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A Hopf Bifurcation in a Radially Symmetric Interfacial Problem with Global Coupling

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Abstract. We consider an interfacial problem arising in reaction-diffusion models in an inhomogeneous media with global coupling. The purpose of this paper is to analyze the occurrence of Hopf bifurcation in the interfacial problem as the bifurcation parameters vary and to examine the effects of an inhomogeneous media and with the global coupling intensity in two- and three- dimensional system. Conditions for existence of stationary solutions and Hopf bifurcation for a certain class of inhomogeneity and global coupling are obtained analytically in two- and three- dimensional system with radial symmetry.

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1. Introduction

A typical reaction-diffusion system with additive space-dependent term is given by [1, 13, 21, 22]:

$$\left\{ \begin{array}{l} \sigma\varepsilon w_t = \varepsilon^2 \nabla^2 w + F(w,v) + \kappa(\mathbf{x}) \\ v_t = \nabla^2 v + G(w,v), \quad t>0, \quad \mathbf{x} \in \mathbb{R}^n, \end{array} \right. \tag{1.1}$$

where ε and σ are positive constant parameters and ∇ is a gradient operator. Here w and v measure the levels of two diffusing quantities and the reaction terms are piecewise linear functions [14],

$$F(w,v) = H(w-a) - w - v \quad \text{and} \quad G(w,v) = \mu w - v, \tag{1.2}$$

where $H(z)$ is a Heaviside step function satisfying $H(z) = 1$ for $z \geq 0$ and $H(z) = 0$ for $z < 0$ and a constant μ satisfies the bistable condition. The term of bistable means that the nullclines of F and G posses three intersection points, which determine all of the interactions between w and v. The term bistable refers to the fact that these points of intersection correspond to equilibria of the system (1.1), two of which are stable, the third unstable.

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The function a is an instantaneous inhibitory feedback in the form of global coupling as introduced by Krischer and Mikhailov [10]:

$$\frac{1}{\alpha}\Big(\frac{da}{dt}+\frac{a-a_0}{\sigma\varepsilon}\Big)=\int_{\mathbb{R}^n}\Big(\frac{1}{\sigma\varepsilon}H(w-a)+\mu w-v\Big)d\mathbf{x}-\frac{1}{\sigma\varepsilon}s_0,\quad 0<a_0<\frac{1}{2}. \tag{1.3}$$

This equation can be written in the equivalent form

$$a=a_0+\alpha\Big(\int_{\mathbb{R}^n}(w+v-\kappa(\mathbf{x}))d\mathbf{x}-s_0\Big),\quad 0<a_0<\frac{1}{2}. \tag{1.4}$$

The excitation threshold a depends on the total activator concentration in the medium. The positive constant s_0 is the value of the integral $\int_{\mathbb{R}^n}(w+v-\kappa(\mathbf{x}))d\mathbf{x}$ for the spatial distribution corresponding to a stationary solution; the coefficient α characterizes the intensity of global coupling. For a sufficiently large value of α in (1.4), Krischer and Mikhailov [10] proved that the system (1.1) for the homogeneous case ($\kappa(\mathbf{x})=0$) has a supercritical bifurcation from a motionless pulse to a propagating pulse, Kawaguchi and Mimura [11] examined the collision between two traveling waves and Ohta [18] investigated the reflection of interacting pulses in a reaction-diffusion system using singular perturbation methods to extend in the interface dynamics.

For the homogeneous media without global coupling, the singular limit analysis $\varepsilon\to 0$ is applied to show the existence and the stability of localized radially symmetric equilibrium solutions [17, 19]. In one-dimensional space, such equilibrium solutions should undergo certain instabilities and the loss of stability resulting from a Hopf bifurcation produces a kind of periodic oscillation in the location of the internal layers [2, 3, 16, 17]. Moreover, for $\varepsilon=0$, a free boundary problem of (1.1) has been obtained and a Hopf bifurcation of this problem has been examined in [4, 6, 12], as a parameter σ varies. The $\kappa(\mathbf{x})$ can be regarded as an external forcing. Its role is very similar to the case of without inhomogeneity, since it also causes the nonlinear w-nullcline to move up and down in the (w,v) phase plane.

A variety of wave phenomena including front oscillation, front pinning, and front reflection are shown to occur in inhomogeneous media [21]. The effect of an inhomogeneous diffusion process has been studied for one component system in [20, 23] and a two component system close to a singular limit in [7, 8]. For the inhomogeneous media without global coupling, a Hopf bifurcation in one-dimensional space has been examined in [5] as a parameter σ varies. However, the free boundary problem for this case in two- and three-dimensional space has not been studied. This gap will be filled in this paper.

We first deduce an equation of interface. Suppose that there is only one $(n-1)$ - dimensional hypersurface $\eta(t)$ which is simply single closed curve given in $\mathbb{R}^n\times(0,\infty)$ in such a way that $\mathbb{R}^n\times(0,\infty)=\Omega_1(t)\cup\eta(t)\cup\Omega_0(t)$, where $\Omega_1(t)=\{(\mathbf{x},t)\in\mathbb{R}^n\times(0,\infty)\quad : w(\mathbf{x},t)>a\}$ and $\Omega_0(t)=\{(\mathbf{x},t)\in\mathbb{R}^n\times(0,\infty)\quad : w(\mathbf{x},t)<a\}$. The interface equation of $\eta(t)$ can be represented from the analysis of [11, 15, 19]:

$$\frac{d\eta(t)}{dt}\cdot\nu=C(v_i;a(t)),\quad \mathbf{x}\in\eta(t),$$

where ν is the outward normal vector on $\eta(t)$, v_i is the value of v on the interface $\eta(t)$ and $C(\cdot)$ is the velocity of the interface. The nullcline $F(w,v)+\kappa=0$ is the triple valued function of w which are called $h^+=1-v+\kappa$, $h^-=-v+\kappa$ and $h^0=a$. From [3, 9, 15], the velocity of the interface C is given by $-(h^+-2h^0+h^-)$ and is a continuously differentiable function defined on an interval $I:=(-a,1-a)$. Thus the velocity of the interface can be well-defined on $(-a, 1-a)$ with

$$C(v,\eta,;a) = -\frac{1}{\sigma}\frac{1-2a-2(v(\eta)-\kappa(\eta))}{\sqrt{(v(\eta)-\kappa(\eta)+a)(1-a-v(\eta)+\kappa(\eta))}}. \tag{1.5}$$

