BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **36***(1) (2013), 107–116*

A Hybrid Extragradient Method for Pseudomonotone Equilibrium Problems and Fixed Point Problems

PHAM NGOC ANH

Department of Scientific Fundamentals, Posts and Telecommunications Institute of Technology, Hanoi, Vietnam
anhpn@ptit.edu.vn

Abstract. In this paper, we introduce a new hybrid extragradient iteration method for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of equilibrium problems for pseudomonotone and Lipschitz-type continuous bifunctions. The iterative process is based on two well-known methods: Hybrid and extragradient. We show that the iterative sequences generated by this algorithm converge strongly to the common element in a real Hilbert space.

2010 Mathematics Subject Classification: 65K10, 65K15, 90C25, 90C33

Keywords and phrases: Equilibrium problems, pseudomonotone, Lipschitz-type continuous, strong convergence, nonexpansive mapping, extragradient method.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H and f be a bifunction from $C \times C$ to $\mathbb{R}$. We consider the equilibrium problems given as:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0 \ \ \forall y \in C. \qquad EP(f,C)$$

The set of solutions of $EP(f,C)$ is denoted by $\mathrm{Sol}(f,C)$.

If $f(x,y) = \langle F(x), y-x \rangle$ for every $x, y \in C$, where F is a mapping from C to H, then Problem $EP(f,C)$ becomes the following variational inequalities:

$$\text{Find } x^* \in C \text{ such that } \langle F(x^*), y - x^* \rangle \geq 0 \ \ \forall y \in C. \qquad VI(F,C)$$

We denote $\mathrm{Sol}(F,C)$ which is the set of solutions of $VI(F,C)$.

In recent years, equilibrium problems become an attractive field for many researchers both theory and applications [1, 2, 4, 11, 17]. There are myriad of literature related to equilibrium problems and their applications in electricity market, transportation, economics and network [3, 5].

For solving $VI(F,C)$ in the Euclidean space $\mathbb{R}^n$ under the assumption that a subset $C \subseteq \mathbb{R}^n$ is nonempty closed convex, F is monotone, L-Lipschitz continuous and $\mathrm{Sol}(F,C) \neq \emptyset$,

Communicated by Ahmad Izani Md. Ismail.
Received: November 18, 2010; *Revised:* March 1, 2011.

Korpelevich in [7] introduced the following extragradient method:

$$\begin{cases} x^0 \in C, \\ y^n = \Pr_C\left(x^n - \lambda F(x^n)\right), \\ x^{n+1} = \Pr_C\left(x^n - \lambda F(y^n)\right), \end{cases}$$

for all $n \geq 0$, where $\lambda \in (0, \frac{1}{L})$. The author showed that the sequences $\{x^n\}$ and $\{y^n\}$ converge to the same point $z \in \mathrm{Sol}(F,C)$.

For each $x, y \in C$, $f_\varphi(x,y) := f(x,y) + \varphi(y) - \varphi(x)$, motivated by the results of Peng in [11] introduced a new iterative scheme for finding a common element of the sets $\mathrm{Sol}(f_\varphi, C)$, $\mathrm{Sol}(F,C)$ and $\mathrm{Fix}(T)$ in a real Hilbert space. Let sequences $\{x^n\}, \{y^n\}, \{t^n\}$ and $\{z^n\}$ be defined by

$$\begin{cases} x^0 \in H, \\ f_\varphi(u^n, y) + \frac{1}{r_n}\langle y - u^n, u^n - x^n\rangle \geq 0 \ \ \forall y \in C, \\ y^n := \Pr_C\left(u^n - \lambda_n F(u^n)\right), \\ t^n := \Pr_C\left(u^n - \lambda_n F(y^n)\right), \\ z^n := \alpha_n t^n + (1-\alpha_n)T(t^n), \\ C_n := \{z \in C : \ \|z^n - z\|^2 \leq \|x^n - z\|^2 - (1-\alpha_n)(\alpha_n - \varepsilon)\|t^n - T(t^n)\|\}, \\ Q_n := \{z \in H : \ \langle x^n - z, x - x^n\rangle \geq 0\}, \\ x^{n+1} := \Pr_{C_n \cap Q_n}(x^0). \end{cases}$$

Then, the author showed that under certain appropriate conditions imposed on $\{\alpha_n\}, \{\lambda_n\}$ and ε, the sequences $\{x^n\}, \{u^n\}, \{t^n\}, \{y^n\}$ and $\{z^n\}$ converge strongly to $\Pr_\Omega(x^0)$, where $\Omega := \mathrm{Sol}(f_\varphi, C) \cap \mathrm{Sol}(F,C) \cap \mathrm{Fix}(T)$.

