BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **35***(4) (2012), 885–896*

The Cardinal of Various Monoids of Transformations That Preserve a Uniform Partition

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Abstract. In this paper we give formulas for the number of elements of the monoids $\mathscr{OR}_{m\times n}$ of all full transformations on a finite chain with mn elements that preserve a uniform m-partition and preserve or reverse the orientation and for its submonoids $\mathscr{OD}_{m\times n}$ of all order-preserving or order-reversing elements, $\mathscr{OP}_{m\times n}$ of all orientation-preserving elements, $\mathscr{O}_{m\times n}$ of all order-preserving elements, $\mathscr{O}^+_{m\times n}$ of all extensive order-preserving elements and $\mathscr{O}^-_{m\times n}$ of all co-extensive order-preserving elements.

2010 Mathematics Subject Classification: 20M20, 05A10

Keywords and phrases: Order-preserving/reversing, orientation-preserving/reversing, extensive, equivalence-preserving, transformations.

1. Introduction and preliminaries

For $n \in \mathbb{N}$, let $X_n = \{1,2,\ldots,n\}$. Following the standard notation, we denote by $\mathscr{PT}_n$ the monoid (under composition) of all partial transformations on X_n and by $\mathscr{T}_n$ and $\mathscr{I}_n$ its submonoids of all full transformations and of all injective partial transformations, respectively. Now, consider the usual linear order on X_n, i.e. $X_n = \{1 < 2 < \cdots < n\}$. A transformation $\alpha \in \mathscr{PT}_n$ is said to be *extensive* [resp., *co-extensive*] if $x \le x\alpha$ [resp., $x\alpha \le x$], for all $x \in \mathrm{Dom}(\alpha)$. We denote by $\mathscr{T}_n^+$ [resp., $\mathscr{T}_n^-$] the submonoid of $\mathscr{T}_n$ of all extensive [resp., co-extensive] transformations.

A transformation $\alpha \in \mathscr{PT}_n$ is said to be *order-preserving* [resp., *order-reversing*] if $x \le y$ implies $x\alpha \le y\alpha$ [resp., $y\alpha \le x\alpha$], for all $x,y \in \mathrm{Dom}(\alpha)$. We denote by $\mathscr{PO}_n$ the submonoid of $\mathscr{PT}_n$ of all order-preserving partial transformations. As usual, we denote by $\mathscr{O}_n$ the monoid $\mathscr{PO}_n \cap \mathscr{T}_n$ of all full transformations that preserve the order. This monoid has been extensively studied since the sixties (e.g. see [1, 2, 3, 7, 9, 21, 32, 35]). In particular, in 1971, Howie [22] showed that the cardinal of $\mathscr{O}_n$ is $\binom{2n-1}{n-1}$ and in [19], jointly with

Communicated by Rosihan M. Ali, Dato'.
Received: August 6, 2010; *Revised:* November 10, 2010.

Gomes, they proved that

$$|\mathscr{PO}_n| = \sum_{i=1}^{n} \binom{n}{i}\binom{n+i-1}{i} + 1.$$

See also Laradji and Umar [28, 29].

Next, denote by $\mathscr{O}_n^+$ [resp., by $\mathscr{O}_n^-$] the monoid $\mathscr{T}_n^+ \cap \mathscr{O}_n$ [resp., $\mathscr{T}_n^- \cap \mathscr{O}_n$] of all extensive [resp., co-extensive] order-preserving full transformations. The monoids $\mathscr{O}_n^+$ and $\mathscr{O}_n^-$ are isomorphic and their cardinal is the n^{th}-Catalan number, i.e.

$$|\mathscr{O}_n^+| = |\mathscr{O}_n^-| = \frac{1}{n+1}\binom{2n}{n}$$

(see [33]). Moreover, the family $\{\mathscr{O}_n^+ \mid n \in \mathbb{N}\}$ generates the pseudovariety of $\mathscr{J}$-trivial monoids. Notice that, this pseudovariety is also generated by the syntactic monoids of the piecewise testable languages (see e.g. [31]). Regarding the injective counterpart of $\mathscr{O}_n$, i.e. the inverse monoid $\mathscr{POI}_n = \mathscr{PO}_n \cap \mathscr{I}_n$ of all injective order-preserving partial transformations, we have $|\mathscr{POI}_n| = \binom{2n}{n}$. This result was first presented by Garba in [18] (see also [7]).

Now, being $\mathscr{POD}_n$ the submonoid of $\mathscr{PT}_n$ of all partial transformations that preserve or reverse the order, $\mathscr{OD}_n = \mathscr{POD}_n \cap \mathscr{T}_n$ and $\mathscr{PODI}_n = \mathscr{POD}_n \cap \mathscr{I}_n$ (the full and partial injective counterparts of $\mathscr{POD}_n$, respectively), Fernandes *et al.* [10, 11] proved that

$$|\mathscr{POD}_n| = \sum_{i=1}^{n} \binom{n}{i}\left(2\binom{n+i-1}{i} - n\right) + 1, \quad |\mathscr{OD}_n| = 2\binom{2n-1}{n-1} - n$$

and

$$|\mathscr{PODI}_n| = 2\binom{2n}{n} - n^2 - 1.$$

Wider classes of monoids are obtained when we consider transformations that either preserve or reverse the orientation. Let $a = (a_1, a_2, \ldots, a_t)$ be a sequence of t, $t \geq 0$, elements from the chain X_n. We say that a is *cyclic* [resp., *anti-cyclic*] if there exists no more than one index $i \in \{1, \ldots, t\}$ such that $a_i > a_{i+1}$ [resp., $a_i < a_{i+1}$], where a_{t+1} denotes a_1. Let $\alpha \in \mathscr{T}_n$ and suppose that $\mathrm{Dom}(\alpha) = \{a_1, \ldots, a_t\}$, with $t \geq 0$ and $a_1 < \cdots < a_t$. We say that α is *orientation-preserving* [resp., *orientation-reversing*] if the sequence of its images $(a_1\alpha, a_2\alpha, \ldots, a_t\alpha)$ is cyclic [resp., anti-cyclic]. This notions were introduced by McAlister in [30] and independently by Catarino and Higgins in [6].