A free boundary problem of (1.1)–(1.2) can be obtained:

$$\begin{cases} v_t = \nabla^2 v - (\mu+1)v + \mu + \mu\,\kappa(\eta), & (\mathbf{x},t) \in \Omega_1(t) \\ v_t = \nabla^2 v - (\mu+1)v + \mu\,\kappa(\eta), & (\mathbf{x},t) \in \Omega_0(t) \\ v(\mathbf{x},0) = v_0(\mathbf{x}) \\ v(\eta(t)-0,t) = v(\eta(t)+0,t) \\ \frac{d}{dv}v(\eta(t)-0,t) = \frac{d}{dv}v(\eta(t)+0,t) \\ \lim_{|\mathbf{x}|\to\infty} v(|\mathbf{x}|,t) = 0,\, t > 0. \end{cases} \tag{1.6}$$

Our aim is to explore the dynamics of interfaces in the problem (1.6) with global coupling (1.4) in order to investigate the existence of time periodic solutions as global coupling intensity and the bifurcation parameter σ vary. Also, we shall investigate the effect of an inhomogeneous media in two and three dimensions. In Section 2, a change of variables is given which regularizes problem (1.6) in such a way that results from the theory of nonlinear evolution equations can be applied. In this way, we obtain enough regularity of the solution for an analysis of the bifurcation. In Section 3, we show the existence of radially symmetric localized equilibrium solutions for (1.6) and obtain the linearization of problem (1.6). In the last section we investigate the conditions of the inhomogeneous media to obtain the periodic solutions and the bifurcation of the interface problem as global coupling intensity and the parameter σ vary in two and three dimensions.

2. Regularization of the interface equation

We look for an existence problem of radially symmetric equilibrium solutions of (1.6) with $|\mathbf{x}| = r$ where the center and the interface are located at the origin and $r = \eta$, respectively. The problem is given by:

$$\begin{cases} v_t = \dfrac{\partial^2 v}{\partial r^2} + \dfrac{n-1}{r}\dfrac{\partial v}{\partial r} - (\mu+1)v + \mu H(\eta - r) + \mu\,\kappa(\eta), & r \in (0,\infty), \quad t > 0 \\ v(r,0) = v_0(r) \\ \dfrac{\partial v}{\partial r}v(0,t) = 0 = v(\infty,t), \quad t > 0 \\ \eta'(t)\cdot \mathrm{v} = C(\chi(v,\eta);a(t)), \quad t > 0, \end{cases} \tag{2.1}$$

where $\chi(v,\eta) = v(\eta) - \kappa(\eta)$. The global coupling of (1.4) is given by

$$a(t) = a_0 + \alpha(|\Omega_1(t)| - s_0), \tag{2.2}$$

where $|\Omega_1(t)|$ is a measure of $\Omega_1(t)$, that is $|\Omega_1(t)| = \pi\eta^2(t)$ for $n = 2$ and $|\Omega_1(t)| = (4/3)\pi\eta^3(t)$ for $n = 3$.

Let A be a differential operator

$$A := -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} + \mu + 1$$

with the domain $D(A) = \{v \in H^{2,2}(\mathbb{R}) : (\partial v/\partial r)v(0,t) = 0, \lim_{r\to\infty} v(r,t) = 0\}$. For the application of semigroup theory to (2.1), we choose the space $X := L_2(\mathbb{R})$ with norm $\|\cdot\|_2$.

We define $g : \mathbb{R}^2 \to \mathbb{R}$,

$$g(r,\eta) := A^{-1}(\mu H(\eta - \cdot)(r) + \mu\kappa(\eta)) = \int_0^\infty G(r,y)(\mu H(\eta - y) + \mu\kappa(\eta))\,dy,$$

where $G : \mathbb{R}^2 \to \mathbb{R}$ is a Green's function of A satisfying the boundary conditions:

$$G(r,z) = \begin{cases} zK_0(z\sqrt{1+\mu})\,I_0(r\sqrt{1+\mu}), & 0 < r < z \\ zI_0(z\sqrt{1+\mu})\,K_0(r\sqrt{1+\mu}), & z < r \qquad (n = 2), \end{cases}$$

where I_0 and K_0 are modified Bessel functions and

$$G(r,z) = \begin{cases} z e^{-z\sqrt{1+\mu}} \quad \frac{\sinh(r\sqrt{1+\mu})}{r\sqrt{1+\mu}}, & 0 < r < z \\ z\sinh(z\sqrt{1+\mu}) \quad \frac{e^{-r\sqrt{1+\mu}}}{r\sqrt{1+\mu}}, & z < r \qquad (n = 3). \end{cases}$$

Applying the transformation $u(r,t) = v(r,t) - g(r,\eta(t))$ we obtain an equivalent abstract evolution equation of (2.1) :

$$\begin{cases} \frac{d}{dt}(u,\eta) + \widetilde{A}(u,\eta) = f(u,\eta) \\ (u,\eta)(0) = (u_0(r),\eta_0), \end{cases} \tag{2.3}$$

where $\widetilde{A}$ is a 2×2 matrix whose (1,1)-entry is an operator A and all the others are zero. The nonlinear forcing term f is

$$f(u,\eta) = \begin{pmatrix} \left(\mu G(r,\eta) - \frac{\mu}{\mu+1}\kappa'(\eta)\right) C(\chi(u(\eta) + \gamma(\eta),\eta);a) \\ C(\chi(u(\eta) + \gamma(\eta),\eta);a) \end{pmatrix},$$

where the function $\gamma : \mathbb{R} \to \mathbb{R}$ is defined by $\gamma(\eta) := g(\eta,\eta)$.

