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Artinianness of Local Cohomology Modules Defined by a Pair of Ideals

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Abstract. Let R be a commutative Noetherian ring and I, J two ideals of R. Let M be a finitely generated R-module; it is shown that (1) if $\dim R/(I+J)=0$, then $H^i_{I,J}(M)/JH^i_{I,J}(M)$ is I-cofinite Artinian for all $i\geq 0$; let $\dim_R M/JM = d$ (2) if R is local and S is a non-zero Serre subcategory of the category of R-modules satisfying the condition C_I, then $H^d_{I,J}(M)/JH^d_{I,J}(M)\in S$ (3) if M has finite Krull dimension, then $H^{d+1}_{I,J}(M)/JH^{d+1}_{I,J}(M)=0$. Furthermore, notion of (I,J)-relative Goldie dimension of modules is defined and it is shown that $H^n_{I,J}(M)/JH^n_{I,J}(M)$ is Artinian, whenever M is a ZD-module of dimension n such that the (I,J)-relative Goldie dimension of any quotient of M is finite.

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1. Introduction

Throughout this paper, R is a commutative Noetherian ring with non-zero identity, I, J are two ideals of R and M is an R-module. For notations and terminologies not given in this paper, the reader is referred to [5,6] and [12], if necessary.

The local cohomology theory has been an significant tool in commutative Algebra and Algebraic Geometry. As a generalization of the ordinary local cohomology modules, in [12], the authors introduced the local cohomology modules with respect to a pair of ideals. To be more precise, let $\mathrm{W}(I,J)=\{\mathfrak{p}\in\mathrm{Spec}(R): I^t\subseteq J+\mathfrak{p}$ for some positive integer $t\}$. The set of elements x of M such that $\mathrm{Supp}_R Rx\subseteq \mathrm{W}(I,J)$ is said to be (I,J)-torsion submodule of M and is denoted by $\Gamma_{I,J}(M)$. It is easy to see that $\Gamma_{I,J}$ is a covariant, R-linear functor from the category of R-modules to itself. For an integer i, the local cohomology functor $H^i_{I,J}$ with respect to (I,J) is defined to be the i-th right derived functor of $\Gamma_{I,J}$. Also $H^i_{I,J}(M)$ is called the i-th local cohomology module of M with respect to (I,J). If $J=0$, then $H^i_{I,J}$ coincides with the ordinary local cohomology functor H^i_I.

Recently, some authors approached the study of local cohomology modules by means of Serre subcategories and it is noteworthy that their approach enables us to deal with several important problems on local cohomology modules comprehensively; see, for example [1–

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4]. In this direction, we study the local cohomology modules with respect to a pair of ideals by the notion of Serre subcategory. One of the main results of this paper (Theorem 2.1) is a generalization of [9, Theorem 3.1], and shows that if M is a ZD-module and S a Serre subcategory of the category of R-modules satisfying the condition C_I, then the following statements are equivalent: (i) $\Gamma_{I,J}(M/N)/J\Gamma_{I,J}(M/N) \in S$ for any submodule N of M; (ii) $H^i_{I,J}(M/N)/JH^i_{I,J}(M/N) \in S$ for any submodule N of M and all $i \geq 0$.

Theorem 2.2 in [7] shows that if $(R,\mathfrak{m})$ is local, $\sqrt{I+J} = \mathfrak{m}$, M is a finitely generated R-module and t is an integer such that $H^i_{I,J}(M)$ is Artinian for all $i > t$, then $H^t_{I,J}(M)/JH^t_{I,J}(M)$ is Artinian. In Corollary 2.2, we improve this theorem by using the above mentioned result and we show that, for any finitely generated R-module M, $H^i_{I,J}(M)/JH^i_{I,J}(M)$ is Artinian for all $i \geq 0$, when R is an arbitrary (not necessary local) ring and $\dim R/(I+J) = 0$.

