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Automatic Continuity of Higher Derivations

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Abstract. Let $\mathscr{A}$ and $\mathscr{B}$ be two algebras. A sequence $\{d_n\}$ of linear mappings from $\mathscr{A}$ into $\mathscr{B}$ is called a higher derivation if $d_n(a_1a_2) = \sum_{k=0}^n d_k(a_1)d_{n-k}(a_2)$ for each $a_1, a_2 \in \mathscr{A}$ and each nonnegative integer n. In this paper, we show that if $\{d_n\}$ is a higher derivation from $\mathscr{A}$ into $\mathscr{B}$ such that d_0 is onto and $\ker(d_0) \subseteq \ker(d_n)$ $(n \in \mathbb{N})$, then there is a sequence $\{\delta_n\}$ of derivations on $\mathscr{B}$ such that

$$d_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \ldots + r_i} \right) \delta_{r_1} \ldots \delta_{r_i} d_0 \right).$$

As a corollary we prove that a higher derivation $\{d_n\}$ from a Banach algebra into a semisimple Banach algebra is continuous provided that d_0 is onto and $\ker(d_0) \subseteq \ker(d_n)$ $(n \in \mathbb{N})$. We also deduce that if $\mathscr{A}$ is a semisimple Jordan Banach algebra and $\{d_n\}$ is a higher derivation on $\mathscr{A}$ with $d_0(\mathscr{A}) = \mathscr{A}$ and $\ker(d_0) \subseteq \ker(d_n)$ $(n \in \mathbb{N})$ then $\{d_n\}$ is continuous.

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1. Introduction

Let $\mathscr{A}$ and $\mathscr{B}$ be two algebras, $\mathscr{X}$ be a $\mathscr{B}$-bimodule and $\sigma : \mathscr{A} \to \mathscr{B}$ be a linear mapping. A linear mapping $\delta : \mathscr{A} \to \mathscr{X}$ is called a σ-derivation if it satisfies the generalized Leibniz rule $\delta(a_1a_2) = \delta(a_1)\sigma(a_2) + \sigma(a_1)\delta(a_2)$ for each $a_1, a_2 \in \mathscr{A}$. In the case $\mathscr{A} = \mathscr{B} = \mathscr{X}$ and $\sigma = I_{\mathscr{A}}$, the identity mapping on $\mathscr{A}$, a σ-derivation is called a derivation. (For other approaches to generalized derivations and their applications see [2, 3, 5, 14, 13] and references therein. In particular, an automatic continuity problem for (σ, τ)-derivations is considered in [12] and an achievement of continuity of (σ, τ)-derivations without linearity is given in [7].)

A sequence $\{d_n\}$ of linear mappings from $\mathscr{A}$ into $\mathscr{B}$ is called a higher derivation if $d_n(a_1a_2) = \sum_{k=0}^n d_k(a_1)d_{n-k}(a_2)$ for each $a_1, a_2 \in \mathscr{A}$ and each nonnegative integer n. Higher derivations were introduced by Hasse and Schmidt [6], and algebraists sometimes call them

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Hasse-Schmidt derivations. For an account on higher derivations the reader is referred to the book [4].