The well posedness of solutions of (2.3) are shown in [6] with the help of the semigroup theory using domains of fractional powers $\theta \in (3/4,1]$ of A and $\widetilde{A}$. Moreover, the nonlinear term f is a continuously differentiable function from $W \cap \widetilde{X}^\theta$ to $\widetilde{X}$ where

$$W := \{(u,\eta) \in C^1(\mathbb{R}) \times \mathbb{R} : u(\eta) + \gamma(\eta) \in I\} \subset_{\text{open}} C^1(\mathbb{R}) \times \mathbb{R},$$
$$\widetilde{X} := D(\widetilde{A}) = D(A) \times \mathbb{R},\ X^\theta := D(A^\theta) \quad \text{and} \quad \widetilde{X}^\theta := D(\widetilde{A}^\theta) = X^\theta \times \mathbb{R}.$$

3. Radially symmetric equilibrium solutions and linearization of the interface equation

In this section, we shall examine the existence of radially symmetric equilibrium solutions of (2.3) in $\mathbb{R}^n$ $(n = 2,3)$. We look for $(u^*,\eta^*) \in D(\widetilde{A}) \cap W$ satisfying the following problem:

(3.1)
$$\begin{cases} Au = \left(\mu G(r,\eta) - \frac{\mu}{\mu+1}\kappa'(\eta)\right) C(\chi(u(\eta)+\gamma(\eta),\eta);a) \\ 0 = C(\chi(u(\eta)+\gamma(\eta),\eta);a) \\ u'(0) = 0 = u(\infty), \end{cases}$$

where $\chi(u(\eta)+\gamma(\eta),\eta) = u(\eta)+\gamma(\eta)-\kappa(\eta)$.

Theorem 3.1. *In case of without or with global coupling ($0 \le \alpha < \infty$): Suppose that*

$$\kappa'(\eta) < 0, \quad \frac{\mu}{\mu+1} > \frac{1}{\mu+1}\kappa(0) + \frac{1}{2} - a_0 + \alpha s_0 \quad \textit{and} \quad \gamma'(\eta) - \kappa'(\eta) + \alpha b \pi \eta < 0,$$

where $b = 2$ for $n = 2$ and $b = 4\eta$ for $n = 3$. Then equation (2.3) has at least one radially symmetric equilibrium solution $(0,\eta^*)$ for all $\sigma \neq 0$.

In case of strong global coupling ($\alpha = \infty$): Equation (2.3) has only one radially symmetric equilibrium solution $(0,\eta^*)$ for all $\sigma \neq 0$ with $\eta^* = \sqrt{s_0/\pi}$ $(n = 2)$ and $\eta^* = (3/(4\pi))s_0^{1/3}$ $(n = 3)$.

The linearization of f at the stationary solution $(0,\eta^*)$ is

$$Df(0,\eta^*)(\hat{u},\hat{\eta}) = \begin{pmatrix} \frac{4}{\sigma}\left(\hat{u}(\eta^*) + \gamma'(\eta^*)\hat{\eta} - \kappa'(\eta)\hat{\eta} + \alpha b \pi \eta^* \hat{\eta}\right)\left(\mu G(\cdot,\eta^*) - \frac{\mu}{\mu+1}\kappa'(\eta)\right) \\ \frac{4}{\sigma}\left(\hat{u}(\eta^*) + \gamma'(\eta^*)\hat{\eta} - \kappa'(\eta)\hat{\eta} + \alpha b \pi \eta^* \hat{\eta}\right) \end{pmatrix}.$$

The pair $(0,\eta^*)$ corresponds to a unique steady state (v^*,η^*) of (1.6) for $\sigma \neq 0$ with $v^*(r) = g(r,\eta^*)$.

Proof. System (3.1) is equivalent to the pair of equations:

(3.2)
$$\begin{cases} 0 = -u''(r) - \frac{n-1}{r}u'(r) + (\mu+1)u \qquad \text{with} \quad u'(0) = 0 = u(\infty) \\ 0 = \frac{1}{2} - a_0 - \alpha\,(|\Omega_1| - s_0) - u(\eta) - \gamma(\eta) + \kappa(\eta). \end{cases}$$

For $n = 2$, the general solution of (3.2) is given by

$$u(r) = c_1 I_0(r\sqrt{1+\mu}) + c_2 K_0(r\sqrt{\mu+1})$$

for some constants c_1 and c_2. The boundary condition $u(\infty) = 0$ implies that $c_1 = 0$ since $\lim_{r\to\infty} K_0(r\sqrt{\mu+1}) = 0$. Moreover, $u'(0) = 0$ implies that $c_2 = 0$. Hence we have $u^* = 0$. For $n = 3$, the general solution of (3.2) is given by

$$u(r) = c_1 \frac{e^{-r\sqrt{\mu+1}}}{r} + c_2 \frac{e^{r\sqrt{\mu+1}}}{2r\sqrt{\mu+1}}$$

for some constants c_1 and c_2. Apply the boundary conditions, we obtain $u^* = 0$. In order to show existence of η^* we define

$$\Gamma(\eta) := \gamma(\eta) - \kappa(\eta) - (\tfrac{1}{2} - a_0 - \alpha\,(|\Omega_1| - s_0)).$$