Denote by $\mathscr{POP}_n$ [resp., $\mathscr{POR}_n$] the submonoid of $\mathscr{PT}_n$ of all orientation-preserving [resp., orientation-preserving or orientation-reversing] transformations. The cardinalities of $\mathscr{POP}_n$ and $\mathscr{POR}_n$ were calculated by Fernandes *et al.* [12] and are $1 + (2^n - 1)n + \sum_{k=2}^{n} k\binom{n}{k}^2 2^{n-k}$ and $1 + (2^n - 1)n + 2\binom{n}{2}^2 2^{n-2} + \sum_{k=3}^{n} 2k\binom{n}{k}^2 2^{n-k}$, respectively. As usual, $\mathscr{OP}_n$ denotes the monoid $\mathscr{POP}_n \cap \mathscr{T}_n$ of all full transformations that preserve the orientation, $\mathscr{OR}_n$ denotes the monoid $\mathscr{POR}_n \cap \mathscr{T}_n$ of all full transformations that preserve or reserve the orientation and $\mathscr{POPI}_n$ and $\mathscr{PORI}_n$ denote the submonoids of $\mathscr{POP}_n$ and $\mathscr{POR}_n$, respectively, whose elements are the injective transformations. McAlister in [30], and independently Catarino and Higgins in [6], proved that

$$|\mathscr{OP}_n| = n\binom{2n-1}{n-1} - n(n-1) \quad \text{and} \quad |\mathscr{OR}_n| = n\binom{2n}{n} - \frac{n^2}{2}(n^2 - 2n + 5) + n.$$

The monoids $\mathcal{OP}_n$ and $\mathcal{OR}_n$ were also studied by Arthur and Ruškuc in [5]. Regarding their injective counterparts, in [8], Fernandes established that $|\mathcal{POPI}_n| = 1 + \frac{n}{2}\binom{2n}{n}$ and, in [10], Fernandes *et al.* showed that

$$|\mathcal{PORI}_n| = 1 + n\binom{2n}{n} - \frac{n^2}{2}(n^2 - 2n + 3).$$

All these results are summarized in [13].

Now, let X be a set and denote by $\mathcal{T}(X)$ the monoid (under composition) of all full transformations on X. Let ρ be an equivalence relation on X and denote by $\mathcal{T}_\rho(X)$ the submonoid of $\mathcal{T}(X)$ of all transformations that preserve the equivalence relation ρ, i.e. $\mathcal{T}_\rho(X) = \{\alpha \in \mathcal{T}(X) \mid (a\alpha, b\alpha) \in \rho, \text{for all } (a,b) \in \rho\}$. This monoid was studied by Huisheng in [24] who determined its regular elements and described its Green's relations.

Let $m, n \in \mathbb{N}$. Of particular interest is the submonoid $\mathcal{T}_{m\times n} = \mathcal{T}_\rho(X_{mn})$ of $\mathcal{T}_{mn}$, with ρ the equivalence relation on X_{mn} defined by $\rho = (A_1 \times A_1) \cup (A_2 \times A_2) \cup \cdots \cup (A_m \times A_m)$, where $A_i = \{(i-1)n+1, \ldots, in\}$, for $i \in \{1, \ldots, m\}$. Notice that the ρ-classes A_i, with $1 \le i \le m$, form a uniform m-partition of X_{mn}. Regarding the rank of $\mathcal{T}_{m\times n}$, first, Huisheng [23] proved that it is at most 6 and, later, Araújo and Schneider [4] improved this result by showing that, for $m \ge 2$ and $n \ge 2$, the rank of $\mathcal{T}_{m\times n}$ is precisely 4.

Finally, denote by $\mathcal{OR}_{m\times n}$ the submonoid of $\mathcal{T}_{m\times n}$ of all orientation-preserving or orientation-reversing transformations, i.e. $\mathcal{OR}_{m\times n} = \mathcal{T}_{m\times n} \cap \mathcal{OR}_{mn}$. Similarly, let $\mathcal{OD}_{m\times n} = \mathcal{T}_{m\times n} \cap \mathcal{OD}_{mn}$, $\mathcal{OP}_{m\times n} = \mathcal{T}_{m\times n} \cap \mathcal{OP}_{mn}$ and $\mathcal{O}_{m\times n} = \mathcal{T}_{m\times n} \cap \mathcal{O}_{mn}$. Consider also the submonoids $\mathcal{O}^+_{m\times n} = \mathcal{O}_{m\times n} \cap \mathcal{T}^+_{mn}$ and $\mathcal{O}^-_{m\times n} = \mathcal{O}_{m\times n} \cap \mathcal{T}^-_{mn}$ of $\mathcal{O}_{m\times n}$ whose elements are the extensive transformations and the co-extensive transformations, respectively.

Example 1.1. Consider the following transformations of $\mathcal{T}_{12}$:

$$\alpha_1 = \left(\begin{array}{cccc|cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 9 & 11 & 10 & 12 & 1 & 3 & 3 & 2 & 5 & 5 & 7 & 8 \end{array}\right),$$

$$\alpha_2 = \left(\begin{array}{cccc|cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 8 & 8 & 8 & 6 & 6 & 5 & 5 & 5 & 12 & 12 & 11 & 10 \end{array}\right),$$

$$\alpha_3 = \left(\begin{array}{cccc|cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 11 & 11 & 10 & 10 & 10 & 9 & 9 & 9 & 4 & 3 & 3 & 1 \end{array}\right),$$

$$\alpha_4 = \left(\begin{array}{cccc|cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 7 & 7 & 7 & 8 & 8 & 8 & 5 & 5 & 5 & 6 & 6 & 7 \end{array}\right),$$

$$\alpha_5 = \left(\begin{array}{cccc|cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 1 & 1 & 2 & 3 & 3 & 4 & 4 & 10 & 11 & 11 & 11 \end{array}\right),$$

$$\alpha_6 = \left(\begin{array}{cccc|cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 5 & 5 & 6 & 6 & 6 & 7 & 7 & 8 & 10 & 11 & 11 & 12 \end{array}\right),$$

$$\alpha_7 = \left(\begin{array}{cccc|cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 1 & 2 & 3 & 5 & 5 & 6 & 8 & 9 & 9 & 10 & 11 \end{array}\right),$$

$$\alpha_8 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 1 & 2 & 3 & 5 & 5 & 6 & 9 & 9 & 10 & 10 & 11 \end{pmatrix}.$$

Then, we have: $\alpha_1 \in \mathcal{T}_{3\times 4}$, but $\alpha_1 \notin \mathcal{OR}_{3\times 4}$; $\alpha_2 \in \mathcal{OR}_{3\times 4}$, but $\alpha_2 \notin \mathcal{OP}_{3\times 4}$; $\alpha_3 \in \mathcal{OD}_{3\times 4}$, but $\alpha_3 \notin \mathcal{O}_{3\times 4}$; $\alpha_4 \in \mathcal{OP}_{3\times 4}$, but $\alpha_4 \notin \mathcal{O}_{3\times 4}$; $\alpha_5 \in \mathcal{O}_{3\times 4}$, but $\alpha_5 \notin \mathcal{O}^+_{3\times 4}$ and $\alpha_5 \notin \mathcal{O}^-_{3\times 4}$; $\alpha_6 \in \mathcal{O}^+_{3\times 4}$; $\alpha_7 \in \mathcal{O}^-_{3\times 4}$; and, finally, $\alpha_8 \notin \mathcal{T}_{3\times 4}$.

