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Some Mathematical Characteristics of the Beta Density Function of Two Variables

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Abstract. Some well known results on the bivariate beta distribution have been reviewed. Corrected product moments are derived. These moments will be important for studying further characteristics of the distribution. The distribution of the ratio of two correlated beta variables has been derived and used to obtain a new reliability expression. Other interesting distributions stemming from the correlated beta variables are also derived.

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1. Introduction

The bivariate beta distribution has applications in areas such as voting analysis of political issues of two competing candidates and research on soil strength (see Hutchinson and Lai [2]). In this paper we derive some centered moments that are important in studying further properties of the distribution. We also review some well known results. In addition, we derive some new results for which some applications are offered.

The beta distribution is given by

$$f(x)=\frac{1}{B(m,n)}x^{m-1}(1-x)^{n-1}, \tag{1.1}$$

where $m,n>0$, $0\le x\le 1$ and $B(m,n)=\Gamma(m)\Gamma(n)/\Gamma(m+n)$. A distribution is said to be a beta distribution of a second kind BetaII(m,n) if its density function is given by

$$f(y)=\frac{1}{B(m,n)}\frac{y^{m-1}}{(1+y)^{m+n}}, \tag{1.2}$$

where $m,n>0$, $0\le y<\infty$. A transformation $Z=1/(1+Y)$ of the random variable Y in (1.2) will result in a Beta(m,n) distribution. It is interesting to note that if $m=n$, the density function in (1.2) is also the density function of an $F(2m,2m)$ distribution.

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The product moments of order a and b for two random variables X and Y are defined by $\mu'_{a,b} = E\left(X^a Y^b\right)$ while the centered product moments (sometimes called central product moments, corrected moments or central mixed moments) are defined by

$$\mu_{a,b} = E\left[(X - E(X))^a (Y - E(Y))^b\right].$$

The former moments $\mu'_{a,b}$ are often called product moments of order zero or raw product moments. Evidently $\mu'_{a,0} = E(X^a)$ is the a-th moment of X, and $\mu'_{0,b} = E\left(Y^b\right)$ is the b-th moment of Y. In case X and Y are independent $\mu'_{a,b} = E\left(X^a)E(Y^b\right) = \mu'_{a,0}\mu'_{0,b}$. Interested readers may go through Johnson *et al.* [4, p. 46].

The correlation coefficient ρ $(-1 < \rho < 1)$ between X and Y is denoted by

$$\rho_{X,Y} = \frac{\mu_{1,1}}{\sqrt{\mu_{2,0}\mu_{0,2}}}. \tag{1.3}$$

Note that $\mu_{2,0} = E(X - E(X))^2 = \sigma_{20}$ which is popularly denoted by σ_1^2 while the central product moment, $\mu_{1,1} = E\left[(X - E(X))(Y - E(Y))\right]$ denoted popularly by σ_{12}, is in fact the covariance between X and Y.

The importance of evaluating central moments of a bivariate distribution cannot be overlooked. In a series of papers, Mardia [7–11] defined and discussed the properties of moments based on Mahalanobis distance. As it is difficult to derive distribution of Mahalanobis distance for many distributions, bivariate or multivariate, and calculate moments thereof, Joarder [3] derived Mahalanobis moments (or simply, standardized moments) in terms of central product moments. He showed that the central moments can be used as an alternative way to describe further important characteristics of a bivariate distribution, for example, bivariate kurtosis coefficient.

It should be mentioned that the central moments derived in Section 4 is a formidable task requiring meticulous calculation. Some discusssions on the product moment correlation is given in Section 5. In Section 6, the distribution of the ratio of two correlated beta variables has been derived and used to obtain a new reliability expression. In addition, some other interesting distributions are also derived.