Suppose $0 \le \alpha < \infty$. Let η_c be a solution of $\gamma(\eta) - \kappa(\eta) = 1/2 - a_0$ and η_d be a solution of $h(\eta) := \alpha(|\Omega_1| - s_0) = 0$. Thus $\eta_d = \sqrt{s_0/\pi}$ for $n = 2$ and $\eta_d = (3/(4\pi))s_0^{1/3}$ for $n = 3$. Suppose that $\Gamma'(\eta) < 0$ and $\eta_c < \eta_d$. Then $\Gamma(\eta_d) < \Gamma(\eta_c) = h(\eta_c) < h(\eta_d) = 0$. Hence if $\Gamma(0) > 0$, then there exists a unique $\eta^* \in (0,\eta_d)$.

For the case of strong global coupling, equation (2.2) is equivalent to $\alpha = (a(t) - a_0)/(|\Omega_1(t)| - s_0)$. As $\alpha \uparrow \infty$, we have $|\Omega_1(t)| \to s_0$ and thus $\eta^* = \sqrt{s_0/\pi}$ for $n = 2$

and $\eta^* = (3/(4\pi))s_0^{1/3}$ for $n = 3$. The formula for $Df(0,\eta^*)$ follows from the relation $C'(1/2 - a_0 + \alpha s_0) = 4/\sigma$ and the corresponding steady state (v^*, η^*) for (1.6) is obtained using the transformation and Theorem 2.1 in [6]. ∎

4. A Hopf bifurcation

We shall show that there exists a Hopf bifurcation as a parameter σ varies. Now, let us introduce the following definition.

Definition 4.1. *Under the assumptions of Theorem 3.1, define (for $1 \geq \theta > 3/4$) the linear operator B from $\widetilde{X}^\theta$ to $\widetilde{X}$ by*

$$B := \frac{\sigma}{4} Df(0,\eta^*)\,.$$

We then define $(0,\eta^*)$ to be a Hopf point *for (2.3) if and only if there exists an $\varepsilon_0 > 0$ and a C^1-curve*

$$(-\varepsilon_0 + \tau^*, \tau^* + \varepsilon_0) \mapsto (\lambda(\tau), \phi(\tau)) \in \mathbb{C} \times \widetilde{X}_{\mathbf{C}}$$

($Y_{\mathbf{C}}$ denotes the complexification of the real space Y) of eigendata for $-\widetilde{A} + \tau B$ with

(i) $(-\widetilde{A} + \tau B)(\phi(\tau)) = \lambda(\tau)\phi(\tau), \quad (-\widetilde{A} + \tau B)(\overline{\phi(\tau)}) = \overline{\lambda(\tau)}\,\overline{\phi(\tau)}$*;*

(ii) $\lambda(\tau^*) = i\beta$ *with* $\beta > 0$*;*

(iii) $\mathrm{Re}\,(\lambda) \neq 0$ *for all* $\lambda \in \sigma(-\widetilde{A} + \tau^* B) \setminus \{\pm i\beta\}$*;*

(iv) $\mathrm{Re}\,\lambda'(\tau^*) \neq 0$ *(transversality);*

where $\tau = 4/\sigma$.

We next have to check (2.3) for Hopf points. For this we have to solve the eigenvalue problem:

$$-\widetilde{A}(u,\eta) + \tau B(u,\eta) = \lambda(u,\eta)$$

which is equivalent to

(4.1)
$$\begin{cases} (A+\lambda)u &= \tau\left(u(\eta^*) + \gamma'(\eta^*)\,\eta - \kappa'(\eta)\eta + \alpha\, b\, \pi\, \eta^* \eta\right)\left(\mu G(\cdot,\eta^*) - \frac{\mu}{\mu+1}\kappa'(\eta)\right) \\ \quad\lambda \quad \eta &= \tau(u(\eta^*) + \gamma'(\eta^*)\,\eta - \kappa'(\eta)\eta + \alpha\, b\, \pi\, \eta^* \eta). \end{cases}$$

We shall show that radially symmetric equilibrium solution become a Hopf point.

Theorem 4.1. *Assume*

$$0 \leq \alpha < \infty, \quad \frac{1}{\mu+1}\kappa(0) + \frac{1}{2} - a_0 + \alpha s_0 < \frac{\mu}{\mu+1} \quad \text{and} \quad \gamma'(\eta^*) - \kappa'(\eta^*) + \alpha\, b\, \pi \eta^* < 0.$$

Moreover, assume that $\kappa'(\eta^*) < 0$. Suppose the operator $-\widetilde{A} + \tau^* B$ has a unique pair $\{\pm i\beta\}$ of purely imaginary eigenvalues for some $\tau^* > 0$. Then $(0,\eta^*,\tau^*)$ is a Hopf point for (2.3).

Proof. We assume without loss of generality that $\beta > 0$, and ϕ^* is the (normalized) eigenfunction of $-\widetilde{A} + \tau^* B$ with eigenvalue $i\beta$. We have to show that $(\phi^*, i\beta)$ can be extended to a C^1-curve $\tau \mapsto (\phi(\tau), \lambda(\tau))$ of eigendata for $-\widetilde{A} + \tau B$ with $\mathrm{Re}(\lambda'(\tau^*)) \neq 0$.