Observe that, as happens with $\mathscr{O}_n^-$ and $\mathscr{O}_n^+$, the monoids $\mathscr{O}_{m\times n}^-$ and $\mathscr{O}_{m\times n}^+$ are isomorphic [15]. Recall that in [26] Kunze proved that the monoid $\mathscr{O}_n$ is a quotient of a bilateral semidirect product of its subsemigroups $\mathscr{O}_n^-$ and $\mathscr{O}_n^+$. This result was generalized by the authors [15] by showing that $\mathscr{O}_{m\times n}$ is also a quotient of a bilateral semidirect product of its subsemigroups $\mathscr{O}_{m\times n}^-$ and $\mathscr{O}_{m\times n}^+$. See also [27, 14]. In [25] Huisheng and Dingyu described the regular elements and the Green relations of $\mathscr{O}_{m\times n}$. On the other hand, the ranks of the monoids $\mathscr{O}_{m\times n}$, $\mathscr{O}_{m\times n}^+$ and $\mathscr{O}_{m\times n}^-$ were calculated by the authors in [15]. Regarding $\mathscr{O}\mathscr{P}_{m\times n}$, a description of the regular elements and a characterization of the Green relations were given by Sun *et al.* in [34]. Its rank was determined by the authors in [16], who also computed in the same paper the ranks of the monoids $\mathscr{O}\mathscr{D}_{m\times n}$ and $\mathscr{O}\mathscr{R}_{m\times n}$.

In this paper we calculate the cardinals of the monoids $\mathscr{O}\mathscr{R}_{m\times n}$, $\mathscr{O}\mathscr{P}_{m\times n}$, $\mathscr{O}\mathscr{D}_{m\times n}$, $\mathscr{O}_{m\times n}$, $\mathscr{O}_{m\times n}^+$ and $\mathscr{O}_{m\times n}^-$. In order to achieve this goal we use a wreath product description of $\mathscr{T}_{m\times n}$, due to Araújo and Schneider [4], that we recall in Section 2.

2. Wreath products of transformation semigroups

In [4] Araújo and Schneider proved that the rank of $\mathscr{T}_{m\times n}$ is 4, by using the concept of wreath product of transformation semigroups. This approach will also be very useful in this paper. Next, we recall some facts from [4, 15, 16]. First, we define the wreath product $\mathscr{T}_n \wr \mathscr{T}_m$ of $\mathscr{T}_n$ and $\mathscr{T}_m$ as being the monoid with underlying set $\mathscr{T}_n^m \times \mathscr{T}_m$ and multiplication defined by $(\alpha_1,\ldots,\alpha_m;\beta)(\alpha'_1,\ldots,\alpha'_m;\beta') = (\alpha_1\alpha'_{1\beta},\ldots,\alpha_m\alpha'_{m\beta};\beta\beta')$, for all $(\alpha_1,\ldots,\alpha_m;\beta),(\alpha'_1,\ldots,\alpha'_m;\beta') \in \mathscr{T}_n^m \times \mathscr{T}_m$. Now, let $\alpha \in \mathscr{T}_{m\times n}$ and let $\beta = \alpha/\rho \in \mathscr{T}_m$ be the *quotient* map of α by ρ, i.e. for all $j \in \{1,\ldots,m\}$, we have $A_j\alpha \subseteq A_{j\beta}$. For each $j \in \{1,\ldots,m\}$, define $\alpha_j \in \mathscr{T}_n$ by $k\alpha_j = ((j-1)n+k)\alpha - (j\beta-1)n$, for all $k \in \{1,\ldots,n\}$. Let $\overline{\alpha} = (\alpha_1,\alpha_2,\ldots,\alpha_m;\beta) \in \mathscr{T}_n^m \times \mathscr{T}_m$. With these notations, the function $\psi : \mathscr{T}_{m\times n} \longrightarrow \mathscr{T}_n \wr \mathscr{T}_m$, $\alpha \longmapsto \overline{\alpha}$, is an isomorphism (see [4, Lemma 2.1]).

Observe that, from this fact, we can immediately conclude that the cardinal of $\mathscr{T}_{m\times n}$ is $n^{nm}m^m$.

Example 2.1. Consider the transformation

$$\alpha = \left(\begin{array}{cccc|cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 5 & 5 & 7 & 6 & 10 & 10 & 9 & 12 & 1 & 1 & 2 & 3 \end{array}\right) \in \mathscr{T}_{3\times 4}.$$

Since

$$\beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},\ \alpha_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 2 \end{pmatrix},$$

$$\alpha_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 \end{pmatrix},\ \alpha_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix},$$

we have $\overline{\alpha} = (\alpha_1,\alpha_2,\alpha_3;\beta)$.

Next, consider

$$\overline{\mathscr{O}}_{m\times n} = \{(\alpha_1,\ldots,\alpha_m;\beta) \in \mathscr{O}_n^m \times \mathscr{O}_m \mid j\beta = (j+1)\beta \text{ implies}$$

$$n\alpha_j \le 1\alpha_{j+1},\ \text{for all } j \in \{1,\ldots,m-1\}\}.$$

Notice that, if $(\alpha_1,\ldots,\alpha_m;\beta) \in \overline{\mathscr{O}}_{m\times n}$ and $1 \le i < j \le m$ are such that $i\beta = j\beta$, then $n\alpha_i \le 1\alpha_j$.