In this paper we characterize all higher derivations from an algebra $\mathscr{A}$ into another algebra $\mathscr{B}$ in terms of derivations on $\mathscr{B}$, provided that d_0 is onto and $\ker(d_0) \subseteq \ker(d_n)$ $(n \in \mathbb{N})$. A characterization of higher derivations on an algebra $\mathscr{A}$ into itself can be found in [11]. Indeed, we show that each higher derivation is a linear combination of compositions of d_0 and some derivations on $\mathscr{B}$. The importance of our work is to transfer the problems such as innerness (for a definition and discussion see [15]) and automatic continuity (see [8, 10]) of higher derivations into the same problems concerning derivations. As a corollary, using the facts that each homomorphism from a Banach algebra onto a semisimple Banach algebra is automatically continuous [4, 5.1.5] and each derivation on a semisimple Banach algebra is automatically continuous [9], we prove a theorem of Jewell [8], which asserts that a higher derivation $\{d_n\}$ from a Banach algebra into a semisimple Banach algebra is continuous provided that d_0 is onto and $\ker(d_0) \subseteq \ker(d_n)$ $(n \in \mathbb{N})$. We also deduce that if $\mathscr{A}$ is a semisimple Jordan Banach algebra and $\{d_n\}$ is a higher derivation on $\mathscr{A}$ with $d_0(\mathscr{A}) = \mathscr{A}$ and $\ker(d_0) \subseteq \ker(d_n)$ $(n \in \mathbb{N})$ then $\{d_n\}$ is continuous. Throughout the paper, all algebras are assumed over the filed of complex numbers.

2. The results

Let $\mathscr{A}$ and $\mathscr{B}$ be two algebras, $\mathscr{X}$ be a $\mathscr{B}$-bimodule and $\sigma : \mathscr{A} \to \mathscr{B}$ be a linear mapping. A linear mapping $\delta : \mathscr{A} \to \mathscr{X}$ is called a σ-derivation if it satisfies the generalized Leibniz rule $\delta(a_1a_2) = \delta(a_1)\sigma(a_2) + \sigma(a_1)\delta(a_2)$ for all $a_1, a_2 \in \mathscr{A}$. A sequence $\{d_n\}$ of linear mappings from $\mathscr{A}$ into $\mathscr{B}$ is called a higher derivation if $d_n(a_1a_2) = \sum_{k=0}^{n} d_k(a_1)d_{n-k}(a_2)$ for each $a_1, a_2 \in \mathscr{A}$ and each nonnegative integer n.

Let $\{d_n\}$ be a higher derivation. Then d_0 is a homomorphism and d_1 is a d_0-derivation. Thus if d_0 is onto then $\tilde{d}_0 : \mathscr{A}/\ker(d_0) \to \mathscr{B}$ defined by $\tilde{d}_0(a + \ker(d_0)) = d_0(a)$ is an isomorphism. Moreover, for each $n \in \mathbb{N}$, $\tilde{d}_n : \mathscr{A}/\ker(d_0) \to \mathscr{B}$ defined by $\tilde{d}_n(a + \ker(d_0)) = d_n(a)$ is a well-defined linear mapping provided that $\ker(d_0) \subseteq \ker(d_n)$.

Proposition 2.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be two algebras (associative or not) and $\{d_n\}$ be a higher derivation from $\mathscr{A}$ into $\mathscr{B}$ with $d_0(\mathscr{A}) = \mathscr{B}$ and $\ker(d_0) \subseteq \ker(d_n)$ $(n \in \mathbb{N})$. Then there is a sequence $\{\delta_n\}$ of derivations on $\mathscr{B}$ such that for each nonnegative integer n*

$$(n+1)\tilde{d}_{n+1} = \sum_{k=0}^{n} \delta_{k+1}\tilde{d}_{n-k}. \tag{2.1}$$

Proof. We use induction on n. For $n = 0$, let $\delta_1 : \mathscr{B} \to \mathscr{B}$ be defined by $\delta_1 = \tilde{d}_1\tilde{d}_0^{-1}$ and $b_1, b_2 \in \mathscr{B}$. Since $\tilde{d}_0$ is an isomorphism, there exist $a_1, a_2 \in \mathscr{A}$ such that $d_0(a_1) = b_1$ and $d_0(a_2) = b_2$. We thus have

$$\begin{aligned}
\delta_1(b_1b_2) &= \tilde{d}_1\tilde{d}_0^{-1}(d_0(a_1)d_0(a_2)) \\
&= d_1(a_1a_2) \\
&= d_0(a_1)d_1(a_2) + d_1(a_1)d_0(a_2) \\
&= d_0(a_1)\tilde{d}_1\tilde{d}_0^{-1}(d_0(a_2)) + \tilde{d}_1\tilde{d}_0^{-1}(d_0(a_1))d_0(a_2) \\
&= b_1\delta_1(b_2) + \delta_1(b_1)b_2.
\end{aligned}$$

So δ_1 is a derivation. Note that $\tilde{d}_1 = \delta_1\tilde{d}_0$.