For this let $\phi^* = (\psi_0, \eta_0) \in D(A) \times \mathbb{R}$. First, we see that $\eta_0 \neq 0$, for otherwise, by (4.1),

$$(A + i\beta)\psi_0 = \mu\, i\beta\, \eta_0 \left(G(\cdot,\eta^*) - \frac{1}{\mu+1}\kappa'(\eta^*)\right) = 0,$$

which is not possible because A is symmetric. So without loss of generality, let $\eta_0 = 1$. Then $E(\psi_0, i\beta, \tau^*) = 0$ by (4.1), where

$$E : D(A)_{\mathbf{C}} \times \mathbb{C} \times \mathbb{R} \longrightarrow X_{\mathbf{C}} \times \mathbb{C},$$

$$E(u,\lambda,\tau) := \begin{pmatrix} (A+\lambda)u - \tau\big(u(\eta^*) + \gamma'(\eta^*) - \kappa'(\eta^*) + \alpha b \pi \eta^*\big)\big(\mu G(\cdot,\eta^*) - \frac{\mu}{\mu+1}\kappa'(\eta^*)\big) \\ \lambda - \tau\big(u(\eta^*) + \gamma'(\eta^*) - \kappa'(\eta^*) + \alpha b \pi \eta^*\big) \end{pmatrix}.$$

The equation $E(u,\lambda,\tau) = 0$ is equivalent to λ being an eigenvalue of $-\widetilde{A} + \tau B$ with eigenfunction $(u,1)$. We shall here apply the implicit function theorem to E. For this it is necessary to E is of C^1-class and that

$$D_{(u,\lambda)}E(\psi_0, i\beta, \tau^*) \in L(D(A)_{\mathbf{C}} \times \mathbb{C}, X_{\mathbf{C}} \times \mathbb{C}) \text{ is an isomorphism.} \tag{4.2}$$

It is easy to see that E is of C^1-class. In addition, the mapping

$$D_{(u,\lambda)}E(\psi_0, i\beta, \tau^*)(\hat{u}, \hat{\lambda}) = \begin{pmatrix} (A+i\beta)\hat{u} - \tau^*\hat{u}(\eta^*)\big(\mu G(\cdot,\eta^*) - \frac{\mu}{\mu+1}\kappa'(\eta^*)\big) + \hat{\lambda}\,\psi_0 \\ \hat{\lambda} - \tau^*\,\hat{u}(\eta^*) \end{pmatrix}$$

is a compact perturbation of the mapping

$$(\hat{u}, \hat{\lambda}) \longmapsto \big((A+i\beta)\hat{u},\, \hat{\lambda}\big)$$

which is invertible. Thus $D_{(u,\lambda)}E(\psi_0, i\beta, \tau^*)$ is a Fredholm operator of index 0. Therefore in order to verify (4.2), it suffices to show that the system

$$D_{(u,\lambda)}E(\psi_0, i\beta, \tau^*)(\hat{u}, \hat{\lambda}) = 0$$

which is equivalent to

$$\begin{cases} (A+i\beta)\hat{u} + \hat{\lambda}\,\psi_0 = \tau^*\,\hat{u}(\eta^*)\,\big(\mu G(\cdot,\eta^*) - \frac{\mu}{\mu+1}\kappa'(\eta^*)\big) \\ \hat{\lambda} = \tau^*\,\hat{u}(\eta^*) \end{cases} \tag{4.3}$$

necessarily implies that $\hat{u} = 0$ and $\hat{\lambda} = 0$. We define $\psi_1 := \psi_0 - \mu\, G(\cdot,\eta^*) + \frac{\mu}{\mu+1}\kappa'(\eta^*)$ then the first equation of (4.3) is given by

$$(A+i\beta)\hat{u} + \hat{\lambda}\,\psi_1 = 0. \tag{4.4}$$

On the other hand, since $E(\psi_0, i\beta, \tau^*) = 0$, we have

$$(A+i\beta)\psi_0 = i\beta\,\Big(\mu G(\cdot,\eta^*) - \frac{\mu}{\mu+1}\kappa'(\eta^*)\Big)$$

and so, ψ_1 is a solution to the equation

$$(A+i\beta)\psi_1 = -\mu\,\delta_{\eta^*} + \mu\,\kappa'(\eta^*) \tag{4.5}$$

and

$$i\beta = \tau^*\Big(\psi_1(\eta^*) + \mu G(\eta^*,\eta^*) + \gamma'(\eta^*) - \frac{\mu}{\mu+1}\kappa'(\eta^*) - \kappa'(\eta^*) + \alpha b \pi \eta^*\Big), \tag{4.6}$$

where δ_{η^*} is a Dirac delta function at η^*, that is $\delta_{\eta^*} = \delta(r-\eta^*)$. Multiplying (4.4) by $G(\cdot,\eta^*)\psi_1(\cdot)$ and (4.5) by $G(\cdot,\eta^*)\hat{u}(\cdot)$ and integrating, and then subtracting the resultants from each other,

$$0 = \hat{\lambda}\int G(r,\eta^*)\psi_1^2(r)dr - \mu G(\eta^*,\eta^*)\hat{u}(\eta^*) + \mu\kappa'(\eta^*)\int G(r,\eta^*)\hat{u}(r)dr. \tag{4.7}$$

Multiplying (4.5) by $G(\cdot,\eta^*)\psi_1(\cdot)$ and integrating, we obtain

$$\begin{aligned}&\psi_1^2(\eta^*) + i\beta\int \psi_1^2(r)G(r,\eta^*)dr \\ &\quad = -\mu G(\eta^*,\eta^*)\psi_1(\eta^*) + \mu\kappa'(\eta^*)\int G(r,\eta^*)\psi_1(r)dr.\end{aligned} \tag{4.8}$$

From (4.4), we have

$$\hat{u}(\eta^*) + i\beta\int G(r,\eta^*)\hat{u}(r)dr + \hat{\lambda}\int G(r,\eta^*)\psi_1(r)dr = 0$$

and thus (4.7) and (4.8) become

$$0 = \hat{\lambda}\left(\psi_1^2(\eta^*) + \mu G(\eta^*,\eta^*)\psi_1(\eta^*)\right) + (i\beta\mu\, G(\eta^*,\eta^*) + \mu\kappa'(\eta^*))\hat{u}(\eta^*). \tag{4.9}$$