Proposition 2.1. [15] *The set $\overline{\mathscr{O}}_{m\times n}$ is a submonoid of $\mathscr{T}_n \wr \mathscr{T}_m$ (and of $\mathscr{O}_n \wr \mathscr{O}_m$) isomorphic to $\mathscr{O}_{m\times n}$.*

On the other hand, since

$$\overline{\mathscr{O}}^{+}_{m\times n} = \{(\alpha_1,\dots,\alpha_m;\beta) \in \mathscr{O}_n^{m-1}\times\mathscr{O}_n^{+}\times\mathscr{O}_m^{+} \mid j\beta=(j+1)\beta \text{ implies } n\alpha_j \le 1\alpha_{j+1} \text{ and } j\beta = j \text{ implies } \alpha_j \in \mathscr{O}_n^{+}, \text{ for all } j\in\{1,\dots,m-1\}\}$$

and

$$\overline{\mathscr{O}}^{-}_{m\times n} = \{(\alpha_1,\dots,\alpha_m;\beta) \in \mathscr{O}_n^{-}\times\mathscr{O}_n^{m-1}\times\mathscr{O}_m^{-} \mid (j-1)\beta=j\beta \text{ implies } n\alpha_{j-1} \le 1\alpha_{j} \text{ and } j\beta = j \text{ implies } \alpha_j \in \mathscr{O}_n^{-}, \text{ for all } j\in\{2,\dots,m\}\},$$

we have:

Proposition 2.2. [15] *The set $\overline{\mathscr{O}}^{+}_{m\times n}$ [resp. $\overline{\mathscr{O}}^{-}_{m\times n}$] is a submonoid of $\mathscr{T}_n \wr \mathscr{T}_m$ (and of $\mathscr{O}_n \wr \mathscr{O}_m$) isomorphic to $\mathscr{O}^{+}_{m\times n}$ [resp. $\mathscr{O}^{-}_{m\times n}$].*

A description of $\mathscr{OP}_{m\times n}$ in terms of wreath products is more elaborate. In fact, considering addition modulo m (in particular, $m+1=1$), we have:

Proposition 2.3. [16] *A $(m+1)$-tuple $(\alpha_1,\alpha_2,\dots,\alpha_m;\beta)$ of $\mathscr{T}_n^m\times\mathscr{T}_m$ belongs to $\mathscr{OP}_{m\times n}\psi$ if and only if it satisfies one of the following conditions:*

(1) (a) *β is a non-constant transformation of $\mathscr{OP}_m$,*
 (b) *for all $i\in\{1,\dots,m\}$, $\alpha_i\in\mathscr{O}_n$ and,*
 (c) *for all $j\in\{1,\dots,m\}$, $j\beta=(j+1)\beta$ implies $n\alpha_j\le 1\alpha_{j+1}$;*

(2) (a) *β is a constant transformation,*
 (b) *for all $i\in\{1,\dots,m\}$, $\alpha_i\in\mathscr{O}_n$ and*
 (c) *there exists at most one index $j\in\{1,\dots,m\}$ such that $n\alpha_j > 1\alpha_{j+1}$;*

(3) (a) *β is a constant transformation,*
 (b) *there exists one index $i\in\{1,\dots,m\}$ such that $\alpha_i\in\mathscr{OP}_n\setminus\mathscr{O}_n$ and, for all $j\in\{1,\dots,m\}\setminus\{i\}$, $\alpha_j\in\mathscr{O}_n$*
 (c) *and, for all $j\in\{1,\dots,m\}$, $n\alpha_j\le 1\alpha_{j+1}$.*

Let $\alpha\in\mathscr{OP}_{m\times n}$. We say that α is of *type i* if $\alpha\psi$ satisfies the condition (i) of the previous proposition, for $i\in\{1,2,3\}$.

3. The cardinals

In this section we use the previous bijections to obtain formulas for the number of elements of the monoids $\mathscr{O}_{m\times n}$, $\mathscr{O}^{+}_{m\times n}$, $\mathscr{O}^{-}_{m\times n}$, $\mathscr{OD}_{m\times n}$, $\mathscr{OP}_{m\times n}$ and $\mathscr{OR}_{m\times n}$. In order to count the elements of $\mathscr{O}_{m\times n}$, on one hand, for each transformation $\beta\in\mathscr{O}_m$, we determine the number of sequences $(\alpha_1,\dots,\alpha_m)\in\mathscr{O}_n^m$ such that $(\alpha_1,\dots,\alpha_m;\beta)\in\overline{\mathscr{O}}_{m\times n}$ and, on the other hand, we notice that this last number just depends of the kernel of β (and not of β itself).

With this purpose, let $\beta\in\mathscr{O}_m$. Suppose that $\mathrm{Im}(\beta)=\{b_1<b_2<\cdots<b_t\}$, for some $1\le t\le m$, and define $k_i=|b_i\beta^{-1}|$, for $i=1,\dots,t$. Being β an order-preserving transformation, the sequence $(k_1,\dots,k_t)$ determines the kernel of β: we have $\{k_1+\cdots+k_{i-1}+1,\dots,k_1+\cdots+k_i\}\beta=\{b_i\}$, for $i=1,\dots,t$ (considering $k_1+\cdots+k_{i-1}+1=1$, with $i=1$). We define the *kernel type* of β as being the sequence $(k_1,\dots,k_t)$. Notice that $1\le k_i\le m$, for $i=1,\dots,t$, and $k_1+k_2+\cdots+k_t=m$. Now, recall that the number of non-decreasing sequences of length k whose terms are taken from a chain with n elements is equal to $\binom{n+k-1}{k}=\binom{n+k-1}{n-1}$,

i.e. the number of k-combinations with repetitions from a set with n elements (see [20], for example). Therefore, since $(\alpha_1,\ldots,\alpha_k)\in\mathscr{O}_n^k$ satisfies the condition $n\alpha_j\le 1\alpha_{j+1}$, for all $1\le j\le k-1$, if and only if the sequence obtained by concatenating the sequences of images of $\alpha_1,\ldots,\alpha_k$ (by this order) is non-decreasing, it follows that the set $\{(\alpha_1,\ldots,\alpha_k)\in\mathscr{O}_n^k \mid n\alpha_j\le 1\alpha_{j+1}, \text{for all } 1\le j\le k-1\}$ has size $\binom{n+kn-1}{n-1}$.

Since $(\alpha_1,\ldots,\alpha_m;\beta)\in\overline{\mathscr{O}}_{m\times n}$ if and only if, for all $1\le i\le t$, $\alpha_{k_1+\cdots+k_{i-1}+1},\ldots,\alpha_{k_1+\cdots+k_i}$ are k_i order-preserving transformations such that the concatenation sequence of their images (by this order) is still a non-decreasing sequence, then we have $\prod_{i=1}^{t}\binom{k_in+n-1}{n-1}$ elements in $\overline{\mathscr{O}}_{m\times n}$ whose $(m+1)$-component is β.

Finally it is clear that if β and β' are two elements of $\mathscr{O}_m$ with the same kernel type then $(\alpha_1,\ldots,\alpha_m;\beta)\in\overline{\mathscr{O}}_{m\times n}$ if and only if $(\alpha_1,\ldots,\alpha_m;\beta')\in\overline{\mathscr{O}}_{m\times n}$. Thus, as the number of transformations $\beta\in\mathscr{O}_m$ with kernel type of length t $(1\le t\le m)$ coincides with the number of t-combinations (without repetition) from a set with m elements, it follows:

Theorem 3.1. $|\mathscr{O}_{m\times n}| = \displaystyle\sum_{\substack{1\le k_1,\ldots,k_t\le m\\ k_1+\cdots+k_t=m\\ 1\le t\le m}} \binom{m}{t}\prod_{i=1}^{t}\binom{k_in+n-1}{n-1}.$

The table below shows the size of the monoid $\mathscr{O}_{m\times n}$ for several values of m and n. These calculations were performed by using GAP [17].