Now suppose that δ_k is defined and is a derivation for $k \le n$ satisfying (2.1). Putting $\delta_{n+1} = [(n+1)\tilde{d}_{n+1} - \sum_{k=0}^{n-1} \delta_{k+1}\tilde{d}_{n-k}]\tilde{d_0}^{-1}$, we show that δ_{n+1} is a derivation. For $b_1, b_2 \in \mathscr{B}$ there are $a_1, a_2 \in \mathscr{A}$ such that $d_0(a_1) = b_1$ and $d_0(a_2) = b_2$. Hence

$$\begin{aligned}
\delta_{n+1}(b_1 b_2) &= \left[(n+1)\tilde{d}_{n+1} - \sum_{k=0}^{n-1} \delta_{k+1}\tilde{d}_{n-k}\right] \tilde{d_0}^{-1}(d_0(a_1)d_0(a_2)) \\
&= \left[(n+1)\tilde{d}_{n+1} - \sum_{k=0}^{n-1} \delta_{k+1}\tilde{d}_{n-k}\right] (a_1 a_2 + \ker(d_0)) \\
&= (n+1)d_{n+1}(a_1 a_2) - \sum_{k=0}^{n-1} \delta_{k+1} d_{n-k}(a_1 a_2) \\
&= (n+1)\sum_{k=0}^{n+1} d_k(a_1)d_{n+1-k}(a_2) - \sum_{k=0}^{n-1} \delta_{k+1}\left(\sum_{\ell=0}^{n-k} d_\ell(a_1)d_{n-k-\ell}(a_2)\right).
\end{aligned}$$

Since $\delta_1, \dots, \delta_n$ are derivations,

$$\begin{aligned}
\delta_{n+1}(b_1 b_2) = &\sum_{k=0}^{n+1} k d_k(a_1)d_{n+1-k}(a_2) + \sum_{k=0}^{n+1} d_k(a_1)(n+1-k)d_{n+1-k}(a_2) \\
&- \sum_{k=0}^{n-1}\sum_{\ell=0}^{n-k} [\delta_{k+1}(d_\ell(a_1))d_{n-k-\ell}(a_2) + d_\ell(a_1)\delta_{k+1}(d_{n-k-\ell}(a_2))].
\end{aligned}$$

Writing

$$\begin{aligned}
K &= \sum_{k=0}^{n+1} k d_k(a_1)d_{n+1-k}(a_2) - \sum_{k=0}^{n-1}\sum_{\ell=0}^{n-k} \delta_{k+1}(d_\ell(a_1))d_{n-k-\ell}(a_2), \\
L &= \sum_{k=0}^{n+1} d_k(a_1)(n+1-k)d_{n+1-k}(a_2) - \sum_{k=0}^{n-1}\sum_{\ell=0}^{n-k} d_\ell(a_1)\delta_{k+1}(d_{n-k-\ell}(a_2))
\end{aligned}$$

we have $\delta_{n+1}(b_1 b_2) = K + L$. Let us compute K and L. In the summation $\sum_{k=0}^{n-1}\sum_{\ell=0}^{n-k}$ we have $0 \le k+\ell \le n$ and $k \ne n$. Thus if we put $r = k+\ell$ then we can write it as the form $\sum_{r=0}^{n}\sum_{k+\ell=r, k\ne n}$. Putting $\ell = r-k$ we indeed have

$$\begin{aligned}
K &= \sum_{k=0}^{n+1} k d_k(a_1)d_{n+1-k}(a_2) - \sum_{r=0}^{n}\sum_{0\le k\le r, k\ne n} \delta_{k+1}(d_{r-k}(a_1))d_{n-r}(a_2) \\
&= \sum_{k=0}^{n+1} k d_k(a_1)d_{n+1-k}(a_2) - \sum_{r=0}^{n-1}\sum_{k=0}^{r} \delta_{k+1}(d_{r-k}(a_1))d_{n-r}(a_2) - \sum_{k=0}^{n-1} \delta_{k+1}(d_{n-k}(a_1))d_0(a_2).
\end{aligned}$$