Apply (4.3) to (4.9), we get

$$0 = \hat{\lambda}\left(\frac{i\beta}{\tau^*}\mu G(\eta^*,\eta^*) + \frac{\mu}{\tau^*}\kappa'(\eta^*) + \psi_1^2(\eta^*) + \mu G(\eta^*,\eta^*)\psi_1(\eta^*)\right). \tag{4.10}$$

Suppose that $\hat{\lambda} \neq 0$. Then the real part and the imaginary part of (4.10) are given by

$$\begin{cases} (\gamma'(\eta^*) - \frac{\mu}{\mu+1}\kappa'(\eta^*) - \kappa'(\eta^*) + \alpha b\pi\eta^*)\cdot \\ \quad \left(\mu G(\eta^*,\eta^*) + \gamma'(\eta^*) - \frac{\mu}{\mu+1}\kappa'(\eta^*) - \kappa'(\eta^*) + \alpha b\pi\eta^*\right) \\ -\frac{\beta^2}{\tau^{*2}} + \frac{\mu}{\tau^*}\kappa'(\eta^*) = 0 \\ 2\frac{\beta}{\tau^*}\left(\mu G(\eta^*,\eta^*) + \gamma'(\eta^*) - \frac{\mu}{\mu+1}\kappa'(\eta^*) - \kappa'(\eta^*) + \alpha b\pi\eta^*\right) = 0. \end{cases}$$

We have

$$\mu G(\eta^*,\eta^*) + \gamma'(\eta^*) - \frac{\mu}{\mu+1}\kappa'(\eta^*) - \kappa'(\eta^*) + \alpha b\pi\eta^* = 0$$

and thus

$$\frac{\mu}{\tau^*}\kappa'(\eta^*) = \frac{\beta^2}{\tau^{*2}}.$$

This leads to a contradiction by the assumption $\kappa'(\eta^*) < 0$. Hence $\hat{\lambda} = 0$ and so $\hat{u} = 0$. ∎

Theorem 4.2. *Under the same condition as in Theorem 4.1, $(0,\eta^*,\tau^*)$ satisfies the transversality condition. Hence this is a Hopf point for (2.3).*

Proof. By implicit differentiation of $E(\psi_0(\tau),\lambda(\tau),\tau) = 0$,

$$\begin{aligned}&D_{(u,\lambda)}E(\psi_0, i\beta,\tau^*)(\psi_0'(\tau^*),\lambda'(\tau^*)) \\ &\quad = \begin{pmatrix} \left(\mu G(r,\eta) - \frac{\mu}{\mu+1}\kappa'(\eta^*)\right)\left(\psi_0(\eta^*) + \gamma'(\eta^*) - \kappa'(\eta^*) + \alpha b\pi\eta^*\right) \\ \psi_0(\tau) + \gamma'(\eta^*) - \kappa'(\eta^*) + \alpha b\pi\eta^* \end{pmatrix}.\end{aligned}$$

This means that the function $\tilde{u} := \psi_0'(\tau^*)$ and $\tilde{\lambda} := \lambda'(\tau^*)$ satisfy the equations

(4.11)
$$\begin{cases} (A+i\beta)\tilde{u}+\tilde{\lambda}\psi_0-\tau^*\big(\mu G(r,\eta)-\frac{\mu}{\mu+1}\kappa'(\eta^*)\big)\tilde{u}(\eta^*) \\ \qquad\qquad = \big(\mu G(r,\eta)-\frac{\mu}{\mu+1}\kappa'(\eta^*)\big(\psi_0(\tau)+\gamma'(\eta^*)-\kappa'(\eta^*)+\alpha b\pi\eta^*\big), \\ \tilde{\lambda}-\tau^*\tilde{u}(\eta^*) = \psi_0(\tau)+\gamma'(\eta^*)-\kappa'(\eta^*)+\alpha b\pi\eta^*. \end{cases}$$

By letting

$$\psi_1 := \psi_0-\mu G(r,\eta)+\frac{\mu}{\mu+1}\kappa'(\eta^*)$$

as before, we obtain

$$(A+i\beta)\tilde{u}+\tilde{\lambda}\psi_1 = 0. \tag{4.12}$$

Multiplying (4.5) by $G(\cdot,\eta^*)\tilde{u}$ and (4.12) by $G(\cdot,\eta^*)\psi_1$, and integrating, and subtracting the resultants from each other,

$$0 = \tilde{\lambda}\int G(r,\eta^*)\psi_1^2(r)dr-\mu G(\eta^*,\eta^*)\tilde{u}(\eta^*)+\mu\kappa'(\eta^*)\int G(r,\eta^*)\tilde{u}(r)dr. \tag{4.13}$$

Multiplying (4.5) by $G(\cdot,\eta^*)\psi_1(\cdot)$ and integrating, we obtain

(4.14)
$$\psi_1^2(\eta^*)+i\beta\int\psi_1^2(r)G(r,\eta^*)dr = -\mu G(\eta^*,\eta^*)\psi_1(\eta^*)+\mu\kappa'(\eta^*)\int G(r,\eta^*)\psi_1(r)dr$$

which implies that

$$0 = i\beta\mu\hat{u}(\eta^*)G(\eta^*,\eta^*)+\tilde{\lambda}(\psi_1^2(\eta^*)+\mu G(\eta^*,\eta^*)\psi_1(\eta^*))+\mu\kappa'(\eta^*)\tilde{u}(\eta^*). \tag{4.15}$$