$m\setminus n$	1	2	3	4	5	6
1	1	3	10	35	126	462
2	3	19	156	1555	17878	225820
3	10	138	2845	78890	2768760	115865211
4	35	1059	55268	4284451	454664910	61824611940
5	126	8378	1109880	241505530	77543615751	34003513468232
6	462	67582	22752795	13924561150	13556873588212	19134117191404027

In view of Theorem 3.1, finding the cardinal of $\mathscr{OD}_{m\times n}$ is not difficult. Indeed, consider the reflexion permutation $h=\begin{pmatrix}1 & 2 & \cdots & mn-1 & mn\\ mn & mn-1 & \cdots & 2 & 1\end{pmatrix}$. Observe that $h\in\mathscr{OD}_{m\times n}$ and, given $\alpha\in\mathscr{T}_{m\times n}$, we have $\alpha\in\mathscr{OD}_{m\times n}$ if and only if $\alpha\in\mathscr{O}_{m\times n}$ or $h\alpha\in\mathscr{O}_{m\times n}$. On the other hand, as clearly $|\mathscr{O}_{m\times n}|=|h\mathscr{O}_{m\times n}|$ and $|\mathscr{O}_{m\times n}\cap h\mathscr{O}_{m\times n}|=|\{\alpha\in\mathscr{O}_{m\times n}\mid |\mathrm{Im}(\alpha)|=1\}|=mn$, it follows immediately that

Theorem 3.2. $|\mathscr{OD}_{m\times n}| = 2|\mathscr{O}_{m\times n}|-mn = 2\displaystyle\sum_{\substack{1\le k_1,\ldots,k_t\le m\\ k_1+\cdots+k_t=m\\ 1\le t\le m}} \binom{m}{t}\prod_{i=1}^{t}\binom{k_in+n-1}{n-1}-mn.$

Next, we describe a process to count the number of elements of $\mathscr{O}^+_{m\times n}$. First, recall that the cardinal of $\mathscr{O}^+_n$ is the n^{th}-Catalan number, i.e. $|\mathscr{O}^+_n|=\frac{1}{n+1}\binom{2n}{n}$. See [33]. It is also useful to consider the following numbers: $\theta(n,i)=|\{\alpha\in\mathscr{O}^+_n \mid 1\alpha=i\}|$, for $1\le i\le n$. Clearly, we have $|\mathscr{O}^+_n|=\sum_{i=1}^{n}\theta(n,i)$. Moreover, for $2\le i\le n-1$, we have $\theta(n,i)=\theta(n,i+1)+\theta(n-1,i-1)$. In fact, $\{\alpha\in\mathscr{O}^+_n \mid 1\alpha=i\}=\{\alpha\in\mathscr{O}^+_n \mid 1\alpha=i<2\alpha\}\,\dot\cup\,\{\alpha\in\mathscr{O}^+_n \mid 1\alpha=2\alpha=i\}$ and it is easy to show that the function which maps each transformation $\beta\in\{\alpha\in\mathscr{O}^+_n \mid 1\alpha=i<2\alpha\}$ into the transformation

$$\begin{pmatrix}1 & 2 & \ldots & n\\ i+1 & 2\beta & \ldots & n\beta\end{pmatrix}\in\{\alpha\in\mathscr{O}^+_n \mid 1\alpha=i+1\}$$

and the function which maps each transformation $\beta \in \{\alpha \in \mathscr{O}_{n-1}^+ \mid 1\alpha = i-1\}$ into the transformation

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ i & i & 2\beta+1 & \dots & (n-2)\beta+1 & (n-1)\beta+1 \end{pmatrix} \in \{\alpha \in \mathscr{O}_n^+ \mid 1\alpha = 2\alpha = i\}$$

are bijections. Thus

$$\begin{aligned} \theta(n,i) &= |\{\alpha \in \mathscr{O}_n^+ \mid 1\alpha = i < 2\alpha\}| + |\{\alpha \in \mathscr{O}_n^+ \mid 1\alpha = 2\alpha = i\}| \\ &= |\{\alpha \in \mathscr{O}_n^+ \mid 1\alpha = i+1\}| + |\{\alpha \in \mathscr{O}_{n-1}^+ \mid 1\alpha = i-1\}| \\ &= \theta(n,i+1) + \theta(n-1,i-1). \end{aligned}$$

Also, it is not hard to prove that $\theta(n,2) = \theta(n,1) = \sum_{i=1}^{n-1} \theta(n-1,i) = |\mathscr{O}_{n-1}^+|$.

Now, we can prove:

Lemma 3.1. *For all* $1 \le i \le n$,

$$\theta(n,i) = \frac{i}{n}\binom{2n-i-1}{n-i} = \frac{i}{n}\binom{2n-i-1}{n-1}.$$

Proof. We prove the lemma by induction on n. For $n=1$, it is clear that $\theta(1,1) = 1 = \frac{1}{1}\binom{2-1-1}{1-1}$. Let $n \ge 2$ and suppose that the formula is valid for $n-1$. Next, we prove the formula for n by induction on i. For $i = 1$, as observed above, we have

$$\theta(n,1) = |\mathscr{O}_{n-1}^+| = \frac{1}{n}\binom{2n-2}{n-1}.$$

For $i = 2$, we have

$$\begin{aligned} \theta(n,2) = \theta(n,1) &= \frac{1}{n}\binom{2n-2}{n-1} = \frac{2}{n}\frac{(2n-2)!}{(n-1)!(n-1)!}\frac{n-1}{2n-2} \\ &= \frac{2}{n}\frac{(2n-3)!}{(n-1)!(n-2)!} = \frac{2}{n}\binom{2n-3}{n-1}. \end{aligned}$$