Putting $r+1$ instead of k in the first summation we have

$$\begin{aligned}
&K + \sum_{k=0}^{n-1} \delta_{k+1}(d_{n-k}(a_1))d_0(a_2) \\
&= \sum_{r=0}^{n} (r+1)d_{r+1}(a_1)d_{n-r}(a_2) - \sum_{r=0}^{n-1}\sum_{k=0}^{r} \delta_{k+1}(d_{r-k}(a_1))d_{n-r}(a_2) \\
&= \sum_{r=0}^{n-1}\left[(r+1)d_{r+1}(a_1) - \sum_{k=0}^{r} \delta_{k+1}(d_{r-k}(a_1))\right] d_{n-r}(a_2) + (n+1)d_{n+1}(a_1)d_0(a_2)
\end{aligned}$$

$$= \sum_{r=0}^{n-1} \left[(r+1)\tilde{d}_{r+1} - \sum_{k=0}^{r} \delta_{k+1}\tilde{d}_{r-k} \right] (\tilde{d}_0^{-1} d_0(a_1)) d_{n-r}(a_2) + (n+1) d_{n+1}(a_1) d_0(a_2).$$

By our assumption

$$(r+1)\tilde{d}_{r+1} - \sum_{k=0}^{r} \delta_{k+1}\tilde{d}_{r-k} = 0$$

for $r = 0, \ldots, n-1$. We can therefore deduce that

$$\begin{aligned} K &= \left[(n+1) d_{n+1}(a_1) - \sum_{k=0}^{n-1} \delta_{k+1}(d_{n-k}(a_1)) \right] d_0(a_2) \\ &= \left[(n+1)\tilde{d}_{n+1} - \sum_{k=0}^{n-1} \delta_{k+1}\tilde{d}_{n-k} \right] (\tilde{d}_0^{-1} d_0(a_1)) d_0(a_2) \\ &= \delta_{n+1}(b_1) b_2. \end{aligned}$$

By a similar argument we have

$$\begin{aligned} L &= d_0(a_1) \left[(n+1)\tilde{d}_{n+1} - \sum_{k=0}^{n-1} \delta_{k+1}\tilde{d}_{n-k} \right] (\tilde{d}_0^{-1} d_0(a_2)) \\ &= b_1 \delta_{n+1}(b_2). \end{aligned}$$

Thus

$$\delta_{n+1}(b_1 b_2) = K + L = \delta_{n+1}(b_1) b_2 + b_1 \delta_{n+1}(b_2).$$

Whence δ_{n+1} is a derivation on $\mathscr{A}$. ∎

Theorem 2.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be two algebras (associative or not) and $\{d_n\}$ be a higher derivation from $\mathscr{A}$ into $\mathscr{B}$ with $d_0(\mathscr{A}) = \mathscr{B}$ and $\ker(d_0) \subseteq \ker(d_n)$ $(n \in \mathbb{N})$. Then there is a sequence $\{\delta_n\}$ of derivations on $\mathscr{B}$ such that*

$$d_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \ldots + r_i} \right) \delta_{r_1} \ldots \delta_{r_i} d_0 \right),$$

where the inner summation is taken over all positive integers $r_1, \ldots, r_i$ with $\sum_{j=1}^{i} r_j = n$.