Apply (4.11) to (4.15), we get

$$-\frac{\beta^2}{\tau^{*2}}\mu G(\eta^*,\eta^*) = \hat{\lambda}\Big(\frac{i\beta}{\tau^*}\mu G(\eta^*,\eta^*)+\frac{\mu}{\tau^*}\kappa'(\eta^*)+\psi_1^2(\eta^*)+\mu G(\eta^*,\eta^*)\psi_1(\eta^*)\Big)$$

which is given by

$$-\frac{\beta^2}{\tau^{*2}}\mu G(\eta^*,\eta^*) = \tilde{\lambda}\Big(\frac{\beta^2}{\tau^{*2}}-2\frac{i\beta}{\tau^*}\big(P-\mu G(\eta^*,\eta^*)\big)-P\mu G(\eta^*,\eta^*)\Big)$$

where

$$P = \mu G(\eta^*,\eta^*)+\gamma'(\eta^*)-\frac{\mu}{1+\mu}\kappa'(\eta^*)-\kappa'(\eta^*)+\alpha b\pi\eta^*.$$

The real part of the above equation is given by

$$\mathrm{Re}\,\tilde{\lambda}\cdot\frac{\beta^2}{\tau^{*2}}\mu G(\eta^*,\eta^*) = |\tilde{\lambda}|^2\Big(\frac{\beta^2}{\tau^{*2}}+P\mu G(\eta^*,\eta^*)\Big).$$

Since $\kappa'(\eta^*) < 0$, we have $P > 0$ and thus $\mathrm{Re}\,\tilde{\lambda} > 0$. Therefore, $\mathrm{Re}\lambda'(\tau^*) > 0$ for $\beta > 0$ and thus by the Hopf-bifurcation theorem in [6], there exists a family of periodic solutions which bifurcates from the stationary solution as τ passes τ^*. ∎

We shall show that there exists a unique $\tau^* > 0$ such that $(0,\eta^*,\tau^*)$ is a Hopf point, thus τ^* is the origin of a branch of nontrivial periodic orbits.

Theorem 4.3. *Under the same condition as in Theorem 4.1, then for a unique critical point $\tau^* > 0$, there exists a unique, purely imaginary eigenvalue $\lambda = i\beta$ of (4.1) with $\beta > 0$.*

Proof. We only need to show that the function $(u,\beta,\tau) \mapsto E(u,i\beta,\tau)$ has a unique zero with $\beta > 0$ and $\tau > 0$. This means solving the system (4.1) with $\lambda = i\beta$ and $u = v - \mu G(\cdot,\eta^*) + (\mu/(\mu+1))\kappa'(\eta^*)$,

$$(A + i\beta)v = -\mu\,\delta_{\eta^*} + \mu\,\kappa'(\eta^*) \tag{4.16}$$

and

$$\frac{i\beta}{\tau^*} = v(\eta^*) + \mu G(\eta^*,\eta^*) + \gamma'(\eta^*) - \frac{\mu}{\mu+1}\,\kappa'(\eta^*) - \kappa'(\eta^*) + \alpha\, b\,\pi\,\eta^*. \tag{4.17}$$

The real and imaginary parts of the above equation are given by

$$\begin{cases} \frac{\beta}{\tau^*} = -\mu\,\mathrm{Im}\big(G_\beta(\eta^*,\eta^*)\big) - \frac{\mu\beta}{(\mu+1)^2+\beta^2}\,\kappa'(\eta^*) \\ 0 = -\mu\,\mathrm{Re}\big(G_\beta(\eta^*,\eta^*)\big) + \mu G(\eta^*,\eta^*) + \gamma'(\eta^*) + \frac{\mu(\mu+1)}{(\mu+1)^2+\beta^2}\,\kappa'(\eta^*) \\ \qquad -(\frac{\mu}{\mu+1}+1)\kappa'(\eta^*) + \alpha\, b\,\pi\,\eta^*. \end{cases}$$

Since $\mu\,\mathrm{Im}\big(G_\beta(\eta^*,\eta^*)\big)$ is negative by [6, Lemma 12], there is a critical point τ^* provided the existence of β. We now define

$$\begin{aligned} T(\beta) := &-\mu\mathrm{Re}\big(G_\beta(\eta^*,\eta^*)\big) + \mu G(\eta^*,\eta^*) + \gamma'(\eta^*) \\ &+ \frac{\mu(\mu+1)}{(\mu+1)^2+\beta^2}\,\kappa'(\eta^*) - \big(\frac{\mu}{\mu+1}+1\big)\kappa'(\eta^*) + \alpha\, b\,\pi\,\eta^*. \end{aligned} \tag{4.18}$$

Suppose $\kappa'(\eta^*) < 0$, then $T'(\beta) > 0$ since $\mathrm{Re}G_\beta(\eta^*,\eta^*)$ is a decreasing function of $\beta > 0$ by [6, Lemma 12]. Moreover,

$$\begin{aligned} &\lim_{\beta\to\infty} T(\beta) = \mu G(\eta^*,\eta^*) + \gamma'(\eta^*) - (\tfrac{\mu}{\mu+1}+1)\,\kappa'(\eta^*) + \alpha\, b\,\pi\,\eta^* \\ &= \begin{cases} \mu\,\eta^* I_1(\eta^*\sqrt{\mu+1})\,K_1(\eta^*\sqrt{\mu+1}) - (\frac{\mu}{\mu+1}+1)\kappa'(\eta^*) + 2\alpha\,\pi\eta^* & (n=2), \\ \mu\,\eta^* I_{\frac{3}{2}}(\eta^*\sqrt{\mu+1})\,K_{\frac{3}{2}}(\eta^*\sqrt{\mu+1}) - (\frac{\mu}{\mu+1}+1)\kappa'(\eta^*) + 4\alpha\,\pi\eta^{*2} & (n=3) \end{cases} \end{aligned}$$

which are positive. Therefore, there exists a unique $\beta > 0$ if $T(0) = \gamma'(\eta^*) - \kappa'(\eta^*) + \alpha\, b\,\pi\,\eta^* < 0$. ∎

We shall show that there exists a unique $\tau^* > 0$ such that $(0,\eta^*,\tau^*)$ is a Hopf point, thus τ^* is the origin of a branch of nontrivial periodic orbits.

