta)}$. Let $\Phi_{\alpha,\beta} = \{\phi_{d(\alpha,\beta)} : d(\alpha,\beta) \in D(\alpha,\beta)\}$. Then, we have the following properties:

(i) *for $\alpha \in Y$, the homomorphism $\phi_{D(\alpha,\alpha)} : S_\alpha \longrightarrow S_\alpha$ is the identity automorphism of the semigroup S_α.*
(ii) *for $\alpha \geq \beta \geq \gamma$ on Y, $\Phi_{\alpha,\beta}\Phi_{\beta,\gamma} \subseteq \Phi_{\alpha,\gamma}$, where*

$$\Phi_{\alpha,\beta}\Phi_{\beta,\gamma} = \{\phi_{d(\alpha,\beta)}\phi_{d(\beta,\gamma)} : d(\alpha,\beta) \in D(\alpha,\beta), d(\beta,\gamma) \in D(\beta,\gamma)\}.$$

(iii) *for* $\alpha \geq \beta$ *on* Y *and* $a \in S_\alpha$, $x \in S_{d(\alpha,\beta)}$, $ax = (a\phi_{d(\alpha,\beta)})x$ *and* $xa = x(a\phi_{d(\alpha,\beta)})$.

We call the semigroup S *a quasi-strong semilattice of subsemigroups* $S_\alpha (\alpha \in Y)$ *and is denoted by* $S = [Y; S_\alpha, \rho_{\alpha,\beta}, \Phi_{\alpha,\beta}]$.

If $S = [Y; S_\alpha, \Phi_{\alpha,\beta}]$ is a strong semilattice of subsemigroups S_α with structure homomorphism $\Phi_{\alpha,\beta}$. Then, we let every band congruence $\rho_{\alpha,\beta}$ be the universal relation on S_β, and hence, every $D(\alpha,\beta)$ is a singleton.Thus, a strong semilattice of semigroups must be a quasi-strong semilattice of semigroups, but from our results,the converse of the above statement does not hold.

4. Regular cryptic super r-ample semigroups

Following Theorem 2.1, we can easily see that a cryptic super r-ample semigroup S is a semilattice of a completely $\mathscr{J}^{*,\sim}$-simple semigroups and so S=$S_\alpha(\alpha \in Y)$.

In this section, we consider the band congruence $\rho_{\alpha,\beta}$ on a regular cryptic super r-ample semigroup $S = (Y; S_\alpha)$ with the structure homomorphisms set $\Phi_{\alpha,\beta}$.

Finally, we show that a cryptic super r-ample semigroup is a regular cryptic super r-ample semigroup if and only if it is a quasi-strong semilattice of completely $\mathscr{J}^{*,\sim}$-simple semigroups.

As a corollary of the above result, we deduce that a cryptic super r-ample semigroup is a normal cryptic super r-ample semigroup if and only if it is a strong semilattice of completely $\mathscr{J}^{*,\sim}$-simple semigroups. This is the main result of this paper.

We now prove the following lemma on regular cryptic super r-ample semigroups.

Lemma 4.1. *Let* $S = (Y; S_\alpha)$ *be a regular cryptic super r-ample semigroup, that is, S is a cryptic super r-ample semigroup with* $(*,\sim)$*-Green's relation* $\mathscr{H}^{*,\sim}$ *is a regular band congruence. For any* $\alpha, \beta \in Y$*, we define* $\rho_{\alpha,\beta}$ *on* S_β *as the followings:*

$$(x, y \in S_\beta)(x,y) \in \rho_{\alpha,\beta} \Leftrightarrow (axa)^0 = (aya)^0$$

for some $a \in S_\alpha$. *Then the following statements hold:*

(i) $\rho_{\alpha,\beta}$ *is a band congruence on* S_β *and for* $x, y \in S_\beta$, $(x,y) \in \rho_{\alpha,\beta}$ *if and only if for any* $b \in S_\alpha$, $(bxb)^0 = (byb)^0$.

(ii) *for* $\alpha \geq \beta \geq \gamma$ *on* Y, $\rho_{\alpha,\gamma} \subseteq \rho_{\beta,\gamma}$ *and* $\rho_{\alpha,\alpha}$ *is the universal relation* ω_{S_α} *on* S_α.