Proof. We show that if d_n is of the above form then $\tilde{d}_n$ satisfies the recursive relation of Proposition 2.1. Since the solution of the recursive relation is unique, this proves the theorem.

Simplifying the notation we put

$$a_{r_1,\ldots,r_i} = \prod_{j=1}^{i} \frac{1}{r_j + \ldots + r_i}.$$

Note that if $r_1 + \ldots + r_i = n+1$ then $(n+1) a_{r_1,\ldots,r_i} = a_{r_2,\ldots,r_i}$. Moreover, $a_{n+1} = 1/(n+1)$.

Now for each $a \in \mathscr{A}$ we have

$$\begin{aligned} &(n+1)\tilde{d}_{n+1}(a + \ker(d_0)) \\ &= (n+1) d_{n+1}(a) \end{aligned}$$

$$= \sum_{i=2}^{n+1}\left(\sum_{\sum_{j=1}^{i} r_j=n+1}(n+1)a_{r_1,\ldots,r_i}\delta_{r_1}\ldots\delta_{r_i}d_0\right)(a)+\delta_{n+1}d_0(a)$$

$$= \sum_{i=2}^{n+1}\left(\sum_{r_1=1}^{n+2-i}\delta_{r_1}\sum_{\sum_{j=2}^{i} r_j=n+1-r_1}a_{r_2,\ldots,r_i}\delta_{r_2}\ldots\delta_{r_i}d_0\right)(a)+\delta_{n+1}d_0(a)$$

$$= \sum_{r_1=1}^{n}\delta_{r_1}\sum_{i=2}^{n-(r_1-2)}\left(\sum_{\sum_{j=2}^{i} r_j=n-(r_1-2)}a_{r_2,\ldots,r_i}\delta_{r_2}\ldots\delta_{r_i}d_0\right)(a)+\delta_{n+1}d_0(a)$$

$$= \sum_{r_1=1}^{n}\delta_{r_1}d_{n-(r_1-1)}(a)+\delta_{n+1}d_0(a)$$

$$= \sum_{k=0}^{n}\delta_{k+1}\tilde{d}_{n-k}(a+\ker(d_0)).$$

We therefore have $(n+1)\tilde{d}_{n+1}=\sum_{k=0}^{n}\delta_{k+1}\tilde{d}_{n-k}$. ∎

Example 2.1. We evaluate the coefficients $a_{r_1,\ldots,r_i}$ for the case $n=4$.

$$d_4=\Big(\frac{1}{4}\delta_4+\frac{1}{12}\delta_1\delta_3+\frac{1}{4}\delta_3\delta_1+\frac{1}{8}\delta_2\delta_2+\frac{1}{24}\delta_1\delta_1\delta_2+\frac{1}{12}\delta_1\delta_2\delta_1+\frac{1}{8}\delta_2\delta_1\delta_1+\frac{1}{24}\delta_1\delta_1\delta_1\delta_1\Big)d_0.$$

Theorem 2.2. *Let $\mathscr{A}$ and $\mathscr{B}$ be two algebras (associative or not), $\sigma:\mathscr{A}\to\mathscr{B}$ be a surjective homomorphism and D be the set of all higher derivations $\{d_n\}_{n=0,1,\ldots}$ from $\mathscr{A}$ into $\mathscr{B}$ with $d_0=\sigma$ and $\ker(d_0)\subseteq\ker(d_n)$ $(n\in\mathbb{N})$. Suppose also that Δ is the set of all sequences $\{\delta_n\}_{n=1,2,\ldots}$ of derivations on $\mathscr{B}$. Then there is a one to one correspondence between D and Δ.*

Proof. Let $\{\delta_n\}\in\Delta$. Define $d_n:\mathscr{A}\to\mathscr{B}$ by $d_0=\sigma$ and

$$d_n=\sum_{i=1}^{n}\left(\sum_{\sum_{j=1}^{i} r_j=n}\left(\prod_{j=1}^{i}\frac{1}{r_j+\ldots+r_i}\right)\delta_{r_1}\ldots\delta_{r_i}d_0\right).$$