Theorem 4.4. *Under the same condition as in Theorem 4.1, then for a unique critical point $\tau^* > 0$, there exists a unique, purely imaginary eigenvalue $\lambda = i\beta$ of (4.1) with $\beta > 0$.*

Proof. We only need to show that the function $(u,\beta,\tau) \mapsto E(u,i\beta,\tau)$ has a unique zero with $\beta > 0$ and $\tau > 0$. This means solving the system (4.1) with $\lambda = i\beta$ and $u = v - \mu G(\cdot,\eta^*) + (\mu/(\mu+1))\kappa'(\eta^*)$,

$$(A + i\beta)v = -\mu\,\delta_{\eta^*} + \mu\,\kappa'(\eta^*) \tag{4.19}$$

and

$$\frac{i\beta}{\tau^*} = v(\eta^*) + \mu G(\eta^*,\eta^*) + \gamma'(\eta^*) - \frac{\mu}{\mu+1}\,\kappa'(\eta^*) - \kappa'(\eta^*) + \alpha\, b\,\pi\,\eta^*. \tag{4.20}$$

The real and imaginary parts of the above equation are given by

$$\begin{cases} \frac{\beta}{\tau^*} = -\mu \operatorname{Im}\big(G_\beta(\eta^*,\eta^*)\big) - \frac{\mu\beta}{(\mu+1)^2+\beta^2}\kappa'(\eta^*) \\ 0 = -\mu \operatorname{Re}\big(G_\beta(\eta^*,\eta^*)\big) + \mu G(\eta^*,\eta^*) + \gamma'(\eta^*) + \frac{\mu(\mu+1)}{(\mu+1)^2+\beta^2}\kappa'(\eta^*) \\ \quad -(\frac{\mu}{\mu+1}+1)\kappa'(\eta^*) + \alpha b \pi \eta^*. \end{cases}$$

Since $\mu \operatorname{Im}\big(G_\beta(\eta^*,\eta^*)\big)$ is negative by [6, Lemma 12], there is a critical point τ^* provided the existence of β. We now define

$$T(\beta) := -\mu\operatorname{Re}\big(G_\beta(\eta^*,\eta^*)\big) + \mu G(\eta^*,\eta^*) + \gamma'(\eta^*) + \frac{\mu(\mu+1)}{(\mu+1)^2+\beta^2}\kappa'(\eta^*) \\ -\big(\frac{\mu}{\mu+1}+1\big)\kappa'(\eta^*) + \alpha b \pi \eta^*. \tag{4.21}$$

Suppose $\kappa'(\eta^*) < 0$, then $T'(\beta) > 0$ since $\operatorname{Re}G_\beta(\eta^*,\eta^*)$ is a decreasing function of $\beta > 0$ by [6, Lemma 12]. Moreover,

$$\lim_{\beta\to\infty} T(\beta) = \mu G(\eta^*,\eta^*) + \gamma'(\eta^*) - (\tfrac{\mu}{\mu+1}+1)\kappa'(\eta^*) + \alpha b \pi \eta^*$$

$$= \begin{cases} \mu\eta^* I_1(\eta^*\sqrt{\mu+1})K_1(\eta^*\sqrt{\mu+1}) - (\frac{\mu}{\mu+1}+1)\kappa'(\eta^*) + 2\alpha\pi\eta^* & (n=2), \\ \mu\eta^* I_{\frac{3}{2}}(\eta^*\sqrt{\mu+1})K_{\frac{3}{2}}(\eta^*\sqrt{\mu+1}) - (\frac{\mu}{\mu+1}+1)\kappa'(\eta^*) + 4\alpha\pi\eta^{*2} & (n=3) \end{cases}$$

which are positive. Therefore, there exists a unique $\beta > 0$ if $T(0) = \gamma'(\eta^*) - \kappa'(\eta^*) + \alpha b \pi \eta^* < 0$. ∎

The following theorem summarizes the things we have proved.

Theorem 4.5. *Case without and with global coupling ($0 \le \alpha < \infty$): Assume that*

$$\kappa'(\eta^*) < 0,\ \frac{1}{\mu+1}\kappa(0) + (\frac{1}{2} - a_0) + \alpha s_0 < \frac{\mu}{\mu+1} \quad \text{and} \quad \gamma'(\eta^*) - \kappa'(\eta^*) + \alpha b \pi \eta^* < 0.$$

Then there exists a unique τ^* such that the linearization $-\widetilde{A} + \tau^* B$ has a purely imaginary pair of eigenvalues. The point $(0,\eta^*,\tau^*)$ is then a Hopf point for (2.3) and there exists a C^0-curve of nontrivial periodic orbits for (2.3) and (1.6), bifurcating from $(0,\eta^*,\tau^*)$ and (v^*,η^*,τ^*), respectively.

Case of strong global coupling ($\alpha \uparrow \infty$): Equation (2.3) has only one radially symmetric equilibrium solutions $(0,\eta^*)$ for all $\sigma \neq 0$ with $\eta^* = \sqrt{s_0/\pi}$ $(n=2)$ and $\eta^* = (3/4\pi)s_0^{1/3}$ $(n=3)$. Hence there occurs no oscillatory bifurcation.

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