As a generalization of the concept of I-relative Goldie dimension, that is introduced in [9], we say that M has finite (I,J)-relative Goldie dimension if the Goldie dimension of (I,J)-torsion submodule of M is finite. Let M be a ZD-module with finite Krull dimension n. It is shown that $H^n_{I,J}(M)/JH^n_{I,J}(M)$ is Artinian, whenever (I,J)-relative Goldie dimension of any quotient of M is finite.

2. Artinianness of $H^i_{I,J}(M)$

Recall that R is a Noetherian ring, I, J are two ideals of R and M is an R-module. Let $Z_R(M)$ denote the set of zero-divisors of M.

Definition 2.1. *An R-module M is said to be zero-divisor module if for any submodule N of M, the set $Z_R(M/N)$ is a finite union of prime ideals in $\mathrm{Ass}_R(M/N)$.*

According to [9, Example 2.2], the class of zero-divisor modules (ZD-modules) contains finitely generated, Laskerian [11], weakly Laskerian [10], linearly compact and Matlis reflexive modules. Also it contains modules whose quotients have finite Goldie dimension and modules with finite support, in particular Artinian modules.

Definition 2.2. *A full subcategory of the category of R-modules is said to be Serre subcategory, if it is closed under taking submodules, quotients and extensions. A Serre subcategory S is said to be satisfy the condition C_I if for any I-torsion R-module M, $0 :_M I \in S$ implies that $M \in S$.*

Examples 2.4 and 2.5 in [1] show that the class of zero modules, Artinian modules, I-cofinite Artinian modules, modules with finite support and the class of R-modules M with $\dim_R M \leq t$, where t is a non-negative integer are Serre subcategories of the category of R-modules satisfy the condition C_I.

In the rest of the paper, S denotes a Serre subcategory of the category of R-modules satisfying the condition C_I. The following result is a generalization of [9, Theorem 3.1].

Theorem 2.1. *Let M be a ZD-module such that $\Gamma_{I,J}(M/N)/J\Gamma_{I,J}(M/N) \in S$ for any submodule N of M. Then $H^i_{I,J}(M/N)/JH^i_{I,J}(M/N) \in S$ for any submodule N of M and all $i \geq 0$.*

Proof. We may assume that I is not zero, this can be done simply because $\Gamma_{I,J}$ is identity functor when $I = 0$. We use induction on i. The case $i = 0$ is trivial by assumption. So assume, inductively, that $i > 0$ and we have shown that $H^{i-1}_{I,J}(M'/N')/JH^{i-1}_{I,J}(M'/N') \in S$ for any ZD-module M' and any submodule N' of M'. Now let M be a ZD-module, N a submodule of M and $X = M/N$. Then $H^i_{I,J}(X/\Gamma_{I,J}(X)) \cong H^i_{I,J}(X)$ by [12, Corollary 1.13(4)].

Also $X/\Gamma_{I,J}(X)$ is a (ZD-module) (I,J)-torsion free R-module. We therefore assume in addition that X is an (I,J)-torsion free R-module. We now use [9, Lemma 2.4] to deduce that I contains an element a which is a non zero-divisor on X. The exact sequence $0 \longrightarrow X \xrightarrow{a} X \longrightarrow X/aX \longrightarrow 0$ induces an exact sequence

$$\cdots \longrightarrow H^{i-1}_{I,J}(X/aX) \longrightarrow H^i_{I,J}(X) \xrightarrow{a} H^i_{I,J}(X) \longrightarrow H^i_{I,J}(X/aX) \longrightarrow \cdots$$

of local cohomology modules. So we have the exact sequence

$$H^{i-1}_{I,J}(X/aX)/JH^{i-1}_{I,J}(X/aX) \to H^i_{I,J}(X)/JH^i_{I,J}(X) \xrightarrow{a} aH^i_{I,J}(X)/aJH^i_{I,J}(X) \to 0.$$

Since $X/aX \cong M/(aM+N)$ is a ZD-module, it follows from the inductive hypothesis that $H^{i-1}_{I,J}(X/aX)/JH^{i-1}_{I,J}(X/aX) \in S$. So the above exact sequence shows that the R-module $0 :_{H^i_{I,J}(X)/JH^i_{I,J}(X)} a \in S$. Hence, $H^i_{I,J}(X)/JH^i_{I,J}(X) \in S$ by [1, Lemma 2.3]. This completes the inductive step. The result follows by induction. ∎

The following result is an improvement of [7, Theorem 2.2].