Now, suppose that the formula is valid for $i-1$, with $3 \le i \le n$. Then, using both induction hypotheses on i and on n in the second equality, we have

$$\begin{aligned} \theta(n,i) = \theta(n,i-1) - \theta(n-1,i-2) &= \frac{i-1}{n}\binom{2n-i}{n-1} - \frac{i-2}{n-1}\binom{2n-i-1}{n-2} \\ &= \frac{i-1}{n}\frac{(2n-i)!}{(n-1)!(n-i+1)!} - \frac{i-2}{n-1}\frac{(2n-i-1)!}{(n-2)!(n-i+1)!} \\ &= \frac{i(n-i+1)}{n(2n-i)}\frac{(2n-i)!}{(n-1)!(n-i+1)!} = \frac{i}{n}\binom{2n-i-1}{n-1}, \end{aligned}$$

as required. ∎

Recall that $(\alpha_1,\dots,\alpha_m;\beta) \in \overline{\mathscr{O}}_{m\times n}^+$ if and only if $\beta \in \mathscr{O}_m^+$, $\alpha_m \in \mathscr{O}_n^+$, $\alpha_1,\dots,\alpha_{m-1} \in \mathscr{O}_n$ and, for all $j \in \{1,\dots,m-1\}$, $j\beta = (j+1)\beta$ implies $n\alpha_j \le 1\alpha_{j+1}$ and $j\beta = j$ implies $\alpha_j \in \mathscr{O}_n^+$. Let $\beta \in \mathscr{O}_m^+$. As for the monoid $\mathscr{O}_{m\times n}$, we aim to count the number of sequences $(\alpha_1,\dots,\alpha_m) \in \mathscr{O}_n^m$ such that $(\alpha_1,\dots,\alpha_m;\beta) \in \overline{\mathscr{O}}_{m\times n}^+$. Let $(k_1,\dots,k_t)$ be the kernel type of β. Let $K_i = \{k_1+\dots+k_{i-1}+1,\dots,k_1+\dots+k_i\}$, for $i = 1,\dots,t$. Then, β fixes a point in K_i if and only if it fixes $k_1+\dots+k_i$, for $i = 1,\dots,t$. It follows that $(\alpha_1,\dots,\alpha_m;\beta) \in \overline{\mathscr{O}}_{m\times n}^+$ if and only if, for all $1 \le i \le t$:

(1) If β does not fix a point in K_i, then $\alpha_{k_1+\cdots+k_{i-1}+1},\ldots,\alpha_{k_1+\cdots+k_i}$ are k_i order-preserving transformations such that the concatenation sequence of their images (by this order) is still a non-decreasing sequence (in this case, we have $\binom{k_in+n-1}{n-1}$ subsequences $(\alpha_{k_1+\cdots+k_{i-1}+1},\ldots,\alpha_{k_1+\cdots+k_i})$ allowed);
(2) If β fixes a point in K_i, then $\alpha_{k_1+\cdots+k_{i-1}+1},\ldots,\alpha_{k_1+\cdots+k_i-1}$ are k_i-1 order-preserving transformations such that the concatenation sequence of their images (by this order) is still a non-decreasing sequence, $n\alpha_{k_1+\cdots+k_i-1}\leq 1\alpha_{k_1+\cdots+k_i}$ and $\alpha_{k_1+\cdots+k_i}\in\mathscr{O}_n^+$ (in this case, we have $\sum_{j=1}^n\binom{(k_i-1)n+j-1}{j-1}\theta(n,j)$ subsequences $(\alpha_{k_1+\cdots+k_{i-1}+1},\ldots,\alpha_{k_1+\cdots+k_i})$ allowed).

Define

$$\eth(\beta,i)=\begin{cases}\binom{k_in+n-1}{n-1}, & \text{if } (k_1+\cdots+k_i)\beta\neq k_1+\cdots+k_i\\ \sum_{j=1}^n\frac{j}{n}\binom{2n-j-1}{n-1}\binom{(k_i-1)n+j-1}{j-1}, & \text{if } (k_1+\cdots+k_i)\beta=k_1+\cdots+k_i,\end{cases}$$

for all $1\leq i\leq t$. Thus, we have

Proposition 3.1. $|\mathscr{O}_{m\times n}^+|=\sum_{\beta\in\mathscr{O}_m^+}\prod_{i=1}^t\eth(\beta,i)$.

Next, we obtain a formula for $|\mathscr{O}_{m\times n}^+|$ which does not depend on $\beta\in\mathscr{O}_m^+$. Let β be an element of $\mathscr{O}_m^+$ with kernel type $(k_1,\ldots,k_t)$. Define $s_\beta=(s_1,\ldots,s_t)\in\{0,1\}^{t-1}\times\{1\}$ by $s_i=1$ if and only if $(k_1+\cdots+k_i)\beta=k_1+\cdots+k_i$, for all $1\leq i\leq t-1$. Let $1\leq t,k_1,\ldots,k_t\leq m$ be such that $k_1+\cdots+k_t=m$ and let $(s_1,\ldots,s_t)\in\{0,1\}^{t-1}\times\{1\}$. Let $k=(k_1,\ldots,k_t)$ and $s=(s_1,\ldots,s_t)$. Define $\Delta(k,s)=|\{\beta\in\mathscr{O}_m^+\mid\beta \text{ has kernel type } k \text{ and } s_\beta=s\}|$.

In order to get a formula for $\Delta(k,s)$, we count the number of distinct restrictions to unions of partition classes of the kernel of transformations β of $\mathscr{O}_m^+$ with kernel type k and $s_\beta=s$ corresponding to maximal subsequences of consecutive zeros of s. Let β be an element of $\mathscr{O}_m^+$ with kernel type k and $s_\beta=s$. First, notice that, given $i\in\{1,\ldots,t\}$, if $s_i=1$ then $K_i\beta=\{k_1+\cdots+k_i\}$ and if $s_i=0$ then the (unique) element of $K_i\beta$ belongs to K_j, for some $i<j\leq t$.

Next, let $i\in\{1,\ldots,t\}$ and $r\in\{1,\ldots,t-i\}$ be such that $s_j=0$, for all $j\in\{i,\ldots,i+r-1\}$, $s_{i+r}=1$ and, if $i>1$, $s_{i-1}=1$ (i.e. $(s_i,\ldots,s_{i+r-1})$ is a maximal subsequence of consecutive zeros of s). Then

$$(K_i\cup\cdots\cup K_{i+r-2}\cup K_{i+r-1})\beta\subseteq K_{i+1}\cup\cdots\cup K_{i+r-1}\cup(K_{i+r}\setminus\{k_1+\cdots+k_{i+r}\}).$$

Let $\ell_j=|K_{i+j}\cap(K_i\cup\cdots\cup K_{i+r-1})\beta|$, for $1\leq j\leq r$. Hence, we have $\ell_1,\ldots,\ell_{r-1}\geq 0$, $\ell_r\geq 1$, $\ell_1+\cdots+\ell_r=r$ and $0\leq\ell_1+\cdots+\ell_j\leq j$, for all $1\leq j\leq r-1$.