We show that $\{d_n\}\in D$. By Theorem 2.1, $\{\tilde{d}_n\}$ satisfies the recursive relation

$$(n+1)\tilde{d}_{n+1}=\sum_{k=0}^{n}\delta_{k+1}\tilde{d}_{n-k}.$$

To show that $\{d_n\}$ is a higher derivation we use induction on n. For $n=0$ we have $d_0(a_1a_2)=\sigma(a_1a_2)=d_0(a_1)d_0(a_2)$. Let us assume that $d_k(a_1a_2)=\sum_{i=0}^{k}d_i(a_1)d_{k-i}(a_2)$ for $k\le n$. Thus we have

$$(n+1)d_{n+1}(a_1a_2)$$

$$=(n+1)\tilde{d}_{n+1}(a_1a_2+\ker(d_0))$$

$$=\sum_{k=0}^{n}\delta_{k+1}\tilde{d}_{n-k}(a_1a_2+\ker(d_0))$$

$$=\sum_{k=0}^{n}\delta_{k+1}d_{n-k}(a_1a_2)$$

$$
\begin{aligned}
&= \sum_{k=0}^{n} \delta_{k+1} \sum_{i=0}^{n-k} d_i(a_1) d_{n-k-i}(a_2) \\
&= \sum_{i=0}^{n} \left(\sum_{k=0}^{n-i} \delta_{k+1} d_{n-k-i}(a_1) \right) d_i(a_2) + \sum_{i=0}^{n} d_i(a_1) \left(\sum_{k=0}^{n-i} \delta_{k+1} d_{n-k-i}(a_2) \right) \\
&= \sum_{i=0}^{n} \left(\sum_{k=0}^{n-i} \delta_{k+1} \tilde{d}_{n-k-i}(a_1 + \ker(d_0)) \right) d_i(a_2) \\
&\quad + \sum_{i=0}^{n} d_i(a_1) \left(\sum_{k=0}^{n-i} \delta_{k+1} \tilde{d}_{n-k-i}(a_2 + \ker(d_0)) \right).
\end{aligned}
$$

Using our assumption, we can write

$$
\begin{aligned}
&(n+1) d_{n+1}(a_1 a_2) \\
&= \sum_{i=0}^{n} (n-i+1) \tilde{d}_{n-i+1}(a_1 + \ker(d_0)) d_i(a_2) + \sum_{i=0}^{n} d_i(a_1)(n-i+1) \tilde{d}_{n-i+1}(a_2 + \ker(d_0)) \\
&= \sum_{i=0}^{n} (n-i+1) d_{n-i+1}(a_1) d_i(a_2) + \sum_{i=0}^{n} d_i(a_1)(n-i+1) d_{n-i+1}(a_2) \\
&= \sum_{i=1}^{n+1} i d_i(a_1) d_{n+1-i}(a_2) + \sum_{i=0}^{n} (n+1-i) d_i(a_1) d_{n+1-i}(a_2) \\
&= (n+1) \sum_{k=0}^{n+1} d_k(a_1) d_{n+1-k}(a_2).
\end{aligned}
$$

Thus $\{d_n\} \in D$. Note that for each $n \in \mathbb{N}$, $\ker(d_0) \subseteq \ker(d_n)$.

Conversely, suppose that $\{d_n\} \in D$. Define $\delta_n : \mathscr{B} \to \mathscr{B}$ by $\delta_1 = \tilde{d}_1 \tilde{d}_0^{-1}$ and

$$\delta_n = \left[n\tilde{d}_n - \sum_{k=0}^{n-2} \delta_{k+1} \tilde{d}_{n-1-k} \right] \tilde{d}_0^{-1} \qquad (n \geqslant 2).$$

Then Proposition 2.1 ensures us that $\{\delta_n\} \in \Delta$.