Corollary 2.1. *Let $(R,\mathfrak{m})$ be local, $\sqrt{I+J} = \mathfrak{m}$, S non-zero and M a finitely generated R-module. Then $H^i_{I,J}(M/N)/JH^i_{I,J}(M/N) \in S$ for any submodule N of M and all $i \geq 0$.*

Proof. In view of Theorem 2.1, it is enough to show that $\Gamma_{I,J}(M/N)/J\Gamma_{I,J}(M/N) \in S$ for any submodule N of M. Assume that N is a submodule of M; [12, Proposition 1.4] shows that

$$\Gamma_{I,J}(M/N) = \Gamma_{I+J,J}(M/N) = \Gamma_{\sqrt{I+J},J}(M/N) = \Gamma_{\mathfrak{m},J}(M/N).$$

Since $\Gamma_{\mathfrak{m},J}(M/N)/J\Gamma_{\mathfrak{m},J}(M/N)$ is a finitely generated R-module and annihilated by a power of $\mathfrak{m}$; hence $\Gamma_{\mathfrak{m},J}(M/N)/J\Gamma_{\mathfrak{m},J}(M/N)$ has finite length. So by [4, Lemma 2.11], we have $\Gamma_{\mathfrak{m},J}(M/N)/J\Gamma_{\mathfrak{m},J}(M/N) \in S$. ∎

The following corollary improves Corollary 2.1, when S is considered the class of I-cofinite Artinian modules.

Corollary 2.2. *Let* $\dim R/(I+J) = 0$ *and M be a finitely generated R-module. Then $H^i_{I,J}(M)/JH^i_{I,J}(M)$ is I-cofinite Artinian for all $i \geq 0$.*

Proof. The proof is similar to that of Corollary 2.1. ∎

Let R be local, S non-zero and M a finitely generated R-module of dimension n. Then by using the method of proof of [5, Theorem 7.1.6], one can see that $H^n_I(M) \in S$ by [4, Lemma 2.11]. Having this in mind, we get the following theorem which is a generalization of [7, Theorem 2.3].

Theorem 2.2. *Let R be local, S non-zero and M a finitely generated R-module with* $\dim_R M/JM = d$. *Then $H^d_{I,J}(M)/JH^d_{I,J}(M) \in S$.*

Proof. When $\dim_R M = -1$, there is nothing to prove, as then $M = 0$. We argue by induction on $\dim_R M$. If $\dim_R M = 0$, then M has finite length. Thus $\Gamma_{I,J}(M)/J\Gamma_{I,J}(M)$ has finite length. So the result follows by [4, Lemma 2.11]. Now suppose, inductively, that $\dim_R M = n > 0$, and the result has been proved for all R-modules of dimensions smaller than n satisfying the hypothesis. The exact sequence

$$0 \longrightarrow \Gamma_J(M) \longrightarrow M \longrightarrow M/\Gamma_J(M) \longrightarrow 0 \tag{2.1}$$

induces the long exact sequence

$$(2.2) \qquad \cdots \to H^i_{I,J}(\Gamma_J(M)) \to H^i_{I,J}(M) \to H^i_{I,J}(M/\Gamma_J(M)) \to H^{i+1}_{I,J}(\Gamma_J(M)) \to \cdots .$$

By [12, Corollary 2.5], $H^i_{I,J}(\Gamma_J(M)) \cong H^i_I(\Gamma_J(M))$, for all $i \geq 0$, since $\Gamma_J(M)$ is J-torsion. On the other hand, $\dim_R \Gamma_J(M) \leq \dim_R M/JM = d$. Thus $H^d_I(\Gamma_J(M)) \in S$ by the previous paragraph and $H^{d+1}_I(\Gamma_J(M)) = 0$. Therefore, $H^d_{I,J}(\Gamma_J(M)) \in S$ and $H^{d+1}_{I,J}(\Gamma_J(M)) = 0$. Now by the exact sequence