(iii) *for* $\alpha \geq \beta$ *on* Y *and* $a \in S_\alpha$, $b \in S_\beta$, $ab\rho_{\alpha,\beta}b\rho_{\alpha,\beta}ba$.

Proof. We only prove (i), (ii) as (iii) can be proved similarly. Let $x, y \in S_\beta$ with $(x,y) \in \rho_{\alpha,\beta}$, then there exists $a \in S_\alpha$ such that $(axa)^0 = (aya)^0$. For any element $b \in S_\alpha$, we have $b(axa)^0 b = b(aya)^0 b$. Thus, we have $(b(axa)^0 b)^0 = (b(aya)^0 b)^0$. By the property of regular bands and Lemma 2.1 and 2.6, we easily have $(bxb)^0 = (byb)^0$. Now the proof is completed. ∎

We denote the $\rho_{\alpha,\beta}$-congruence classes by $\{S_{d(\alpha,\beta)} : d(\alpha,\beta) \in D(\alpha,\beta)\}$, following Lemma 2.8, $D(\alpha,\alpha)$ is a singleton.

Lemma 4.2. *Let* $S = (Y; S_\alpha)$ *be a regular cryptic super r-ample semigroup. Then, the following properties hold:*

(i) *For any* $\alpha \geq \beta$ *on* Y *and* $d(\alpha,\beta) \in D(\alpha,\beta)$. *Let* $a \in S_\alpha$, *there exists a unique element* $a_{d(\alpha,\beta)} \in S_{d(\alpha,\beta)}$ *such that* $a \geq a_{d(\alpha,\beta)}$.

(ii) *For any* $\alpha \geq \beta$ *on* Y *and* $a \in S_\alpha$, $x \in S_{d(\alpha,\beta)}$. *If* $a^0 \geq e$ *for some idempotent* $e \in S_{d(\alpha,\beta)}$, *then* $eax = ax, xae = xa$, $ea = ae$ *and* $(ea)^0 = e$.

Proof. (i) By Lemma 2.8 (iii) and Lemma 2.6 (i), for any $c \in S_{d(\alpha,\beta)}$, the element $a_{d(\alpha,\beta)} = a(aca)^0 = (aca)^0 a \in S_{d(\alpha,\beta)}$ such that $a \geq a_{d(\alpha,\beta)}$. It can be easily seen that $a^0_{d(\alpha,\beta)} = (aca)^0$. If there is another $b \in S_{d(\alpha,\beta)}$ such that $a \geq b$, then there are idempotents $g, h \in E(S)$ such that $b = ga = ah$ and so $ba^0 = b = a^0 b$, thus $b^0 a^0 = b^0 = a^0 b^0$ since $b \mathscr{H}^{*,\sim} b^0$, which implies $b^0 \leq a^0$ and hence $b^0 = a^0 b^0 a^0 = (aba)^0 = (aca)^0 = a^0_{d(\alpha,\beta)}$, that is, $b \mathscr{H}^{*,\sim} a_{d(\alpha,\beta)}$. Thus by Lemma 2.6 (ii), $a_{d(\alpha,\beta)} = b$ as required.

(ii) Since $(a^0(ax)^0 a^0) a^0 = a^0 (ax)^0 a^0 = a^0 (a^0 (ax)^0 a^0)$ and $a^0 (ax)^0 a^0 \mathscr{H}^{*,\sim} (a^0 (ax)^0 a^0)^0$, we have $(a^0 (ax)^0 a^0)^0 a^0 = (a^0 (ax)^0 a^0)^0 = a^0 (a^0 (ax)^0 a^0)^0$, that is, $a^0 \geq (a^0 (ax)^0 a^0)^0$. Also, since $a \in S_\alpha$ and $x \in S_{d(\alpha,\beta)}$, we have $ax \in S_{d(\alpha,\beta)}$ and so that $e = (a^0 (ax)^0 a^0)^0$ by (i). Thereby, we have $eax = (a^0 (ax)^0 a^0)^0 ax = (a^0 (ax)^0 a^0)^0 a^0 (ax)^0 a^0 ax = ax$. Similarly, we have $xae = xa$. Since x is an arbitrarily chosen element in $S_{d(\alpha,\beta)}$, we can particularly choose $x = e$. In this manner, we deduce that $ea = ae$ and consequently, by Lemma 2.1, we have $(ea)^0 = (ea^0)^0 = e$. ∎