2. Review of some distributional properties of the bivariate beta distribution

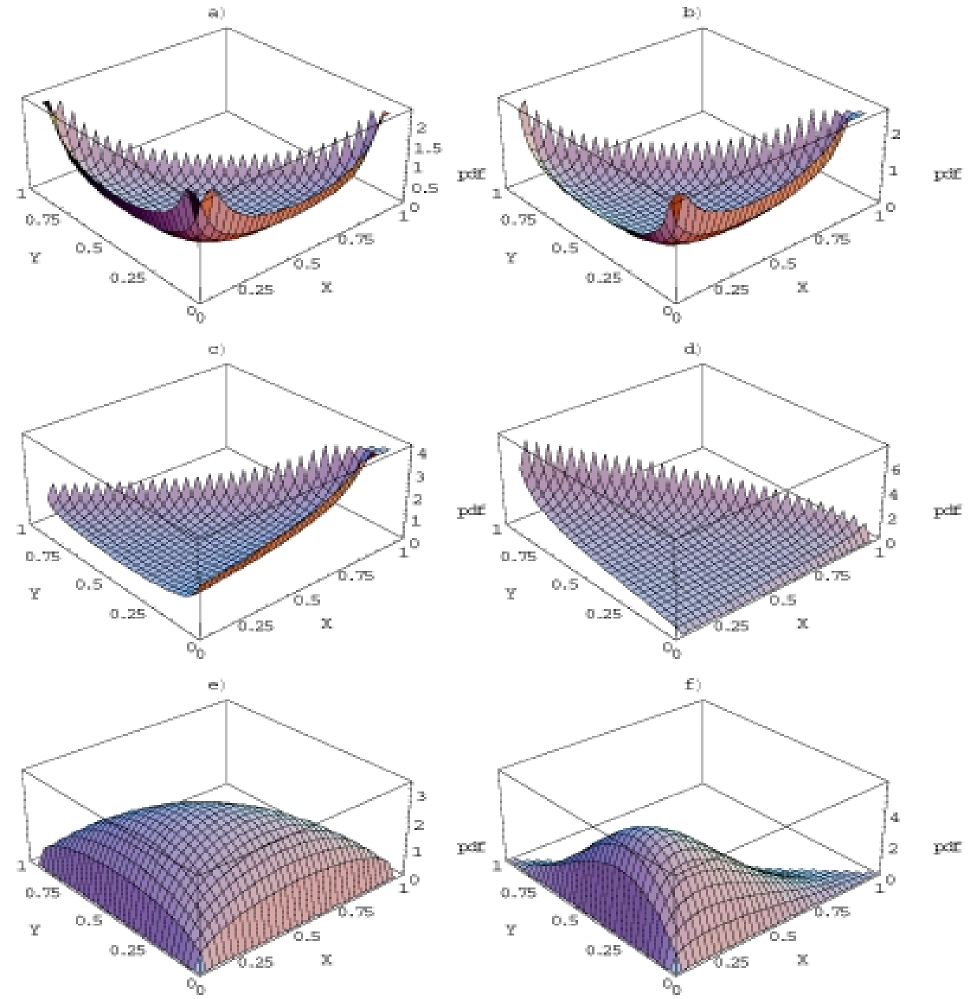
The bivariate beta is a natural extension of a univariate beta distribution. The probability density function of the bivariate beta distribution is given by

$$f(x,y) = \frac{\Gamma(m+n+p)}{\Gamma(m)\Gamma(n)\Gamma(p)} x^{m-1} y^{n-1} (1-x-y)^{p-1}, \tag{2.1}$$

where $m,n,p > 0$, $x \geq 0$, $y \geq 0$, and $x + y \leq 1$. A natural extension of this bivariate beta distribution is the multivariate Dirichlet distribution (see Fang and Zhang [1]). Figure 1 provides a surface graph of the pdf in (2.1) for different values of m, n and p. The following theorem is due to Lee [6].

Theorem 2.1. *Let X and Y have the joint pdf given by (2.1). Then the marginal probability density functions of the bivariate beta distribution with pdf in (2.1) are given by:*

$$X \sim \text{Beta}(m, n+p), \quad Y \sim \text{Beta}(n, m+p). \tag{2.2}$$



a) $m = 0.25$, $n = 0.25$, $p = 0.25$; b) $m = 0.5$, $n = 0.25$, $p = 0.25$; c) $m = 1$, $n = 0.5$, $p = 0.5$; d) $m = 1$, $n = 1.5$, $p = 0.5$; e) $m = 1.5$, $n = 1.5$, $p = 1.5$; and f) $m = 1.5$, $n = 2$, $p = 3$.