$$\begin{aligned} H^d_{I,J}(\Gamma_J(M))/JH^d_{I,J}(\Gamma_J(M)) &\longrightarrow H^d_{I,J}(M)/JH^d_{I,J}(M) \\ &\longrightarrow H^d_{I,J}(M/\Gamma_J(M))/JH^d_{I,J}(M/\Gamma_J(M)) \longrightarrow 0 \end{aligned}$$

we only have to show that $H^d_{I,J}(M/\Gamma_J(M))/JH^d_{I,J}(M/\Gamma_J(M)) \in S$. We have

$$(2.3) \qquad \dim_R(M/\Gamma_J(M))/J(M/\Gamma_J(M)) = \dim_R M/(JM + \Gamma_J(M)) \leq \dim_R M/JM = d.$$

So, in view of [12, Theorem 4.3], we may assume that $\Gamma_J(M) = 0$. So the ideal J contains an element a which is a non zero-divisor on M. The exact sequence $0 \longrightarrow M \xrightarrow{a} M \longrightarrow M/aM \longrightarrow 0$ induces the exact sequence

$$\cdots \longrightarrow H^d_{I,J}(M) \xrightarrow{a} H^d_{I,J}(M) \longrightarrow H^d_{I,J}(M/aM) \longrightarrow 0$$

of local cohomology modules, see [12, Theorem 4.3]. Now the exact sequence

$$H^d_{I,J}(M)/JH^d_{I,J}(M) \xrightarrow{a} H^d_{I,J}(M)/JH^d_{I,J}(M) \longrightarrow H^d_{I,J}(M/aM)/JH^d_{I,J}(M/aM) \longrightarrow 0$$

shows that

$$H^d_{I,J}(M/aM)/JH^d_{I,J}(M/aM) \cong H^d_{I,J}(M)/(J+Ra)H^d_{I,J}(M) = H^d_{I,J}(M)/JH^d_{I,J}(M).$$

We have $\dim_R M/aM = n-1$ and

$$\dim_R(M/aM)/J(M/aM) = \dim_R M/(J+Ra)M = \dim_R M/JM = d.$$

Thus, by the inductive hypothesis $H^d_{I,J}(M/aM)/JH^d_{I,J}(M/aM) \in S$. This completes the inductive step. ∎

Let k be a field and $R = k[x]$ the polynomials ring in an indeterminate x, with coefficients in k. Let $I = (x-1)$ and $J = I \cap (x) = (x^2 - x)$. Then one has $\dim_R R/J = 0$ and $H^1_{I,J}(R) \neq 0$; see [12, Remark 4.6 (2)]. Nevertheless, we have the following result.

Theorem 2.3. *Let M be a finitely generated R-module of finite Krull dimension. If* $\dim_R M/JM = d$*, then* $H^{d+1}_{I,J}(M)/JH^{d+1}_{I,J}(M) = 0$.

Proof. If $JM = M$, then $(1+a)M = 0$ for some $a \in J$ by Nakayama's Lemma. Thus $Jx = Rx$ for all $x \in M$ and so M is (I,J)-torsion. Hence, $\Gamma_{I,J}(M)/J\Gamma_{I,J}(M) = 0$. Now suppose that $d \geq 0$. We use induction on $\dim_R M$. If $\dim_R M = 0$, then $H^1_{I,J}(M)/JH^1_{I,J}(M) = 0$ by [12, Theorem 4.7(1)]. So assume, inductively, that $\dim_R M = n > 0$ and we established the result for R-modules of dimension smaller than n satisfying the hypothesis. By another using of [12, Theorem 4.7(1)], we have $H^{d+1}_{I,J}(\Gamma_J(M)) = H^{d+2}_{I,J}(\Gamma_J(M)) = 0$ since $\dim_R \Gamma_J(M) \leq \dim_R M/JM = d$. Therefore, the exact sequence (2.2) shows that $H^{d+1}_{I,J}(M) \cong H^{d+1}_{I,J}(M/\Gamma_J(M))$. Hence, it is enough to show that $H^{d+1}_{I,J}(M/\Gamma_J(M))/JH^{d+1}_{I,J}(M/\Gamma_J(M)) = 0$. Also by using the exact sequence (2.3) and [12, Theorem 4.7(2)], we may assume $\Gamma_J(M) = 0$. The argument now proceeds like that used in the proof of Theorem 2.2. ∎

Now we get some results on the finiteness of the support of the local cohomology modules.