Recently, iterative algorithms for finding a common element of the set of solutions of equilibrium problems and the set of fixed points of a nonexpansive mapping in a real Hilbert space have further developed by some authors (see [11, 12, 14, 17, 18]). At each iteration n in all of these algorithms, it requires solving approximation auxiliary equilibrium problems.

In this paper, we introduce a new iterative algorithm for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of equilibrium problems for a pseudomonotone, Lipschitz-type continuous bifunction. This method can be considered as an improvement of the iterative method in [11] via an improvement set of extragradient methods in [1, 2]. At each iteration n, we only solve strongly convex problems on C. We obtain a strong convergence theorem for four sequences generated by this process.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. We list some well known definitions and the projection which will be required in our following analysis.

Definition 2.1. *Let C be a closed convex subset in H, we denote the projection on C by $\Pr_C(\cdot)$, i.e.,*

$$\Pr_C(x) = argmin\{\|y - x\| : \ y \in C\} \ \ \forall x \in H.$$

The bifunction $f : C \times C \to \mathbb{R}$ is said to be

(1) *γ-strongly monotone on C if for each $x,y \in C$, we have*

$$f(x,y)+f(y,x) \leq -\gamma\|x-y\|^2;$$

(2) *monotone on C if for each $x,y \in C$, we have*

$$f(x,y)+f(y,x) \leq 0;$$

(3) *pseudomonotone on C if for each $x,y \in C$, we have*

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0;$$

(4) *Lipschitz-type continuous on C with constants $c_1 > 0$ and $c_2 > 0$, if for each $x,y \in C$, we have*

$$f(x,y)+f(y,z) \geq f(x,z) - c_1\|x-y\|^2 - c_2\|y-z\|^2.$$

The mapping $F : C \to H$ is said to be

(5) *monotone on C if for each $x,y \in C$, we have*

$$\langle F(x)-F(y), x-y\rangle \geq 0;$$

(6) *pseudomonotone on C if for each $x,y \in C$, we have*

$$\langle F(y), x-y\rangle \geq 0 \Rightarrow \langle F(x), x-y\rangle \geq 0;$$

(7) *L-Lipschitz continuous on C if for each $x,y \in C$, we have*

$$\|F(x)-F(y)\| \leq L\|x-y\|.$$

If $L=1$, then F is nonexpansive on C.

Note that if F is L-Lipschitz on C, then for each $x,y \in C$, $f(x,y) = \langle F(x), y-x\rangle$ is Lipschitz-type continuous with constants $c_1 = c_2 = \frac{L}{2}$ on C. Indeed,

$$\begin{aligned}
&2f(x,y)+f(y,z)-f(x,z)\\
&= \langle F(x), y-x\rangle + \langle F(y), z-y\rangle - \langle F(x), z-x\rangle\\
&= -\langle F(y)-F(x), y-z\rangle \geq -\|F(x)-F(y)\|\|y-z\| \geq -L\|x-y\|\|y-z\|\\
&\geq -\frac{L}{2}\|x-y\|^2 - \frac{L}{2}\|y-z\|^2 = -c_1\|x-y\|^2 - c_2\|y-z\|^2.
\end{aligned}$$

Thus f is Lipschitz-type continuous on C.

In this paper, for finding a point of the set $\mathrm{Sol}(f,C)\cap \mathrm{Fix}(T)$, we assume that the bifunction f satisfies the following conditions:

(i) f is pseudomonotone on C;
(ii) f is Lipschitz-type continuous on C;
(iii) for each $x \in C$, $y \mapsto f(x,y)$ is convex and subdifferentiable on C;
(iv) $\mathrm{Sol}(f,C)\cap \mathrm{Fix}(T) \neq \emptyset$.

Now we are in a position to describe the extragradient algorithm for finding a common of two sets $\mathrm{Sol}(f,C)$ and $\mathrm{Fix}(T)$.

Algorithm 2.1. Initialization. Choose $x^0 \in C$, positive sequences $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the conditions

$$\begin{cases} \{\lambda_n\} \subset [a,b] & \text{for some } a,b \in \left(0, \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}\right),\\ \{\alpha_n\} \subset [0,c] & \text{for some } c \in (0,1). \end{cases}$$

Step 1. Solve the strongly convex problems:

$$\begin{cases} y^n = \operatorname{argmin}\{\frac{1}{2}\|y-x^n\|^2 + \lambda_n f(x^n,y):\ y \in C\}, \\ t^n = \operatorname{argmin}\{\frac{1}{2}\|t-x^n\|^2 + \lambda_n f(y^n,t):\ t \in C\}, \\ z^n := \alpha_n x^n + (1-\alpha_n)T(t^n). \end{cases}$$

Step 2. Set $P_n = \{z \in C:\ \|z^n - z\| \le \|x^n - z\|\}$ and $Q_n = \{z \in C:\ \langle x^n - z, x^0 - x^n\rangle \ge 0\}$. Compute $x^{n+1} = \mathrm{Pr}_{P_n \cap Q_n}(x^0)$. Increase k by 1 and go to Step 1.