Figure 1. Bivariate Beta probability density function for various values of m, n, and p.

Proof. The marginal pdf of Y follows from the substitution of $u(1-y) = x$ in the following integral

$$g(y) = \int_0^{1-y} \frac{\Gamma(m+n+p)}{\Gamma(m)\Gamma(n)\Gamma(p)} x^{m-1} y^{n-1} (1-x-y)^{p-1} dx.$$

Thus Y follows Beta$(n, m+p)$. Similarly, X follows Beta$(m, n+p)$. ∎

We note that the mean and variance of Y are given by $E(Y) = n/(m+n+p)$, and $V(Y) = (n(m+p))/((t+1)t^2)$ respectively, where $t = m+n+p$. The mean and variance of X are given by $E(X) = m/(m+n+p)$, and $V(X) = (m(n+p))/((t+1)t^2)$ respectively.

Theorem 2.2. *Let X and Y have the joint pdf given by (2.1). Then the conditional pdf of Y given $X = x$ is given by*

$$k(y|x) = \frac{(1-x)}{B(n,p)}\left(\frac{y}{1-x}\right)^{n-1}\left(1-\frac{y}{1-x}\right)^{p-1}, \quad 0<y<1-x,\ 0<x<1. \tag{2.3}$$

Proof. The conditional pdf of Y given $X = x$ is defined as

$$\begin{aligned} f(x,y)/h(x) &= \frac{\Gamma(m+n+p)}{\Gamma(m)\Gamma(n)\Gamma(p)}x^{m-1}y^{n-1}(1-x-y)^{p-1}\left(\frac{1}{B(m,n+p)}x^{m-1}(1-x)^{n+p-1}\right)^{-1} \\ &= \frac{\Gamma(m+n+p)B(m,n+p)}{\Gamma(m)\Gamma(n)\Gamma(p)}y^{n-1}(1-x-y)^{p-1}/(1-x)^{n+p-1} \end{aligned}$$

which can be written as (2.3). ∎

Thus, from (2.3), it can be seen that the conditional distribution of $Y/(1-X)$ given $X = x$ is Beta(n,p) which implies that $E(Y\,|X = x) = (1-x)n/(n+p)$ which can also be written as

$$E(Y \mid X = x) = -\frac{n}{n+p}x + \frac{n}{n+p} \tag{2.4}$$

in the regular regression format. Thus, the regression of Y on X is linear. Also $Var\ (Y \mid X = x) = (1-x)^2 np/((n+p)^2(n+p+1))$ which is not free from x. This means that the conditional variance for the linear regression of Y on X is not homoscedastic. The linear regression also suggests that Y is not independent of X. Lee [6] also proved that $E(X^k \mid Y = y) = (1-y)^k$ and $E(Y^k \mid X = x) = (1-x)^k$.

Theorem 2.3. *Let X and Y have the joint pdf given by (2.1). Then $Y/(1-X)$ and X are independent.*

Proof. Let $u = y/(1-x)$ and $v = x$ with Jacobian $J(x,y \to u,v) = -(1-v)$. The support region is mapped into the region

$$\{(u,v) : v > 0, u(1-v) > 0, v+u(1-v) < 1\} = \{(u,v) : 0 < u < 1, 0 < v < 1\}.$$

Then from (2.1),the joint pdf of U and V is given by

$$g(u,v) = \frac{1}{B(n,p)B(m,n+p)}u^{n-1}v^{m-1}(1-v)^n\left((1-u)(1-v)\right)^{p-1}.$$

That is, $U \sim$ Beta(n,p) and $V \sim$ Beta$(m,n+p)$ and they are independently distributed. ∎

Theorem 2.4. *Let (X,Y) follow the bivariate beta distribution with pdf given by (2.1). Also let $U = X+Y$ and $V = X/(X+Y)$. Then $U \sim$ Beta$(m+n,p)$ is independent of $V \sim$ Beta(m,n).*

Proof. Let us make the transformation $u = x+y$ and $uv = x$. The support region is mapped onto the region $\{(u,v) : uv > 0,\ u(1-v) > 0,\ u < 1\} = \{(u,v) : 0 < u < 1,\ 0 < v < 1\}$ with Jacobian $J(x,y \to u,v) = -u$. The theorem then follows in a straightforward manner. ∎

In what follows we will define

$$\mu_{a,b} = E\left[(X-\xi)^a(Y-\theta)^b\right] \tag{2.5}$$

where $\xi = E(X)$ and $\theta = E(Y)$.