Corollary 2.3. *Let M be a ZD-module such that $\Gamma_{I,J}(M/N)/J\Gamma_{I,J}(M/N)$ has finite support for any submodule N of M. Then $H^i_{I,J}(M/N)/JH^i_{I,J}(M/N)$ has finite support for any submodule N of M and all $i \geq 0$.*

Proof. Apply Theorem 2.1 and the fact that the class of modules with finite support is a Serre subcategory of the category of R-modules satisfying the condition C_I. ∎

Corollary 2.4. *Let R be local and M a finitely generated R-module such that for any submodule N of M and for all $\mathfrak{p} \in \mathrm{Supp}_R \Gamma_I(M/N)$, $\dim_R R/\mathfrak{p} \leq 1$. Then $H^i_I(M)$ has finite support for all $i \geq 0$.*

Proof. In view of [8, Corollary 4.3], $\Gamma_I(M/N)$ has finite support, for any submodules N of M. Now the result follows by Corollary 2.3. ∎

3. Goldie dimension and Artinianness of $H^i_{I,J}(M)$

For an R-module M, the Goldie dimension of M is defined as the cardinal of the set of indecomposable submodules of $E_R(M)$, which appear in a decomposition of $E_R(M)$ into direct sum of indecomposable submodules. We shall use Gdim$M$ to denote the Goldie dimension of M. Let $\mu^0(\mathfrak{p}, M)$ denote the 0-th Bass number of M with respect to prime ideal $\mathfrak{p}$. It is clear that $\mathrm{Gdim} M = \sum_{\mathfrak{p} \in \mathrm{spec(R)}} \mu^0(\mathfrak{p}, M)$. In [9], the authors, offered a generalization of the notion of Goldie dimension and introduced the concept of I-relative Goldie dimension of M as $\mathrm{Gdim}_I M = \sum_{\mathfrak{p} \in \mathrm{V}(I)} \mu^0(\mathfrak{p}, M)$, where $\mathrm{V}(I)$ denotes the set of prime ideals of R which are containing I. We first generalize this concept as follows.

Definition 3.1. *Let I, J be two ideals of R. For an R-module M, we define (I,J)-relative Goldie dimension of M as $\mathrm{Gdim}_{I,J} M = \sum_{\mathfrak{p} \in \mathrm{W}(I,J)} \mu^0(\mathfrak{p}, M)$. Here $\mathrm{W}(I,J)$ denotes the set of prime ideals $\mathfrak{p}$ of R such that $I^t \subseteq \mathfrak{p} + J$ for some positive integer t.*

It is easy to see that finitely generated modules, Artinian modules, quotients of the Matlis reflexive modules and quotients of the linearly compact modules have finite (I,J)-relative Goldie dimension, see [9, Example 2.2]. Also it is clear that if $J = 0$, then $\mathrm{W}(I,J) = \mathrm{V}(I)$ and so $\mathrm{Gdim}_{I,J} M = \mathrm{Gdim}_I M$. Moreover

$$\mathrm{Gdim}_I M \leq \mathrm{Gdim}_{I,J} M \leq \mathrm{Gdim} M.$$

But the following example shows that these inequalities may be strict. Let $I = 2\mathbb{Z}$, $J = 3\mathbb{Z}$ and $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$. Then $\mathrm{V}(I) = \{2\mathbb{Z}\}$, $\mathrm{W}(I,J) \cap \mathrm{Ass}_{\mathbb{Z}} M = \{2\mathbb{Z}, 5\mathbb{Z}\}$ and $E_{\mathbb{Z}}(M) = E_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) \oplus E_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}) \oplus E_{\mathbb{Z}}(\mathbb{Z}/5\mathbb{Z})$. Therefore $\mathrm{Gdim}_I M = 1$, $\mathrm{Gdim}_{I,J} M = 2$ and $\mathrm{Gdim} M = 3$.