Lemma 4.3. *Let* $S = (Y; S_\alpha)$ *be a regular cryptic super r-ample semigroup. For any* $\alpha \geq \beta$ *on* Y *and* $d(\alpha,\beta) \in D(\alpha,\beta)$, *define a mapping* $\phi_{d(\alpha,\beta)}$ *from* S_α *into* $S_{d(\alpha,\beta)}$ *with* $a\phi_{d(\alpha,\beta)} = a_{d(\alpha,\beta)}$, *where* $a_{d(\alpha,\beta)}$ *is defined in Lemma 2.9. Write* $\Phi_{\alpha,\beta} = \{\phi_{d(\alpha,\beta)} : d(\alpha,\beta) \in D(\alpha,\beta)\}$. *Then,the following statements hold:*

(i) $\phi_{d(\alpha,\beta)}$ *is a homomorphism.*
(ii) *for* $\alpha \in Y$, $\phi_{D(\alpha,\alpha)}$ *is the identity homomorphism of* S_α.
(iii) *for* $\alpha \geq \beta \geq \gamma$ *on* Y, $\Phi_{\alpha,\beta}\Phi_{\beta,\gamma} \subseteq \Phi_{\alpha,\gamma}$.
(iv) *for* $\alpha \geq \beta$ *on* Y *and* $a \in S_\alpha$, $x \in S_{d(\alpha,\beta)}$, $ax = (a\phi_{d(\alpha,\beta)})x$ *and* $xa = x(a\phi_{d(\alpha,\beta)})$.

Proof. (i) Following Lemma 2.9, $\phi_{d(\alpha,\beta)}$ is well defined. For $a, b \in S_\alpha$ and $c \in S_{d(\alpha,\beta)}$, by Lemma 2.9 again,

$$(a\phi_{d(\alpha,\beta)})(b\phi_{d(\alpha,\beta)}) = (aca)^0 ab(bcb)^0 = (aca)^0 ab = ab(bcb)^0 \leq ab$$

and so $(ab)\phi_{d(\alpha,\beta)} = (a\phi_{d(\alpha,\beta)})(b\phi_{d(\alpha,\beta)})$.

(ii) This part follows easily since S_α is primitive.

(iii) We only need to show that for any $d(\alpha,\beta) \in D(\alpha,\beta)$ $d(\beta,\gamma) \in D(\alpha,\gamma)$, $S_{d(\alpha,\beta)}\phi_{d(\beta,\gamma)} \subseteq S_{d(\alpha,\gamma)}$ for some $d(\alpha,\gamma) \in D(\alpha,\gamma)$. Let $a \in S_\alpha$, $b_1, b_2 \in S_{d(\alpha,\beta)}$ and $c \in S_\gamma$, we have $(ab_1a)^0 = (ab_2a)^0$ and $b_1\phi_{d(\beta,\gamma)} = b_1(b_1cb_1)^0$, $b_2\phi_{d(\beta,\gamma)} = b_2(b_2cb_2)^0$ and so

$$(a(b_1\phi_{d(\beta,\gamma)})a)^0 = (ab_1(b_1cb_1)^0 a)^0 = (ab_2(b_2cb_2)^0 a)^0 = (a(b_2\phi_{d(\beta,\gamma)})a)^0,$$

which implies $S_{d(\alpha,\beta)}\phi_{d(\beta,\gamma)} \subseteq S_{d(\alpha,\beta)}$ for some $d(\alpha,\gamma) \in D(\alpha,\gamma)$.

(iv) By (i), $a\phi_{d(\alpha,\beta)} = (axa)^0 a$ and so $a\phi_{d(\alpha,\beta)}x = (axa)^0 ax = ax$ by Lemma 2.9 (ii). Similarly $xa = x(a\phi_{d(\alpha,\beta)})$.