In order to prove the main result in Section 3, we shall use the following lemma in the sequel.

Lemma 2.1. [5] *Let C be a convex subset of a real Hilbert space H and $g: C \to \mathbb{R}$ be convex and subdifferentiable on C. Then, x^* is a solution to the following convex problem*

$$\min\{g(x):\ x \in C\}$$

if and only if $0 \in \partial g(x^*) + N_C(x^*)$, where $\partial g(\cdot)$ denotes the subdifferential of g and $N_C(x^*)$ is the (outward) normal cone of C at x^*.

3. Main results

In this section, we show a strong convergence theorem of sequences $\{x^n\}, \{y^n\}, \{z^n\}$ and $\{t^n\}$ defined by Algorithm 2.1 based on the extragradient method which solves the problem of finding a common element of two sets $\mathrm{Sol}(f,C)$ and $\mathrm{Fix}(T)$ for a monotone, Lipschitz-type continuous bifunction f in a real Hilbert space H.

Lemma 3.1. *Suppose that $x^* \in \mathrm{Sol}(f,C)$, $f(x,\cdot)$ is convex and subdifferentiable on C for all $x \in C$, and f is pseudomonotone on C. Then, we have*

$$\|t^n - x^*\|^2 \le \|x^n - x^*\|^2 - (1-2\lambda_n c_1)\|x^n - y^n\|^2 - (1-2\lambda_n c_2)\|t^n - y^n\|^2 \ \ \forall n \ge 0.$$

Proof. Since $f(x,\cdot)$ is convex on C for each $x \in C$ and Lemma 2.1, we obtain

$$t^n = \operatorname{argmin}\{\frac{1}{2}\|t - x^n\|^2 + \lambda_n f(y^n,t):\ t \in C\}$$

if and only if

$$0 \in \partial_2\{\lambda_n f(y^n,t) + \frac{1}{2}\|t-x^n\|^2\}(t^n) + N_C(t^n). \tag{3.1}$$

Since $f(y^n,\cdot)$ is subdifferentiable on C, by the well known Moreau-Rockafellar theorem [5], there exists $w \in \partial_2 f(y^n,t^n)$ such that

$$f(y^n,t) - f(y^n,t^n) \ge \langle w, t - t^n\rangle \ \ \forall t \in C. \tag{3.2}$$

With $t = x^* \in C$, this inequality becomes

$$f(y^n,x^*) - f(y^n,t^n) \ge \langle w, x^* - t^n\rangle. \tag{3.3}$$

It follows from (3.1) that

$$0 = \lambda_n w + t^n - x^n + \bar{w},$$

where $w \in \partial_2 f(y^n,t^n)$ and $\bar{w} \in N_C(t^n)$.
By the definition of the normal cone N_C we have, from the latter equality, that

$$\langle t^n - x^n, t - t^n\rangle \ge \lambda_n \langle w, t^n - t\rangle \ \ \forall t \in C. \tag{3.4}$$

With $t = x^* \in C$ we obtain

$$\langle t^n - x^n, x^* - t^n \rangle \geq \lambda_n \langle w, t^n - x^* \rangle. \tag{3.5}$$

It follows from (3.3) and (3.5) that

$$\langle t^n - x^n, x^* - t^n \rangle \geq \lambda_n \{ f(y^n, t^n) - f(y^n, x^*) \}. \tag{3.6}$$

Since $x^* \in \mathrm{Sol}(f, C), f(x^*, y) \geq 0$ for all $y \in C$, and f is pseudomonotone on C, we have $f(y^n, x^*) \leq 0$. Then, (3.6) implies that

$$\langle t^n - x^n, x^* - t^n \rangle \geq \lambda_n f(y^n, t^n). \tag{3.7}$$

Now applying Lipschitzian of f with $x = x^n, y = y^n$ and $z = t^n$, we get

$$f(y^n, t^n) \geq f(x^n, t^n) - f(x^n, y^n) - c_1 \|y^n - x^n\|^2 - c_2 \|t^n - y^n\|^2. \tag{3.8}$$