Provost and Cheong [14] discussed the distribution of linear combinations of the components of a Dirichlet random vector. The distribution of $ax+by$ is a special case of that.

3. Raw product moments

For any non-negative integer a, Pochhammer factorials are defined as $c_{\{a\}} = c(c+1)(c+2)\cdots(c+a-1)$ and $c^{\{a\}} = c(c-1)(c-2)\cdots(c-a+1)$, with $c_{\{0\}} = 1$, $c^{\{0\}} = 1$. Also, the $(a,b)^{th}$ raw product moment of X and Y of the bivariate beta distribution is given by

$$E(X^a Y^b) = \int_0^1 \int_0^{1-y} x^a y^b f(x,y)dxdy. \tag{3.1}$$

Lemma 3.1. *Let X and Y have the joint pdf given by (2.1). Then*

(1) *the marginal density function of* $X \sim \text{Beta}(m, n+p)$*, has an expected value of* $E(X^a) = m_{\{a\}}/t_{\{a\}}$,
(2) *the marginal density function of* $Y \sim \text{Beta}(n, m+p)$*, has an expected value of* $E(Y^b) = n_{\{b\}}/t_{\{b\}}$,
(3) *and the raw product moment of order* (a,b) *is* $E(X^a Y^b) = m_{\{a\}} n_{\{b\}}/t_{\{a+b\}}$, *where* $t = m+n+p$.

The above lemma gives rise to some useful raw moments that will be used further in this article.

4. Centered moments

The centered product moments of a bivariate beta distribution, $\mu_{a,b}$ can be obtained by directly evaluating the following integral:

$$E\left[(X_1 - E(X_1))^a (X_2 - E(X_2))^b\right] = \int_0^1 \int_0^{1-x_2} (X_1 - E(X_1))^a (X_2 - E(X_2))^b f(x_1,x_2)dx_1 dx_2.$$

For illustration, we derive the central moment $\mu_{1,2}$ below.

$$\begin{aligned}
\mu_{1,2} &= E\left[(X - E(X))^1 (Y - E(Y))^2\right] \\
&= \int_0^1 \int_0^{1-y} \left(x - \frac{m}{t}\right)\left(y - \frac{n}{t}\right)^2 f(x,y)dxdy \\
&= \int_0^1 \int_0^{1-y} \left(xy^2 - 2xy\frac{n}{t} + x\left(\frac{n}{t}\right)^2 - y^2\frac{m}{t} + 2y\frac{n}{t}\frac{m}{t} - \left(\frac{n}{t}\right)^2\frac{m}{t}\right) f(x,y)dxdy \\
&= \int_0^1 \int_0^{1-y} xy^2 f(x,y)dxdy - 2\frac{n}{t}\int_0^1 \int_0^{1-y} xyf(x,y)dxdy \\
&\quad + \left(\frac{n}{t}\right)^2 \int_0^1 \int_0^{1-y} xf(x,y)dxdy - \frac{m}{t}\int_0^1 \int_0^{1-y} y^2 f(x,y)dxdy \\
&\quad + 2\frac{mn}{t^2}\int_0^1 \int_0^{1-y} yf(x,y)dxdy - \frac{n^2 m}{t^3}\int_0^1 \int_0^{1-y} f(x,y)dxdy \\
&= \frac{mn(n+1)}{t(t+1)(t+2)} - 2\left(\frac{n}{t}\right)\frac{mn}{t(t+1)} + \left(\frac{n}{t}\right)^2\left(\frac{m}{t}\right) \\
&\quad - \left(\frac{m}{t}\right)\frac{n(n+1)}{t(t+1)} + 2\left(\frac{m}{t}\right)\left(\frac{n}{t}\right)^2 - \frac{n^2 m}{t^3} \\
&= \frac{mn(n+1)}{t(t+1)(t+2)} - 2\frac{mn^2}{t^2(t+1)} + \frac{mn^2}{t^3} - \frac{mn(n+1)}{t^2(t+1)} + 2\frac{mn^2}{t^3} - \frac{mn^2}{t^3}
\end{aligned}$$