Theorem 3.1. *Let M be a ZD-module such that $\mathrm{Gdim}_{IR_\mathfrak{q}, JR_\mathfrak{q}} M_\mathfrak{q}$ is finite, for any prime ideal $\mathfrak{q}$ which is maximal in $\mathrm{Ass}_R M$. Then $\mathrm{Gdim}_{I,J} M$ is finite.*

Proof. Let $\{\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_t\}$ be the set of all prime ideals with the property being maximal in $\mathrm{Ass}_R M$; note that this set is finite by [9, Lemma 2.3]. It is easy to see that if $\mathfrak{p} \in \mathrm{W}(I,J)$ and $\mathfrak{p} \subseteq \mathfrak{q}$, then $\mathfrak{p}R_\mathfrak{q} \in \mathrm{W}(IR_\mathfrak{q}, JR_\mathfrak{q})$, where $\mathfrak{q}$ is an arbitrary prime ideal of R. Thus

$$\mathrm{Gdim}_{I,J} M = \sum_{\mathfrak{p} \in \mathrm{W}(I,J)} \mu^0(\mathfrak{p}, M) \leq \sum_{i=1}^{t} \sum_{\mathfrak{p} \in \mathrm{W}(I,J), \mathfrak{p} \subseteq \mathfrak{q}_i} \mu^0(\mathfrak{p}, M)$$

$$\leq \sum_{i=1}^{t} \sum_{\mathfrak{p}R_{\mathfrak{q}_i} \in \mathrm{W}(IR_{\mathfrak{q}_i}, JR_{\mathfrak{q}_i})} \mu^0(\mathfrak{p}R_{\mathfrak{q}_i}, M_{\mathfrak{q}_i})$$
$$= \sum_{i=1}^{t} \mathrm{Gdim}_{IR_{\mathfrak{q}_i}, JR_{\mathfrak{q}_i}} M_{\mathfrak{q}_i}$$

so the claim follows. ∎

In the following we show that, for any R-module M, (I,J)-relative Goldie dimension of M is equal to Goldie dimension of its (I,J)-torsion submodule. Precisely, we shall show that:

Lemma 3.1. *If M is an R-module, then* $\mathrm{Gdim}_{I,J} M = \mathrm{Gdim}\Gamma_{I,J}(M)$.

Proof. Let $E_R(M) \cong \oplus_{\mathfrak{p}\in \mathrm{spec(R)}} \mu^0(\mathfrak{p}, M) E_R(R/\mathfrak{p})$ be a decomposition of $E_R(M)$ as the direct sum of indecomposable injective R-modules, where $E_R(R/\mathfrak{p})$ denotes the injective hull of $R/\mathfrak{p}$ and $\mu^0(\mathfrak{p}, M)$ denotes the 0-th Bass number of M with respect to prime ideal $\mathfrak{p}$. Then by using [12, Proposition 1.11], we have $\Gamma_{I,J}(E_R(M)) \cong \oplus_{\mathfrak{p}\in \mathrm{W}(I,J)} \mu^0(\mathfrak{p}, M) E_R(R/\mathfrak{p})$ and so it is an injective R-module. We have to show that $\Gamma_{I,J}(E_R(M))$ is an essential extension of $\Gamma_{I,J}(M)$. Suppose x be a non-zero element of $\Gamma_{I,J}(E_R(M))$. Thus there exists $r \in R$ and a positive integer t such that $I^t x \subseteq Jx$ and $0 \neq rx \in M \cap Rx$. So that $I^t(rx) \subseteq J(rx)$ and $0 \neq rx \in \Gamma_{I,J}(M) \cap Rx$. Hence, $\Gamma_{I,J}(E_R(M))$ is an injective essential extension of $\Gamma_{I,J}(M)$. Therefore we have $E_R(\Gamma_{I,J}(M)) \cong \Gamma_{I,J}(E_R(M))$ and so

$$\mathrm{Gdim}_{I,J} M = \sum_{\mathfrak{p}\in \mathrm{W}(I,J)} \mu^0(\mathfrak{p}, M) = \mathrm{Gdim}\Gamma_{I,J}(M).$$ ∎

The following result is a generalization of [9, Corollary 3.3(ii)].