Combinating (3.7) and (3.8), we have

$$\langle t^n - x^n, x^* - t^n \rangle \geq \lambda_n \{ f(x^n, t^n) - f(x^n, y^n) - c_1 \|y^n - x^n\|^2 - c_2 \|t^n - y^n\|^2 \}. \tag{3.9}$$

Similarly, since y^n is the unique solution to the strongly convex problem

$$\min \left\{ \frac{1}{2} \|y - x^n\|^2 + \lambda_n f(x^n, y) : \ y \in C \right\},$$

we have

$$\lambda_n \{ f(x^n, y) - f(x^n, y^n) \} \geq \langle y^n - x^n, y^n - y \rangle \ \ \forall y \in C. \tag{3.10}$$

As $y = t^n \in C$, we have

$$\lambda_n \{ f(x^n, t^n) - f(x^n, y^n) \} \geq \langle y^n - x^n, y^n - t^n \rangle. \tag{3.11}$$

From (3.9), (3.11) and

$$2\langle t^n - x^n, x^* - t^n \rangle = \|x^n - x^*\|^2 - \|t^n - x^n\|^2 - \|t^n - x^*\|^2,$$

it implies that

$$\begin{aligned} &\|x^n - x^*\|^2 - \|t^n - x^n\|^2 - \|t^n - x^*\|^2 \\ &\qquad \geq 2\langle y^n - x^n, y^n - t^n \rangle - 2\lambda_n c_1 \|x^n - y^n\|^2 - 2\lambda_n c_2 \|t^n - y^n\|^2. \end{aligned}$$

Hence, we have

$$\begin{aligned} &\|t^n - x^*\|^2 \\ \leq &\|x^n - x^*\|^2 - \|t^n - x^n\|^2 - 2\langle y^n - x^n, y^n - t^n \rangle + 2\lambda_n c_1 \|x^n - y^n\|^2 + 2\lambda_n c_2 \|t^n - y^n\|^2 \\ = &\|x^n - x^*\|^2 - \|(t^n - y^n) + (y^n - x^n)\|^2 - 2\langle y^n - x^n, y^n - t^n \rangle \\ &+ 2\lambda_n c_1 \|x^n - y^n\|^2 + 2\lambda_n c_2 \|t^n - y^n\|^2 \\ \leq &\|x^n - x^*\|^2 - \|t^n - y^n\|^2 - \|x^n - y^n\|^2 + 2\lambda_n c_1 \|x^n - y^n\|^2 + 2\lambda_n c_2 \|t^n - y^n\|^2 \\ = &\|x^n - x^*\|^2 - (1 - 2\lambda_n c_1) \|x^n - y^n\|^2 - (1 - 2\lambda_n c_2) \|y^n - t^n\|^2. \end{aligned}$$

The lemma thus is proved. ∎

Lemma 3.2. *Suppose that Assumptions (i)-(iv) hold and T is nonexpansive on C. Then, we have*

(a) $\mathrm{Sol}(f, C) \cap \mathrm{Fix}(T) \subseteq P_n \cap Q_n$ *for all* $n \geq 0$.

(b) $\lim_{n \to \infty} \|x^{n+1} - x^n\| = \lim_{n \to \infty} \|x^n - z^n\| = \lim_{n \to \infty} \|x^n - y^n\| = \lim_{n \to \infty} \|x^n - t^n\| = 0.$

(c) $\lim_{n\to\infty} \|T(t^n) - t^n\| = 0.$

Proof. Since Lemma 3.1 and $z^n = \alpha_n x^n + (1-\alpha_n)T(t^n)$, for each $x^* \in \text{Sol}(f,C) \cap \text{Fix}(T)$ we have

$$
\begin{aligned}
\|z^n - x^*\|^2 =& \|\alpha_n x^n + (1-\alpha_n)T(t^n) - x^*\|^2 \\
=& \|\alpha_n(x^n - x^*) + (1-\alpha_n)\{T(t^n) - x^*\}\|^2 \\
\le& \alpha_n\|x^n - x^*\|^2 + (1-\alpha_n)\|T(t^n) - T(x^*)\|^2 \\
\le& \alpha_n\|x^n - x^*\|^2 + (1-\alpha_n)\|t^n - x^*\|^2 \le \|x^n - x^*\|^2.
\end{aligned} \tag{3.12}
$$