$$= -\frac{2(t-2n)mn}{(t+1)(t+2)t^3}.$$

The higher order moments generally require more meticulous integration and cross-checking of calculations. More details of these centered moments can be found in Omar and Joarder [13]. Some centered product moments of order $a+b=2,3,4,5,6$ are given below:

$$\mu_{0,2} = \frac{(t-n)n}{(t+1)t^2},$$
$$\mu_{0,3} = 2\frac{(t-2n)(t-n)n}{(t+1)(t+2)t^3},$$
$$\mu_{0,4} = 3(t-n)n\frac{t^2(n+2)-n(n+6)t+6n^2}{(t+1)(t+2)(t+3)t^4},$$
$$\mu_{0,5} = 4(t-2n)(t-n)n\frac{t^2(5n+6)-n(5n+12)t+12n^2}{(t+1)(t+2)(t+3)(t+4)t^5},$$
$$\mu_{0,6} = \frac{5(t-n)n}{(t+1)(t+2)(t+3)(t+4)(t+5)t^6}$$
$$\times [t^4\left(3n^2+26n+24\right)-2n\left(3n^2+56n+60\right)t^3$$
$$+n^2\left(3n^2+172n+240\right)t^2-2n^3(43n+120)t+120n^4],$$
$$\mu_{1,1} = -\frac{nm}{(t+1)t^2},$$
$$\mu_{1,2} = -2\frac{(t-2n)mn}{(t+1)(t+2)t^3},$$
$$\mu_{1,3} = 3\frac{mn}{t^4(t+1)(t+2)(t+3)}(-(n+2)t^2+n(n+6)t-6n^2),$$
$$\mu_{1,4} = -4\frac{mn(t-2n)}{t^5(t+1)(t+2)(t+3)(t+4)}\left(12n^2-(12+5n)nt+(6+5n)t^2\right),$$
$$\mu_{1,5} = -5\frac{mn}{t^6(t+1)(t+2)(t+3)(t+4)(t+5)}$$
$$\times [\left(3n^2+26n+24\right)t^4-2n\left(3n^2+56n+60\right)t^3$$
$$+n^2\left(3n^2+172n+240\right)t^2-2n^3(43n+120)t+120n^4],$$
$$\mu_{2,2} = mn\frac{t^3-(m+n)t^2+3(2m+2n+mn)t-18mn}{(t+1)(t+2)(t+3)t^4},$$
$$\mu_{2,3} = 2mn\frac{t^4-t^3(m+6n)+\left(12m+15mn+5n^2\right)t^2-4n(9m+3n+5mn)t+48mn^2}{(t+1)(t+2)(t+3)(t+4)t^5},$$
$$\mu_{2,4} = \frac{mn}{t^6(t+1)(t+2)(t+3)(t+4)(t+5)}$$
$$\times [t^5(3n+6)+t^4\left(-6m-40n-3mn-6n^2\right)$$
$$+t^3\left(120m+164mn+120n^2+3n^3+18mn^2\right)$$
$$+t^2\left(-480mn-86n^3-516mn^2-15mn^3\right)$$

$$+t\left(120n^3+720mn^2+430mn^3\right)-600mn^3],$$

$$\mu_{3,3}=\frac{mn}{(t+1)(t+2)(t+3)(t+4)(t+5)t}$$
$$\times\left[4-3\frac{10m+10n+3mn}{t}+\frac{\left(26n^2+m^2(9n+26)+9mn(n+20)\right)}{t^2}\right.$$
$$-3\frac{2n^2(43m+20)+m^2\left(86n+5n^2+40\right)}{t^3}$$
$$\left.+10\frac{mn(36m+36n+43mn)}{t^4}-600\frac{m^2n^2}{t^5}\right]$$

Similar expressions for moments $\mu(b,a)=E\left[(X-\xi)^b(Y-\theta)^a\right]$ are provided below.