Now define $\varphi : \Delta \to D$ by $\varphi(\{\delta_n\}) = \{d_n\}$, where

$$d_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \ldots + r_i} \right) \delta_{r_1} \ldots \delta_{r_i} d_0 \right). \tag{2.2}$$

Then φ is clearly surjective, we show that it is injective. Let $\{d_n\} = \varphi(\{\delta_n\}) = \varphi(\{\delta'_n\}) = \{d'_n\}$. We use induction on n. For $n = 1$ we have

$$\delta_1 = \tilde{d}_1 \tilde{d}_0^{-1} = \tilde{d}'_1 \tilde{d}'^{-1}_0 = \delta'_1$$

Now suppose that $\delta_k = \delta'_k$ for $k \leqslant n$. As in the proof of Theorem 2.1, the following relations are obtained

$$(n+1)\tilde{d}_{n+1} = \sum_{k=0}^{n} \delta_{k+1} \tilde{d}_{n-k}, \quad (n+1)\tilde{d}'_{n+1} = \sum_{k=0}^{n} \delta'_{k+1} \tilde{d}'_{n-k}.$$

Thus

$$\delta_{n+1}=[(n+1)\tilde{d}_{n+1}-\sum_{k=0}^{n-1}\delta_{k+1}\tilde{d}_{n-k}]\tilde{d_0}^{-1}=[(n+1)\tilde{d}'_{n+1}-\sum_{k=0}^{n-1}\delta'_{k+1}\tilde{d}'_{n-k}]\tilde{d_0'}^{-1}=\delta'_{n+1}.$$ ∎

Corollary 2.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be two normed algebras. If $\{d_n\}$ is a higher derivation from $\mathscr{A}$ into $\mathscr{B}$ with $d_0(\mathscr{A})=\mathscr{B}$ and $\ker(d_0)\subseteq\ker(d_n)$ $(n\in\mathbb{N})$ then $\{d_n\}$ is continuous whenever d_0 and all derivations on $\mathscr{B}$ are continuous.*

Theorem 2.3. [8] *Each higher derivation $\{d_n\}$ from a Banach algebra into a semisimple Banach algebra is continuous provided that d_0 is onto and $\ker(d_0)\subseteq\ker(d_n)$ $(n\in\mathbb{N})$.*

Proof. Since every homomorphism from a Banach algebra onto a semisimple Banach algebra is automatically continuous [4, 5.1.5] and every derivation on a semisimple Banach algebra is automatically continuous [9], the result is obvious. ∎

Our next result is about Jordan algebras. Recall that a Jordan algebra is a commutative non-associative algebra $\mathscr{A}$ satisfying the Jordan identity $(ab)a^2=a(ba^2)$ for all $a,b\in\mathscr{A}$.

Theorem 2.4. *Each higher derivation $\{d_n\}$ from a Jordan Banach algebra into a semisimple Jordan Banach algebra is continuous provided that d_0 is onto and $\ker(d_0)\subseteq\ker(d_n)$ $(n\in\mathbb{N})$.*

Proof. It is known that every homomorphism from a Jordan Banach algebra onto a semisimple Jordan Banach algebra is automatically continuous [1] and every derivation on a semisimple Jordan Banach algebra is automatically continuous [16]. Hence Theorem 2.1 implies that $\{d_n\}$ is continuous. ∎

A higher derivation $\{d_n\}$ on an algebra $\mathscr{A}$ is called normal if $d_0=id_{\mathscr{A}}$. A normal higher derivation clearly satisfies the conditions of Theorem 2.1. Thus we have following corollary.

Corollary 2.2. *Let $\mathscr{A}$ be a semisimple (Jordan) Banach algebra. If $\{d_n\}$ is a higher derivation on $\mathscr{A}$ and d_0 is an isomorphism, then $\{d_n\}$ is continuous. In particular, every normal higher derivation on a semisimple (Jordan) Banach algebra is continuous.*

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