Hence $\|z^n - x^*\| \le \|x^n - x^*\|$ for every $n \ge 0$ and $x^* \in P_n$. So, we have

$$\text{Sol}(f,C) \cap \text{Fix}(T) \subseteq P_n \ \ \forall n \ge 0.$$

Next, we show by mathematical induction that

$$\text{Sol}(f,C) \cap \text{Fix}(T) \subseteq Q_n \ \ \forall n \ge 0.$$

For $n = 0$ we have $Q_0 = C$, hence we have $\text{Sol}(f,C) \cap \text{Fix}(T) \subseteq Q_0$. Now we suppose that $\text{Sol}(f,C) \cap \text{Fix}(T) \subseteq Q_k$ for some $k \ge 0$. From $x^{k+1} = \text{Pr}_{P_k \cap Q_k}(x^0)$, it follows that

$$\langle x^{k+1} - x, x^0 - x^{k+1}\rangle \ge 0 \ \ \forall x \in P_k \cap Q_k.$$

Using this and $\text{Sol}(f,C) \cap \text{Fix}(T) \subseteq Q_k$, we have

$$\langle x^{k+1} - x, x^0 - x^{k+1}\rangle \ge 0 \ \ \forall x \in \text{Sol}(f,C) \cap \text{Fix}(T)$$

and hence $\text{Sol}(f,C) \cap \text{Fix}(T) \subseteq Q_{k+1}$. This proves (a).

It follows from (a) and $x^{n+1} = \text{Pr}_{P_n \cap Q_n}(x^0)$ that

$$\|x^{n+1} - x^0\| \le \|\text{Pr}_{\text{Sol}(f,C)\cap\text{Fix}(T)}(x^0) - x^0\| \ \ \forall n \ge 0. \tag{3.13}$$

Hence, we get that $\{x^n\}$ is bounded. Otherwise, for each $x \in Q_n$, we have

$$\langle x^n - x, x^0 - x^n\rangle \ge 0,$$

and hence $x^n = \text{Pr}_{Q_n}(x^0)$. Using this and $x^{n+1} \in P_n \cap Q_n \subseteq Q_n$, we have

$$\|x^n - x^0\| \le \|x^{n+1} - x^0\| \ \ \forall n \ge 0.$$

Therefore, there exists

$$A = \lim_{n\to\infty} \|x^n - x^0\|. \tag{3.14}$$

Since $x^n = \text{Pr}_{Q_n}(x^0)$ and $x^{n+1} \in Q_n$, using

$$\|\text{Pr}_{Q_n}(x) - x\|^2 \le \|x - y\|^2 - \|\text{Pr}_{Q_n}(x) - y\|^2 \ \ \forall x \in H, y \in Q_n,$$

we have

$$\|x^{n+1} - x^n\|^2 \le \|x^{n+1} - x^0\|^2 - \|x^n - x^0\|^2 \ \ \forall n \ge 0.$$

Combinating this and (3.14), we get

$$\lim_{n\to\infty} \|x^{n+1} - x^n\| = 0.$$

It proves the first apart of (b).

Since $x^{n+1} = \text{Pr}_{P_n \cap Q_n}(x^0)$, we have $x^{n+1} \in P_n$, $\|z^n - x^{n+1}\| \le \|x^n - x^{n+1}\|$ and hence

$$\|x^n - z^n\| \le \|x^n - x^{n+1}\| + \|x^{n+1} - z^n\| \le 2\|x^n - x^{n+1}\| \ \ \forall n \ge 0.$$

From $\lim_{n\to\infty} \|x^{n+1} - x^n\| = 0$, we have

$$\lim_{n\to\infty} \|x^n - z^n\| = 0.$$

This proved the second apart of (b).

From (3.12) and Lemma 3.1, it implies that

$$\begin{aligned}\|z^n - x^*\|^2 \leq& \alpha_n\|x^n - x^*\|^2 + (1-\alpha_n)\|t^n - x^*\|^2\\ \leq& \alpha_n\|x^n - x^*\|^2 + (1-\alpha_n)\{\|x^n - x^*\|^2 - (1-2\lambda_n c_1)\|x^n - y^n\|^2\\ &- (1-2\lambda_n c_2)\|t^n - y^n\|^2\}\\ \leq& \|x^n - x^*\|^2 - (1-\alpha_n)(1-2\lambda_n c_1)\|x^n - y^n\|^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|x^n - y^n\|^2 \leq& \frac{1}{(1-\alpha_n)(1-2\lambda_n c_1)}\{\|x^n - x^*\|^2 - \|z^n - x^*\|^2\}\\ =& \frac{1}{(1-\alpha_n)(1-2\lambda_n c_1)}(\|x^n - x^*\| - \|z^n - x^*\|)(\|x^n - x^*\| + \|z^n - x^*\|)\\ \leq& \frac{1}{(1-\alpha_n)(1-2\lambda_n c_1)}\|x^n - z^n\|(\|x^n - x^*\| + \|z^n - x^*\|)\end{aligned}$$

Since $\lim_{n\to\infty} \|x^n - z^n\| = 0$ and the sequences $\{x^n\}, \{z^n\}$ are bounded, we get

$$\lim_{n\to\infty} \|x^n - y^n\| = 0.$$

This proves the third apart of (b).