$$\mu_{2,0}=\frac{(t-m)m}{(t+1)t^2},$$
$$\mu_{3,0}=2\frac{(t-2m)(t-m)m}{(t+1)(t+2)t^3},$$
$$\mu_{4,0}=3(t-m)m\frac{t^2(m+2)-m(m+6)t+6m^2}{(t+1)(t+2)(t+3)t^4},$$
$$\mu_{5,0}=4(t-m)(t-2m)m\frac{t^2(5m+6)-m(5m+12)t+12m^2}{(t+1)(t+2)(t+3)(t+4)t^5},$$
$$\mu_{6,0}=\frac{5(t-m)m}{(t+1)(t+2)(t+3)(t+4)(t+5)t^6}$$
$$\times[\left(3m^2+26m+24\right)t^4-2m\left(3m^2+56m+60\right)t^3$$
$$+m^2\left(3m^2+172m+240\right)t^2-2m^3(43m+120)t+120m^4],$$
$$\mu_{2,1}=-2\frac{mn(t-2m)}{t^3(t+1)(t+2)},$$
$$\mu_{3,1}=3mn\frac{-(m+2)t^2+m(m+6)t-6m^2}{(t+1)(t+2)(t+3)t^4},$$
$$\mu_{4,1}=-4(t-2m)mn\frac{(5m+6)t^2-\left(12m+5m^2\right)t+12m^2}{(t+1)(t+2)(t+3)(t+4)t^5},$$
$$\mu_{5,1}=-5\frac{mn}{t^6(t+1)(t+2)(t+3)(t+4)(t+5)}$$
$$\times[(3m^2+26m+24)t^4-2m\left(3m^2+56m+60\right)t^3$$
$$+m^2\left(3m^2+172m+240\right)t^2-2m^3(43m+120)t+120m^4],$$
$$\mu_{3,2}=2mn\frac{t^4-t^3(6m+n)+(12n+5m(m+3n))t^2-4m(m(5n+3)+9n)t+48m^2n}{(t+1)(t+2)(t+3)(t+4)t^5},$$
$$\mu_{4,2}=\frac{mn}{t^6(t+1)(t+2)(t+3)(t+4)(t+5)}$$
$$\times[(3m+6)t^5-(6m^2+m(3n+40)+6n)t^4$$
$$+(3m^3+6(3n+20)m^2+4n(41m+30))t^3-m(m^2(15n+86)+12n(43m+40))t^2$$

$$+ 10m^2(12m+72n+43mn)t - 600m^3n].$$

5. Correlation Coefficient for the Bivariate Beta Distribution

Theorem 5.1. *Let X and Y have the joint pdf given by (2.1). Then the product moment correlation coefficient between X and Y is given by*

$$\rho = -\sqrt{\frac{mn}{(n+p)(m+p)}}. \tag{5.1}$$

Proof. Substituting the moments from Section 4 in $\rho\sqrt{\mu_{2,0}\mu_{0,2}} = \mu_{1,1}$ we have (5.1). ∎

Figure 2 provides graphs of values of the correlation coefficient for different values of p. Mainly, the correlation coefficient reaches the limiting value of -1 when p is small as either m or n increases. For a special case of a bivariate beta distribution defined in (2.1) where $m = n$, that is $(t = 2m+p)$, then the product moment correlation coefficient is $\rho = -m/(m+p)$. Note that where $m = n$, the two bivariate marginal probability density functions are identical. Another special case of a bivariate beta distribution defined in (2.1) occurs when $m = n = p$, that is $t = 3m$. In this case, the product moment correlation coefficient $\rho = -1/2$. Note that here we have a special case of (5.1) where the two bivariate marginal pdfs are identical with $m = n = p$.

Alternatively, it is also easy to check that

$$\operatorname{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{-mn}{(t+1)t^2}$$

since

$$E(XY) = E(XE(Y\,|X)) = E\left(X\frac{(1-X)n}{n+p}\right) = \frac{mn}{t(t+1)}.$$

The above shows that the product moment correlation coefficient ρ between X and Y is also given by what we have (5.1).