On the other hand, given $\ell_1,\ldots,\ell_r$ such that $\ell_1,\ldots,\ell_{r-1}\geq 0$, $\ell_r\geq 1$, $\ell_1+\cdots+\ell_r=r$ and $0\leq\ell_1+\cdots+\ell_j\leq j$, for all $1\leq j\leq r-1$, we have precisely

$$\binom{k_{i+1}}{\ell_1}\binom{k_{i+2}}{\ell_2}\cdots\binom{k_{i+r-1}}{\ell_{r-1}}\binom{k_{i+r}-1}{\ell_r}=\binom{k_{i+r}-1}{\ell_r}\prod_{j=1}^{r-1}\binom{k_{i+j}}{\ell_j}$$

distinct restrictions to $K_i\cup\cdots\cup K_{i+r-1}$ of transformations β of $\mathscr{O}_m^+$, with kernel type k and $s_\beta=s$, such that $\ell_j=|K_{i+j}\cap(K_i\cup\cdots\cup K_{i+r-1})\beta|$, for $1\leq j\leq r$. It follows that the number of distinct restrictions to $K_i\cup\cdots\cup K_{i+r-1}$ of transformations β of $\mathscr{O}_m^+$ with kernel type k

and $s_\beta = s$ is

$$\sum_{\substack{\ell_1+\cdots+\ell_r=r \\ 0\le \ell_1+\cdots+\ell_j\le j,\ 1\le j\le r-1 \\ \ell_1,\ldots,\ell_{r-1}\ge 0,\ \ell_r\ge 1}} \binom{k_{i+r}-1}{\ell_r}\prod_{j=1}^{r-1}\binom{k_{i+j}}{\ell_j}.$$

Now, let p be the number of distinct maximal subsequences of consecutive zeros of s. Clearly, if $p=0$ then $\Delta(k,s)=1$. Hence, suppose that $p\ge 1$ and let $1\le u_1<v_1<u_2<v_2<\cdots<u_p<v_p\le t$ be such that

$$\{j\in\{1,\ldots,t\}\mid s_j=0\}=\bigcup_{i=1}^{p}\{u_i,\ldots,v_i-1\}$$

(i.e. $(s_{u_i},\ldots,s_{v_i-1})$, with $1\le i\le p$, are the p distinct maximal subsequences of consecutive zeros of s). Then, being $r_i=v_i-u_i$, for $1\le i\le p$, we have

$$\Delta(k,s)=\prod_{i=1}^{p}\sum_{\substack{\ell_1+\cdots+\ell_{r_i}=r_i \\ 0\le \ell_1+\cdots+\ell_j\le j\ 1\le j\le r_i-1 \\ \ell_1,\ldots,\ell_{r_i-1}\ge 0,\ \ell_{r_i}\ge 1}} \binom{k_{u_i+r_i}-1}{\ell_{r_i}}\prod_{j=1}^{r_i-1}\binom{k_{u_i+j}}{\ell_j}.$$

Finally, notice that, if β and β' are two elements of $\mathscr{O}_m^+$ with kernel type $k=(k_1,\ldots,k_t)$ such that $s_{\beta'}=s_\beta$, then $\mathfrak{d}(\beta,i)=\mathfrak{d}(\beta',i)$, for all $1\le i\le t$. Thus, defining $\Lambda(k,s)=\prod_{i=1}^{t}\mathfrak{d}(\beta,i)$, where β is any transformation of $\mathscr{O}_m^+$ with kernel type k and $s_\beta=s$, we have:

Theorem 3.3. $|\mathscr{O}_{m\times n}^+|=|\mathscr{O}_{m\times n}^-|=\displaystyle\sum_{\substack{k=(k_1,\ldots,k_t) \\ 1\le k_1,\ldots,k_t\le m \\ k_1+\cdots+k_t=m \\ 1\le t\le m}}\ \sum_{s\in\{0,1\}^{t-1}\times\{1\}}\Delta(k,s)\Lambda(k,s).$

The next table gives the size of the monoid $\mathscr{O}_{m\times n}^+$ (or $\mathscr{O}_{m\times n}^-$) for several values of m and n.

$m \setminus n$	1	2	3	4	5	6
1	1	2	5	14	42	132
2	2	8	35	306	2401	21232
3	5	42	569	10024	210765	5089370
4	14	252	8482	410994	25366480	1847511492
5	42	1636	138348	18795636	3547275837	839181666224
6	132	11188	2388624	913768388	531098927994	415847258403464

Despite the unpleasant appearance, the previous formula allows us to calculate the cardinal of $\mathscr{O}_{m\times n}^+$, even for larger m and n. For instance, we have

$$|\mathscr{O}_{10\times 10}^+|=4701675895106986289638897622139264555060675224 4.$$

All these calculations were performed by using GAP [17].

In order to count the number of elements of the monoid $\mathscr{OP}_{m\times n}$, we begin by recalling that, for $k\in\mathbb{N}$, being g_k the k-cycle $\begin{pmatrix}1&2&\cdots&k-1&k\\2&3&\cdots&k&1\end{pmatrix}\in\mathscr{OP}_k$, each element $\alpha\in\mathscr{OP}_k$ admits a factorization $\alpha=g_k^j\gamma$, with $0\le j\le k-1$ and $\gamma\in\mathscr{O}_k$, which is unique unless α is constant [6].

Next, consider the permutations (of $\{1,\dots,mn\}$)

$$g = g_{mn} = \begin{pmatrix} 1 & 2 & \cdots & mn-1 & mn \\ 2 & 3 & \cdots & mn & 1 \end{pmatrix} \in \mathscr{OP}_{mn}$$

and

$$f = g^n = \left(\begin{array}{ccc|ccc|ccc} 1 & \cdots & n & n+1 & \cdots & mn-n & mn-n+1 & \cdots & mn \\ n+1 & \cdots & 2n & 2n+1 & \cdots & mn & 1 & \cdots & n \end{array}\right) \in \mathscr{OP}_{m\times n}.$$

Being α an element of $\mathscr{OP}_{m\times n} \setminus \mathscr{O}_{m\times n}$ of type 1 or 2 (see Proposition 2.3) and $j \in \{1,\dots,m-1\}$ such that $(jn)\alpha > (jn+1)\alpha$, as $(jn+1)\alpha \le \cdots \le (mn)\alpha \le 1\alpha \le \cdots \le (jn)\alpha$, it is clear that $f^j\alpha \in \mathscr{O}_{m\times n}$. Thus, each element α of $\mathscr{OP}_{m\times n}$ of type 1 or 2 admits a factorization $\alpha = f^j\gamma$, with $0 \le j \le m-1$ and $\gamma \in \mathscr{O}_{m\times n}$, which is unique unless α is constant. Notice that, this uniqueness follows immediately from Catarino and Higgins's result mentioned above. Therefore we have precisely $m(|\mathscr{O}_{m\times n}| - mn)$ non-constant transformations of $\mathscr{OP}_{m\times n}$ of types 1 and 2 and mn constant transformations (which are elements of type 2 of $\mathscr{OP}_{m\times n}$).