By similar way, we also obtain that $\lim_{n\to\infty} \|t^n - y^n\| = 0$. Then we have

$$\lim_{n\to\infty} \|x^n - t^n\| \leq \lim_{n\to\infty} \left(\|x^n - y^n\| + \|y^n - t^n\|\right) = 0,$$

and hence $\lim_{n\to\infty} \|x^n - t^n\| = 0$. This proves the last part of (b).

Using (b) and $z^n = \alpha_n x^n + (1-\alpha_n)T(t^n)$, we have

$$\begin{aligned}(1-c)\|T(t^n) - t^n\| \leq& (1-\alpha_n)\|T(t^n) - t^n\|\\ =& \|\alpha_n(t^n - x^n) + (z^n - t^n)\|\\ \leq& \alpha_n\|t^n - x^n\| + \|z^n - t^n\|,\\ \leq& (1+\alpha_n)\|t^n - x^n\| + \|z^n - t^n\|,\end{aligned}$$

and hence $\lim_{n\to\infty} \|t^n - T(t^n)\| = 0$. ∎

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Suppose that Assumptions (i)-(iv) hold and T is nonexpansive on C. Then, the sequences $\{x^n\}, \{y^n\}, \{z^n\}$ and $\{t^n\}$ generated by Algorithm 2.1 converge strongly to the same point x^*, where*

$$x^* = \mathrm{Pr}_{\mathrm{Sol}(f,C)\cap \mathrm{Fix}(T)}(x^0).$$

Proof. Since $\{x^n\}$ is bounded, there exists a subsequence $\{x^{n_j}\}$ of $\{x^n\}$ such that $\{x^{n_j}\}$ converges weakly to some $\bar{x}$ as $j \to \infty$. Then, it follows from (b) of Lemma 3.2 that $\{t^{n_j}\}$ also converges weakly to some $\bar{x}$ as $j \to \infty$. We can obtain that $\bar{x} \in \mathrm{Sol}(f,C) \cap \mathrm{Fix}(T)$. First, we show $\bar{x} \in \mathrm{Fix}(T)$. Assume that $\bar{x} \notin Fix(T)$. Since Opial's condition in [6], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup \bar{x}$ the inequality

$$\liminf_{n\to\infty} \|x_n - \bar{x}\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq \bar{x}$, we have

$$\begin{aligned}\liminf_{j\to\infty} \|t^{n_j} - \bar{x}\| &< \liminf_{j\to\infty} \|t^{n_j} - T(\bar{x})\| \\ &\leq \liminf_{j\to\infty} (\|t^{n_j} - T(t^{n_j})\| + \|T(t^{n_j}) - T(\bar{x})\|) \\ &= \liminf_{j\to\infty} \|T(t^{n_j}) - T(\bar{x})\| \\ &\leq \liminf_{j\to\infty} \|t^{n_j} - \bar{x}\|.\end{aligned}$$

This is contradiction. Thus, $\bar{x} = T(\bar{x})$.

From (b) of Lemma 3.2 and $x^{n_j} \rightharpoonup \bar{x}$ as $j \to \infty$, it follows

$$y^{n_j} \rightharpoonup \bar{x}, t^{n_j} \rightharpoonup \bar{x} \ \text{ as } j \to \infty.$$

Then, using (3.10), $\{\lambda_n\} \subset [a,b] \subset (0,1)$ and assumptions of f, we have

$$\lambda_{n_j}\{f(x^{n_j}, y) - f(x^{n_j}, y^{n_j})\} \geq \langle y^{n_j} - x^{n_j}, y^{n_j} - y\rangle \ \ \forall y \in C.$$

As $j \to \infty$, we get $f(\bar{x}, y) \geq 0$ for all $y \in C$. It means that $\bar{x} \in \mathrm{Sol}(f,C)$. So, we have

$$\bar{x} \in \mathrm{Sol}(f,C) \cap \mathrm{Fix}(T).$$

Since $x^* = \mathrm{Pr}_{\mathrm{Sol}(f,C)\cap\mathrm{Fix}(T)}(x^0)$, $\bar{x} \in \mathrm{Sol}(f,C) \cap \mathrm{Fix}(T)$ and (3.13), we have

$$\|x^* - x^0\| \leq \|\bar{x} - x^0\| \leq \liminf_{j\to\infty} \|x^{n_j} - x^0\| \leq \limsup_{j\to\infty} \|x^{n_j} - x^0\| \leq \|x^* - x^0\|. \tag{3.15}$$