6. Some new distribution of functions of correlated beta variables and their applications to reliability

Theorem 6.1. *Let (X,Y) follow the bivariate beta distribution with pdf given by (2.1). Then the pdf of $W = Y/X$ is given by*

$$g(w) = \frac{w^{n-1}}{B(m,n)(w+1)^{m+n}}, \qquad 0 \le w < \infty. \tag{6.1}$$

Proof. By Theorem 2.4, $V = X/(X+Y) \sim \text{Beta}(m,n)$.With the transformation $W = (1-V)/V$, the support region for bivariate beta distribution is mapped onto the region $\{w : 0 < w < \infty\}$ with absolute value of Jacobian $|J(v \to w)| = (w+1)^{-2}$. Then (6.1) follows in a straightforward manner. ∎

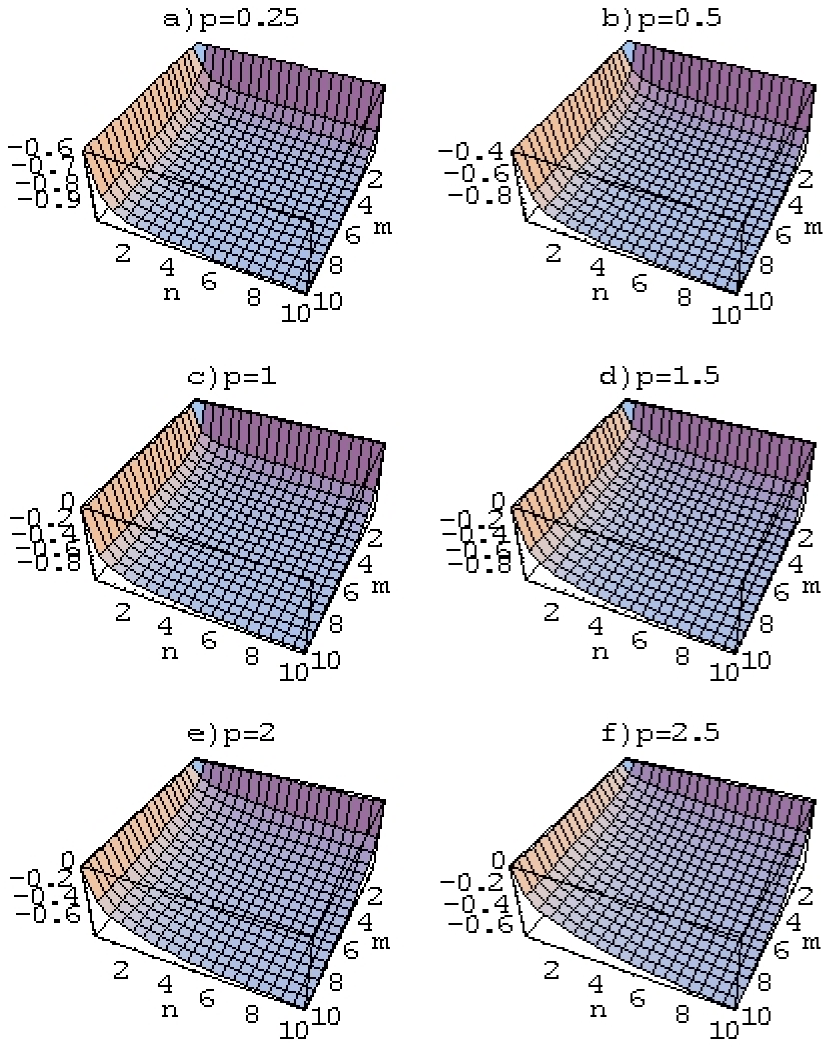


Figure 2. Product moment correlation values for various values of m, n, and p.

Krishnamoorthy *et al.* [5] discussed a special reliability function defined as $R = P(X > Y)$. This type of reliability function is interesting when one wishes to assess the proportion of time the random strength variable X of a component exceeds the random stress variable Y which the component is subjected to. If $X \leq Y$, then either the component fails or the system the component supports malfunctions. The same reliability function R can also be used to estimate the probability that one of the two devices, X and Y, fails before the other. Krishnamoorthy *et al.* [5] also discussed inference on reliability in a stress-strength relationship that follows a two-parameter exponential model.

Theorem 6.1 can immediately be applied to the situations where reliability of a component in a stress-strength relationship follows a bivariate beta distribution. In this situation, the following Theorem can be used.