∎

Theorem 4.1. *A super r-ample semigroup S is a regular cryptic super r-ample semigroup if and only if it is a quasi-strong semilattice of completely* $\mathscr{J}^{*,\sim}$*-simple semigroups.*

Proof. We have already proved the necessity part of the theorem from Lemma 2.8 and 2.9. Now we proceed to prove the sufficiency part of the Theorem. We first let $S =$

$[Y;S_\alpha,\rho_{\alpha,\beta},\Phi_{\alpha,\beta}]$ be a quasi-strong semilattice of completely $\mathscr{J}^{*,\sim}$-simple semigroups S_α. We only need prove that $\mathscr{H}^{*,\sim}$ is a congruence and $S/\mathscr{H}^{*,\sim}$ is a regular band. For $a\in S_\alpha, b\in S_\beta$, we suppose that a^0b^0 falls in $S_{d(\alpha,\alpha\beta)}$, for some $d(\alpha,\alpha\beta)\in D(\alpha,\alpha\beta)$ and also ab falls in $S_{d(\beta,\alpha\beta)}$, for some $d(\beta,\alpha\beta)\in D(\beta,\alpha\beta)$. Then, by Definition 3.1, we have the following equalities:

$$\begin{aligned} ab = aa^0b^0b &= [(a\phi_{d(\alpha,\alpha\beta)})a^0][b^0(b\phi_{d(\beta,\alpha\beta)})] \\ &= [(a\phi_{d(\alpha,\alpha\beta)})(a^0\phi_{d(\alpha,\alpha\beta)})][(b^0\phi_{d(\beta,\alpha\beta)})(b\phi_{d(\beta,\alpha\beta)})] \\ &= (a\phi_{d(\alpha,\alpha\beta)})(b\phi_{d(\beta,\alpha\beta)}). \end{aligned}$$

Hence, we deduce that $(ab)^0=[(a^0\phi_{d(\alpha,\alpha\beta)})(b^0\phi_{d(\beta,\alpha\beta)})]^0=(a^0b^0)^0$ by Lemma 2.7 and its remark. Thus, by Lemma 2.1, S is a cryptic semigroup. To show that $S/\mathscr{H}^{*,\sim}$ is a regular band, let $a\in S_\alpha, x\in S_\beta, y\in S_\gamma$ and a^0x falls in $S_{d(\alpha,\alpha\beta)}$ and ya^0 falls in $S_{d(\beta,\beta\gamma)}$, by a direct computing in a similar fashion as the the above, we have obtain the following equalities $(axya)\mathscr{H}^{*,\sim}=[a(a^0x)(ya^0)a]\mathscr{H}^{*,\sim}=[(a\phi_{d(\alpha,\alpha\beta)})(a\phi_{d(\alpha,\alpha\gamma)})]\mathscr{H}^{*,\sim}$ and $(axaya)\mathscr{H}^{*,\sim}=[a(a^0x)a(ya^0)a]\mathscr{H}^{*,\sim}=[(a\phi_{d(\alpha,\alpha\beta)})(a\phi_{d(\alpha,\alpha\gamma)})]\mathscr{H}^{*,\sim}$.Thus, we deduce that $(axya)\mathscr{H}^{*,\sim}=axaya)\mathscr{H}^{*,\sim}$ and so $S/\mathscr{H}^{*,\sim}$ is a regular band. ∎

Finally, we notice that the normal cryptic super r-ample semigroup $S=(Y;S_\alpha)$, the band congruence $\rho_{\alpha,\beta}$ defined in Lemma 2.8 is the universal relation ω_{S_β} of S_β for all $\alpha,\beta\in Y$. Hence, we deduce the following theorem which characterizes the super r-ample semigroup to be a normal cryptic super r-ample semigroup.

The following theorem is a characterization theorem of a normal super r-ample semigroup, which can be proved easily and it can be regarded as a generalized version of the main result of M. Petrich in r-ample semigroups.

Theorem 4.2. *A super r-ample semigroup is a normal cryptic super r-ample semigroup if and only if it is a strong semilattice of completely $\mathscr{J}^{*,\sim}$-simple semigroups.*

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