So, we get

$$\lim_{j\to\infty} \|x^{n_j} - x^0\| = \|\bar{x} - x^0\|.$$

Since $x^{n_j} - x^0$ converges weakly to $\bar{x} - x^0$ as $j \to \infty$, we have $x^{n_j} - x^0$ converges strongly to $\bar{x} - x^0$ as $j \to \infty$. By $x^n = \mathrm{Pr}_{Q_n}(x^0)$ and $x^* \in \mathrm{Sol}(f,C) \cap \mathrm{Fix}(T) \subset P_n \cap Q_n \subset Q_n$, we have

$$\langle x^* - x^{n_j}, x^0 - x^*\rangle \leq \langle x^* - x^{n_j}, x^0 - x^*\rangle + \langle x^* - x^{n_j}, x^{n_j} - x^0\rangle = -\|x^* - x^{n_j}\|^2.$$

As $j \to \infty$, we have

$$\langle x^* - \bar{x}, x^0 - x^*\rangle \leq -\|x^* - \bar{x}\|^2.$$

Combinating this, $\bar{x} \in \mathrm{Sol}(f,C) \cap \mathrm{Fix}(T)$, $\langle x^* - \bar{x}, x^0 - x^*\rangle \geq 0$ and $x^* = \mathrm{Pr}_{\mathrm{Sol}(f,C)\cap\mathrm{Fix}(T)}(x^0)$, we obtain $\bar{x} = x^*$. This implies that $\lim_{n\to\infty} \|x^n - x^*\| = 0$. From (b) of Lemma 3.2, it follows $\lim_{n\to\infty} \|y^n - x^*\| = 0$ and $\lim_{n\to\infty} \|t^n - x^*\| = 0$. ∎

4. Applications

In this section, we discuss about two applications of Theorem 3.1 to find a common point of the set of fixed points of a nonexpansive mapping and the set of solutions of variational inequality problems for a monotone, Lipschitz continuous mapping.

Let C is a nonempty closed convex subset of a real Hilbert space H, for each pair $x,y \in C$,

$$f(x,y) := \langle F(x), y-x\rangle,$$

where $F : C \to H$.

In Algorithm 2.1, the subproblems needed to solve at Step 1 are of the form

$$\begin{cases} y^n = \text{argmin}\{\frac{1}{2}\|y-x^n\|^2 + \lambda_n \langle F(x^n), y-x^n\rangle : \ y \in C\}, \\ t^n = \text{argmin}\{\frac{1}{2}\|t-x^n\|^2 + \lambda_n \langle F(y^n), t-y^n\rangle : \ t \in C\}. \end{cases}$$

Hence, we have

$$\begin{cases} y^n = \text{argmin}\{\frac{1}{2}\|y-\big(x^n-\lambda_n F(x^n)\big)\|^2 : \ y \in C\} = \text{Pr}_C\big(x^n-\lambda_n F(x^n)\big), \\ t^n = \text{argmin}\{\frac{1}{2}\|t-\big(x^n-\lambda_n F(y^n)\big)\|^2 : \ t \in C\} = \text{Pr}_C\big(x^n-\lambda_n F(y^n)\big). \end{cases}$$

Thus, in this case Algorithm 2.1 and its convergence become the following results:

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \to H$ be a monotone, L-Lipschitz continuous mapping and $T : C \to C$ be a nonexpansive mapping such that* $\text{Fix}(T) \cap \text{Sol}(F,C) \neq \emptyset$*. Let $\{x^n\}, \{y^n\}$ and $\{z^n\}$ be the sequences generated by*

$$\begin{cases} x^0 \in C, \\ y^n = \text{Pr}_C\big(x^n-\lambda_n F(x^n)\big), \\ t^n = \text{Pr}_C\big(x^n-\lambda_n F(y^n)\big), \\ z^n = \alpha_n x^n + (1-\alpha_n)T(t^n), \\ P_n = \{z \in C : \ \|z^n-z\| \le \|x^n-z\|\}, \\ Q_n = \{z \in C : \ \langle x^n-z, x^0-x^n\rangle \ge 0\}, \\ x^{n+1} = \text{Pr}_{P_n\cap Q_n}(x^0), \end{cases}$$

for every $n \ge 0$, where $\{\lambda_n\} \subset [a,b]$ for some $a,b \in (0,\frac{1}{L})$ and $\{\alpha_n\} \subset [0,c]$ for some $c \in$ $[0,1)$. *Then the sequences $\{x^n\}, \{y^n\}$ and $\{z^n\}$ converge strongly to* $\text{Pr}_{Sols(F,C)\cap \text{Fix}(T)}(x^0)$.