Now, let α be a transformation of $\mathscr{OP}_{m\times n}$ of type 3. As α is not constant, it can be factorized in a unique way as $g^r\gamma$, for some $r \in \{0,\dots,mn-1\} \setminus \{jn \mid 0 \le j \le m-1\}$ and some non-constant order-preserving transformation γ from $\{1,\dots,mn\}$ to A_i, for some $1 \le i \le m$. Since only elements of $\mathscr{OP}_{m\times n}$ of type 3 have factorizations of this form and the number of non-constant and non-decreasing sequences of length mn from a chain with n elements is equal to $\binom{mn+n-1}{n-1} - n$, we have precisely $m(mn-m)\left(\binom{mn+n-1}{n-1} - n\right)$ elements of type 3 in $\mathscr{OP}_{m\times n}$. Thus $|\mathscr{OP}_{m\times n}| = m|\mathscr{O}_{m\times n}| + m^2(n-1)\binom{mn+n-1}{n-1} - mn(mn-1)$ and so we obtain:

Theorem 3.4. $$|\mathscr{OP}_{m\times n}| = m \sum_{\substack{1\le k_1,\dots,k_t\le m \\ k_1+\cdots+k_t=m \\ 1\le t\le m}} \binom{m}{t} \prod_{i=1}^{t} \binom{k_i n+n-1}{n-1} + m^2(n-1)\binom{mn+n-1}{n-1} - mn(mn-1).$$

It follows a table with the sizes of the monoids $\mathscr{OP}_{m\times n}$ for some values of m and n. Again, these calculations were performed by using GAP [17].

$m \setminus n$	1	2	3	4	5	6
1	1	4	24	128	610	2742
2	4	46	506	5034	51682	575268
3	24	447	9453	248823	8445606	349109532
4	128	4324	223852	17184076	1819339324	247307947608
5	610	42075	5555990	1207660095	387720453255	170017607919290
6	2742	405828	136530144	83547682248	81341248206546	114804703283314542

We finish this paper computing the cardinal of the monoid $\mathscr{OR}_{m\times n}$. Notice that, as for $\mathscr{OD}_{m\times n}$ and $\mathscr{O}_{m\times n}$, we have a similar relationship between $\mathscr{OR}_{m\times n}$ and $\mathscr{OP}_{m\times n}$. In fact, $\alpha \in \mathscr{OR}_{m\times n}$ if and only if $\alpha \in \mathscr{OP}_{m\times n}$ or $h\alpha \in \mathscr{OP}_{m\times n}$. Hence, since $|\mathscr{OP}_{m\times n}| = |h\mathscr{OP}_{m\times n}|$ and $\mathscr{OP}_{m\times n} \cap h\mathscr{OP}_{m\times n} = \{\alpha \in \mathscr{OP}_{m\times n} \mid |\mathrm{Im}(\alpha)| \le 2\}$, we obtain $|\mathscr{OR}_{m\times n}| = 2|\mathscr{OP}_{m\times n}| - |\{\alpha \in \mathscr{OP}_{m\times n} \mid |\mathrm{Im}(\alpha)| = 2\}| - mn$.

It remains to calculate the number of elements of $A = \{\alpha \in \mathscr{OP}_{m\times n} \mid |\mathrm{Im}(\alpha)| = 2\}$. First, we count the number of elements of A of types 2 and 3. Let α be such a transformation. Then, there exists $k \in \{1,\dots,m\}$ such that $|\mathrm{Im}(\alpha)| \subseteq A_k$. Clearly, in this case, the number of distinct kernels allowed for α coincides with the number of distinct kernels allowed for transformations of $\mathscr{OP}_{mn}$ of rank 2, which is $\binom{mn}{2}$ (see [6]). On the other hand, it is easy to check that we have $m\binom{n}{2}$ distinct images for α. Furthermore, for each such possible kernel and image, we have two distinct transformations of A. Hence, the total number of elements of A of types 2 and 3 is precisely $2m\binom{n}{2}\binom{mn}{2}$.

Finally, we determine the number of elements of A of type 1. Let $\alpha \in A$ be of type 1 and suppose that $\alpha\psi = (\alpha_1,\ldots,\alpha_m;\beta)$. Then β must have rank 2 and so, as $\beta \in \mathscr{OP}_m$, we have $2\binom{m}{2}^2$ distinct possibilities for β (see [6]). Moreover, for each $1 \le i \le m$, α_i must be a constant transformation of $\mathscr{O}_n$ and, for $1 \le i,j \le m$, if $i\beta = j\beta$ then $\alpha_i = \alpha_j$. Thus, for a fixed β, since β as rank 2, we have precisely n^2 sequences $(\alpha_1,\ldots,\alpha_m;\beta)$ allowed. Hence, A has $2n^2\binom{m}{2}^2$ distinct elements of type 1. Therefore,

$$\begin{aligned}|\mathscr{OR}_{m\times n}| &= 2|\mathscr{OP}_{m\times n}| - 2m\binom{n}{2}\binom{mn}{2} - 2n^2\binom{m}{2}^2 - mn\\ &= 2m|\mathscr{O}_{m\times n}| + 2m^2(n-1)\binom{mn+n-1}{n-1} - 2m\binom{n}{2}\binom{mn}{2} - 2n^2\binom{m}{2}^2 - mn(2mn-1)\end{aligned}$$

and so we get:

Theorem 3.5.

$$\begin{aligned}|\mathscr{OR}_{m\times n}| = 2m \sum_{\substack{1\le k_1,\ldots,k_t\le m\\ k_1+\cdots+k_t=m\\ 1\le t\le m}} \binom{m}{t}\prod_{i=1}^{t}\binom{k_i n+n-1}{n-1} + 2m^2(n-1)\binom{mn+n-1}{n-1}\\ - 2m\binom{n}{2}\binom{mn}{2} - 2n^2\binom{m}{2}^2 - mn(2mn-1).\end{aligned}$$

Acknowledgement. The first author gratefully acknowledges support of FCT and PIDDAC, within the projects ISFL-1-143 and PTDC/MAT/69514/2006 of CAUL. The second author gratefully acknowledges support of ISEL and of FCT and PIDDAC, within the projects ISFL-1-143 and PTDC/MAT/69514/2006 of CAUL.

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