Theorem 6.2. *Let (X,Y) follow the bivariate beta distribution with pdf given by (2.1). And let the pdf of $W = Y/X$ is given by*

$$g(w) = \frac{w^{n-1}}{B(m,n)(w+1)^{m+n}}, \qquad 0 < w < \infty$$

Then, the reliability $R = P(X > Y)$ defined in Krishnamoorthy et al. [5] *is given by*

$$R = I_{1/2}(m,n), \tag{6.2}$$

where

$$I_a(m,n) = \int_a^1 \frac{(1-v)^{n-1}v^{m-1}}{B(m,n)}dv.$$

Proof. From the definition we have $R = P(X > Y) = P(W < 1)$. But, $W = Y/X \sim$ BetaII$(n, m+n)$, a Beta distribution of the second kind defined in (1.2). So,

$$R = \int_0^1 \frac{w^{n-1}}{B(m,n)(w+1)^{m+n}}dw$$

Letting

$$W = (1-V)/V, \quad R = \int_{1/2}^1 \frac{(1-v)^{n-1}v^{m-1}}{B(m,n)}dv$$

which is symbolically given in the above Theorem. ∎

Theorem 6.2 provides an alternative simpler expression of reliability than what is reported in Nadarajah [12]. Also, the evaluation of reliability is reduced to a univariate integral instead of a bivariate one.

It is also worth mentioning that with the appropriate transformation that the pdf of $S = W^2 = Y^2/X^2$ is given by

$$f(s) = \frac{1}{2B(m,n)} s^{n/2-1} \left(\frac{1}{s^{1/2}+1}\right)^{m+n}, \qquad 0 < s < \infty.$$

Theorem 6.3. *Let (X,Y) follow the bivariate beta distribution with pdf given by (2.1). And let $X = R^2\cos^2\Theta$ and $Y = R^2\sin^2\Theta$. Then $R^2 \sim$ Beta$(m+n,p)$, $\sin^2\Theta \sim$ Beta(n,m), where Θ follows the distribution given by*

$$h(\theta) = \frac{2}{\text{Beta}(m,n)} (\cos\theta)^{2m-1} (\sin\theta)^{2n-1}.$$

Proof. Let X and Y follow the bivariate beta probability density function given by (2.1). Also, let $X = R^2\cos^2\Theta$ and $Y = R^2\sin^2\Theta$. With this transformation, the support region is mapped onto the region $\{(r^2,\theta) : 0 < r^2 < 1, 0 < \theta < \pi/2\}$ with $J(x,y \to r^2,\theta) = 2r^2\cos\theta\sin\theta$. Substitution into (2.1) gives the following

$$g(r^2,\theta) = \frac{\Gamma(m+n+p)}{\Gamma(m+n)\Gamma(p)} \left(1-r^2\right)^{p-1} \left(r^2\right)^{m+n} 2\frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} (\cos\theta)^{2m-1} (\sin\theta)^{2n-1}.$$

That is, $R^2$and Θ are independent random variables with distributions as mentioned in the theorem.

Now, let $U = \sin^2\Theta$. Hence with this transformation, the support region is mapped onto the region $\{u : 0 \le u \le 1\}$ with $J(u \to \theta) = 2[(1-\sin^2\theta)(\sin^2\theta)]^{1/2} = 2\left((1-u)u\right)^{1/2}$. Substituting u into $h(\theta)$ with the appropriate Jacobian provides the result that $U = \sin^2\Theta \sim \mathrm{Beta}(n,m)$. ∎

Corollary 6.1. *Let X and Y follow the bivariate beta probability density function given by (2.1). Also, let $X = R^2\cos^2\Theta$ and $Y = R^2\sin^2\Theta$. If $m = n$, then*

$$R^2 \sim \mathrm{Beta}(2m, p),\ U = \sin^2\Theta \sim \mathrm{Beta}(m,m) \quad \textit{and} \quad h(\theta) = \frac{2}{\mathrm{Beta}(m,m)}\left(\frac{1}{2}\sin 2\theta\right)^{2m-1}.$$

Proof. Substituting $m = n$ into Theorem 6.3 provides results of Corollary 6.1. ∎

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