Using Theorem 4.1, we prove the the following theorem proposed by Nakajo and Takahashi.

Theorem 4.2. [9] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping such that* $\text{Fix}(T) \neq \emptyset$*. Let $\{x^n\}$ and $\{y^n\}$ be the sequences generated by*

$$\begin{cases} x^0 \in C, \\ y^n = \alpha_n x^n + (1-\alpha_n)T\{\text{Pr}_C(x^n)\}, \\ P_n = \{z \in C : \ \|z^n-z\| \le \|x^n-z\|\}, \\ Q_n = \{z \in C : \ \langle x^n-z, x-x^n\rangle \ge 0\}, \\ x^{n+1} = \text{Pr}_{P_n\cap Q_n}(x^0), \end{cases}$$

for every $n \geq 0$*, where* $\{\alpha_n\} \subset [0, c]$ *for some* $c \in [0, 1)$*. Then, the sequences* $\{x^n\}$ *and* $\{y^n\}$ *converge strongly to* $\mathrm{Pr}_{\mathrm{Fix}(T)}(x^0)$.

Proof. For $f = 0$, by Theorem 4.1, we have the desired results. ∎

References

[1] P. N. Anh, A logarithmic quadratic regularization method for pseudomonotone equilibrium problems, *Acta Math. Vietnam.* **34** (2009), no. 2, 183–200.

[2] P. N. Anh, An LQ regularization method for pseudomonotone equilibrium problems on polyhedra, *Vietnam J. Math.* **36** (2008), no. 2, 209–228.

[3] P. N. Anh, L. D. Muu, V. H. Nguyen and J. J. Strodiot, Using the Banach contraction principle to implement the proximal point method for multivalued monotone variational inequalities, *J. Optim. Theory Appl.* **124** (2005), no. 2, 285–306.

[4] E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Student* **63** (1994), no. 1–4, 123–145.

[5] P. Daniele, F. Giannessi and A. Maugeri, *Equilibrium Problems and Variational Models*, Nonconvex Optimization and its Applications, 68, Kluwer Acad. Publ., Norwell, MA, 2003.

[6] K. Geobel and W. A. Kirk, *Topics on Metric Fixed Point Theory*, Cambridge University Press, Cambridge, England, 1990.

[7] G. M. Korpelevič, An extragradient method for finding saddle points and for other problems, *Èkonom. i Mat. Metody* **12** (1976), no. 4, 747–756.

[8] N. Nadezhkina and W. Takahashi, Strong convergence theorem by a hybrid method for nonexpansive mappings and Lipschitz-continuous monotone mappings, *SIAM J. Optim.* **16** (2006), no. 4, 1230–1241 (electronic).

[9] K. Nakajo and W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, *J. Math. Anal. Appl.* **279** (2003), no. 2, 372–379.

[10] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.* **73** (1967), 591–597.

[11] J. W. Peng, Iterative algorithms for mixed equilibrium problems, strict pseudocontractions and monotone mappings, *J. Optim. Theory Appl.* **144** (2010), no. 1, 107–119.

[12] S. Takahashi and W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, *J. Math. Anal. Appl.* **331** (2007), no. 1, 506–515.

[13] W. Takahashi and M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, *J. Optim. Theory Appl.* **118** (2003), no. 2, 417–428.

[14] S. Wang and B. Guo, New iterative scheme with nonexpansive mappings for equilibrium problems and variational inequality problems in Hilbert spaces, *J. Comput. Appl. Math.* **233** (2010), no. 10, 2620–2630.

[15] H. K. Xu and T. H. Kim, Convergence of hybrid steepest-descent methods for variational inequalities, *J. Optim. Theory Appl.* **119** (2003), no. 1, 185–201.

[16] Y. Yao, Y.-C. Liou and Y.-J. Wu, An extragradient method for mixed equilibrium problems and fixed point problems, *Fixed Point Theory Appl.* **2009**, Art. ID 632819, 15 pp.

[17] Y. Yao, Y.-C. Liou and J.-C. Yao, An extragradient method for fixed point problems and variational inequality problems, *J. Inequal. Appl.* **2007**, Art. ID 38752, 12 pp.

[18] L.-C. Zeng and J.-C. Yao, Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems, *Taiwanese J. Math.* **10** (2006), no. 5, 1293–1303.