Theorem 3.2. *Let M be a ZD-module of dimension n such that (I,J)-relative Goldie dimension of any quotient of M is finite. Then $H^n_{I,J}(M)/JH^n_{I,J}(M)$ is Artinian.*

Proof. The proof, which we include for the reader's convenience, proceeds like that used in the proof of Theorem 2.2. We use induction on n. If $n = 0$, then $\mathrm{Ass}_R \Gamma_{I,J}(M) \subseteq \mathrm{Ass}_R M \subseteq \mathrm{Max}(R)$. Hence $E_R(\Gamma_{I,J}(M))$ is a finite direct sum of $E_R(R/\mathfrak{m})$, where $\mathfrak{m}$ is a maximal ideal of R. Therefore $E_R(\Gamma_{I,J}(M))$ and so $\Gamma_{I,J}(M)/J\Gamma_{I,J}(M)$ is Artinian. We therefore assume, inductively, that $n > 0$ and the result has been proved for any R-module of dimension less than n satisfying the hypothesis. The exact sequence (2.1) induces the long exact sequence

$$\cdots \to H^n_{I,J}(\Gamma_J(M)) \to H^n_{I,J}(M) \to H^n_{I,J}(M/\Gamma_J(M)) \to H^{n+1}_{I,J}(\Gamma_J(M)) \to \cdots.$$

By [12, Corollary 2.5], $H^i_{I,J}(\Gamma_J(M)) \cong H^i_I(\Gamma_J(M))$, for all $i \geq 0$, since $\Gamma_J(M)$ is J-torsion. On the other hand, we have $\dim_R \Gamma_J(M) \leq \dim_R M = n$, thus $H^n_I(\Gamma_J(M))$ is Artinian and $H^{n+1}_I(\Gamma_J(M)) = 0$. Hence $H^n_{I,J}(\Gamma_J(M))$ is Artinian and $H^{n+1}_{I,J}(\Gamma_J(M)) = 0$. Now by the exact sequence

$$H^n_{I,J}(\Gamma_J(M))/JH^n_{I,J}(\Gamma_J(M)) \longrightarrow H^n_{I,J}(M)/JH^n_{I,J}(M)$$
$$\longrightarrow H^n_{I,J}(M/\Gamma_J(M))/JH^n_{I,J}(M/\Gamma_J(M)) \longrightarrow 0$$

we can assume that $\Gamma_J(M) = 0$. Thus J contains an element a which is a non zero-divisor on M, by [9, Lemma 2.4]. Since $\dim_R M/aM \leq n-1$, thus it follows either from inductive hypothesis or from [12, Theorem 3.2], and Grothendieck's Vanishing Theorem [5, Theorem

6.1.2], that $H^{n-1}_{I,J}(M/aM)/JH^{n-1}_{I,J}(M/aM)$ is Artinian. The exact sequence $0 \longrightarrow M \xrightarrow{a} M \longrightarrow M/aM \longrightarrow 0$ induces the exact sequence

$$H^{n-1}_{I,J}(M/aM)/JH^{n-1}_{I,J}(M/aM) \to H^{n}_{I,J}(M)/JH^{n}_{I,J}(M) \xrightarrow{a} H^{n}_{I,J}(M)/JH^{n}_{I,J}(M) \to 0.$$

Now we have $0 :_{H^{n}_{I,J}(M)/JH^{n}_{I,J}(M)} a$ is Artinian and so $H^{n}_{I,J}(M)/JH^{n}_{I,J}(M)$ is Artinian by [1, Lemma 2.3]. This completes the inductive step. ∎

Corollary 3.1. *Let M be a finitely generated R-module of dimension n. Then $H^{n}_{I,J}(M)/JH^{n}_{I,J}(M)$ is Artinian.*

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